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Large Deviations for Lacunary Systems

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Abstract—Let X_i be a Lacunary System, we established large deviations inequality for Lacunary System. Furthermore, we gained Marcinkiewicz Larger Number Law with dependent random variables sequences.

Keywords—Lacunary system, larger deviations, Locally Generalized Gaussian, Strong law of large numbers.

I. Introduction

LACUNARY systems is a class of random variables. Lai and Wei[1] gave independent and identically distributed random variable, martingale differences with L_p bound are Lacunary Systems. Li [2] obtained L_p bounded dependent is a Lacunary System in 1997.

In this paper, we shall establish large deviations inequality for Lacunary System. Further, we shall get Marcinkiewicz Strong law of large numbers with $m-$dependent random variables sequences.

We give defined of Lacunary system as follows:

Definition 1.1 Given $p > 0$, a sequence of real-valued random variables $\{X_n,\ n \geq 1\}$ is called a Lacunary System or an S_p system, if there exists a positive constant K_p such that

$$E|\sum_{i=m}^{n} C_i X_i|^p \leq K_p(\sum_{i=m}^{n} C_i^2)^{p/2} \quad (1)$$

for any sequence of real constant $\{C_i\}$ and all $n \geq m$.

Definition 1.2 Suppose that $\{X_n,\ n \geq 1\}$ is a sequence of random variables on a probability space $(\Omega, \mathscr{F}, P)$, set $\mathscr{F}_a^b = \sigma(X_k, a \leq k \leq b)$. Denote by the σ-field generated by the random variables $X_a, X_{a+1}, \ldots, X_b$.

1) Let $A \in \mathscr{F}_1^k, B \in \mathscr{F}_{k+n}^{\infty}$ and $k, n \geq 1$,$\{X_n, n \geq 1\}$ is called $\phi - mixing$ if

$$|P(A \cap B) - P(A)P(B)| \leq \phi(n)P(A)$$

for some $\phi(n) \downarrow 0$.

2) $\{X_n,\ n \geq 1\}$ is called $\psi - mixing$, if

$$\psi(n) = \sup_{k \in \mathbf{N}} \psi(\mathscr{F}_1^k, \mathscr{F}_{k+n}^{\infty}) \to 0, n \to \infty,$$

where

$$\psi(\mathscr{A}, \mathscr{B}) = \sup_{A \in \mathscr{A}, B \in \mathscr{B}, P(A)P(B) > 0} \frac{|P(A \cap B) - P(A)P(B)|}{P(A)P(B)}.$$

Definition 1.3 Let X be a real-valued random variable, we call a Locally Generalized Gaussian, If there exists $\alpha > 0$such that

$$E(\exp(ux)|\mathscr{F}) \leq \exp(u^2\alpha^2/2) \qquad a.s. \quad (2)$$

for any $u \in R$.

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II. Larger Deviations Inequality

In order to prove larger deviations we need the following lemmas.

Lemma 2.1 Let X_n be a zero-mean $\phi - mixing$ and $\sum_{i=1}^{\infty} \phi^{1/2}(i) < \infty$, for some $p \geq 2$, $\sup_i E|X_i|^p < \infty$. There exists constant $c > 0$ depending only on p for any real-valued sequence $\{a_{ni}\}$, such that

$$E|\sum_{i=1}^{n} a_{ni} X_i|^p \leq c(\sum_{i=1}^{n} a_{ni}^2)^{p/2}. \quad (3)$$

proof Let $a_{ni} = 0,\ i > n$, since $\sum_{i=1}^{\infty} \phi^{1/2}(i) < \infty$, $\sup_i E|X_i|^p < \infty$, from the proof in [3], we have

$$E|\sum_{i=k+1}^{k+m} a_{ni} X_i|^2 \leq c_1 \sum_{i=k+1}^{k+m} a_{ni}^2 \leq c_1 \sum_{i=1}^{n} a_{ni}^2,$$

for any $k \geq 0, n \geq 1, m \leq n$.

Using the corollary 2.1 in [4], we obtain

$$\begin{aligned} E|\sum_{i=1}^{n} a_{ni} X_i|^p &\leq c_2(\sum_{i=1}^{n} E|a_{ni} X_i|^p + (\sum_{i=1}^{n} a_{ni}^2)^{p/2}) \\ &\leq c_3(\sum_{i=1}^{n} |a_{ni}|^p + (\sum_{i=1}^{n} a_{ni}^2)^{p/2}). \quad (4) \end{aligned}$$

Since $p \geq 2$, it follows that

$$(\sum_{i=1}^{n} |a_{ni}|^p)^{1/p} \leq (\sum_{i=1}^{n} |a_{ni}|^2)^{1/2} \Leftrightarrow \sum_{i=1}^{n} |a_{ni}|^p \leq (\sum_{i=1}^{n} |a_{ni}|^2)^{p/2}.$$

Then, we educe (3) from (4).

Remark 1 Lemma 2.1 implies that $\phi - mixing$ is a Lacunary System. If $a_{ni} \equiv 1$, we have $E|\sum_{i=1}^{n} X_i|^p \leq cn^{p/2}$.

Lemma 2.2 If $\{X_n, n \geq 1\}$ is a zero-mean $\psi - mixing$, such that

$$\sum_{i=1}^{\infty} \psi(i) < \infty,\ E|X_i|^p,\ p \geq 2,$$

then for any real-valued sequence a_{ni}, (3) holds.

Proof From lemma 2.1 and the proof in [5], we can obtain lemma 2.2.

Theorem 2.1 Let $\{X_n, n \geq 1\}$ be a Lacunary System, for any $p > 1, x > 0$, sequence of real constant $\{C_i\}$, then

$$P\{|S_n| \geq nx\} \leq C(p)(\sum_{i=1}^{n} C_i^2)^{p/2} n^{-p}, \quad (5)$$

where $S_n = \sum_{i=1}^{n} C_i X_i$, $C_p = K_p/x^p$.

Proof Since $\{X_n, n \geq 1\}$ is a Lacunary System, we have

$$E|S_n|^p \leq K_p(\sum_{i=1}^{n} C_i^2)^{p/2}.$$

By using Markov's inequality,

$$P\{|S_n| \geq nx\} \leq \frac{E|S_n|^p}{(nx)^p}$$

for every $p > 1$, we can obtain (5).

Remark 2 (1) If $\sum_{i=1}^{n} C_i^2 = O(n)$, we have

$$E|S_n|^p \leq C(p)n^{-p/2}.$$

(2)If $C_i \equiv 1, p > 2$, by Borel-Cantelli lemma:

$$\sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} X_i| \geq nx) \leq \sum_{n=1}^{\infty} C(p)n^{-p/2} < \infty,$$

then

$$\lim_{n\to\infty} P(|\sum_{i=1}^{n} X_i| \geq nx) = 0. \quad a.s.,$$

Theorem 2.2 Let $(X_n, \mathcal{F}_n)$ be a Locally Generalized Gaussian sequence, if $\sup_n X_n = k < \infty$, then (5) holds for any $p \geq 2, x \geq 0$.

Proof Theorem 2.2 holds if only we can prove that Locally Generalized Gaussian sequence is a Lacunary System. Let $A_n = \sum_{i=m}^{n} C_i^2, u = x/k^2 A_n$, by lemma 1 in [6], then

$$\begin{aligned} E(\exp(u\sum_{i=m}^{n} C_i X_i)) &= E(\exp(u(S_n - S_{m-1}))) \\ &\leq \exp(u^2 k^2 A_n/2), \end{aligned} \tag{6}$$

where $S_n = \sum_{i=1}^{n} C_i X_i$. Since

$$P(\{|S_n - S_{m-1}| > x\}) \leq 2\exp\ (-x^2/2k^2 A_n)$$

for $p \geq 2$, by Chebyshev's inequality, we get

$$\begin{aligned} E|\sum_{i=m}^{n} C_i X_i|^p &= p\int_0^{\infty} x^{p-1} P(|S_n - S_{m-1}| > x)\mathrm{d}x \\ &\leq 2p\int_0^{\infty} x^{p-1}\exp(-x^2/2k^2 A_n)\mathrm{d}x \\ &= 2^{p/2} p k^p A_n^{p/2} \int_0^{\infty} x^{p/2-1} e^{-x}\mathrm{d}x \\ &= K_p(\sum_{i=m}^{n} C_i^2)^{p/2}. \end{aligned}$$

where $K_p = p2^{p/2}k^p \int_0^{\infty} x^{p/2-1}e^{-x}\mathrm{d}x$.

III. THE STRONG LAW OF LARGER NUMBERS

Theorem 3.1 Assume that $\{X_n, n \geq 1\}$ is a zero-mean $\psi - mixing$, such that

$$\sum_{i=1}^{\infty} \psi(i) < \infty,\ E|X_i|^p, \text{for } p \geq 2.$$

If there exists $1/2 < r \leq 1, \theta = 2r - 1$ and positive constant K such that $\sum_{i=1}^{n} a_{ni}^2 \leq Kn^{\theta},\ i = 1, 2, \ldots, n$, then

$$\frac{\sum_{i=1}^{n} a_{ni} X_i}{n^r} \to 0, \quad a.s.. \tag{7}$$

Proof Denote $\sum_{i=1}^{n} a_{ni} X_i$, by Markov's inequality, we have

$$P(|S_n| \geq n^r \varepsilon) \leq \frac{E(|S_n|^p)}{\varepsilon^p n^{pr}}. \tag{8}$$

From lemma 1.2 and (8), we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} a_{ni} X_i/n^r| \geq \varepsilon) = \sum_{n=1}^{\infty} P(|S_n| \geq \varepsilon n^r) \\ &\leq \sum_{n=1}^{\infty} \tfrac{E(|S_n|^p)}{\varepsilon^p n^{pr}} \leq \sum_{n=1}^{\infty} \frac{c(\sum_{i=1}^{n} a_{ni}^2)^{p/2}}{\varepsilon^p n^{pr}} \\ &\leq \sum_{n=1}^{\infty} \tfrac{cKn^{p\theta/2}}{\varepsilon^p n^{pr}} < \infty. \end{aligned}$$

(3.1) follows from Borel-Cantelli lemma.

Remark 3. This result extends independent and identically distributed Marcinkiewicz Law of large numbers for $\psi - mixing$.

Theorem 3.2 Let $\{X_n\}$ be a zero-mean $\phi - mixing$, and $\sum_{i=1}^{\infty} \phi^{1/2}(i) < \infty,\ \sup_i E|X_i|^p < \infty$ for some $p > 2$. If there exists $1/2 < r \leq 1, \theta = 1 - 2/p$ and positive constant K such that $\sum_{i=1}^{n} a_{ni}^2 \leq Kn^{\theta},\ i = 1, 2, \ldots, n$, then

$$\frac{\sum_{i=1}^{n} a_{ni} X_i}{\sqrt{n\ln n}} \to 0, \quad a.s.. \tag{9}$$

Proof By lemma 2.1 and (8), we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} P(|\sum_{i=1}^{n} a_{ni} X_i/\sqrt{n\ln n}| \geq \varepsilon) \\ &= \sum_{n=1}^{\infty} P(|S_n| \geq \varepsilon\sqrt{n\ln n}) \\ &\leq \sum_{n=1}^{\infty} \tfrac{E(|S_n|^p)}{\varepsilon^p n^{p/2}(\ln n)^{p/2}} \\ &\leq \sum_{n=1}^{\infty} \frac{c(\sum_{i=1}^{n} a_{ni}^2)^{p/2}}{\varepsilon^p n^{p/2}(\ln n)^{p/2}} \\ &\leq \sum_{n=1}^{\infty} \tfrac{cKn^{p\theta/2}}{\varepsilon^p n^{p/2}(\ln n)^{p/2}} \\ &= \sum_{n=1}^{\infty} \tfrac{cK}{\varepsilon^p n(\ln n)^{p/2}} < \infty. \end{aligned}$$

And then, (9) follows from Borel-Cantelli lemma.

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