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Relational Framework and its Applications

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Abstract—This paper has, as its point of departure, the foundational axiomatic theory of E. De Giorgi (1996, Scuola Normale Superiore di Pisa, Preprints di Matematica 26, 1), based on two primitive notions of quality and relation. With the introduction of a unary relation, we develop a system totally based on the sole primitive notion of relation. Such a modification enables a definition of the concept of dynamic unary relation. In this way we construct a simple language capable to express other well known theories such as Robinson's arithmetic or a piece of a theory of concatenation. A key role in this system plays an abstract relation designated by "( )", which can be interpreted in different ways, but in this paper we will focus on the case when we can perform computations and obtain results.

Keywords—language, unary relations, arithmetic, computability

I. Introduction

Mathematics, although known as exact science, is full of uncertainties. Gödel's theorems have demonstrated that these uncertainties are indeed proper to mathematics itself. Logicians, philosophers, and other scholars have begun to re-examine the founding principles of logic and to propose new solutions. It is interesting to note that Edmund Husserl [10], whose writings Gödel repeatedly recommended for study, was also a precursor in this line of thought. Husserl, who before becoming a philosopher had studied mathematics as a student of Weierstrass and had been an interlocutor of Cantor and Frege, proposed a radical reform of logic for the new model of the world emerging from science.

Since mathematics is a language composed of primitive concepts, rules, definitions, symbols, etc.., we can encode it by the use of other symbols, as well. We define everything in frames of mathematics in terms of sets and membership relation. In this paper we would like to encode some of the well known concepts in terms of a primitive relation "( )" [6], [2], [4]. Hence, we propose a kind of a calculus on unary relations. The most important thing is how we interpret the main operator of a system – "( )". Let us begin with several examples.

Examples:

1) Let "( )" indicate a binary inclusion relation — "$\subseteq$", and x, y are unary relations. $(x\,y) \equiv x \subseteq y$
2) Let "( )" indicate a binary operation of intersection — "$\cap$" ($\cap$ is a ternary relation). $(x\,y) \equiv x \cap y$. $x \cap y = y$ would mean that $y \subseteq x$.
3) Let "( )" be a projection function — p (p is a binary relation). $(x\,y) \equiv p[x, y] = y$
4) Let "( )" be a binary operation of addition — "$+$" ($+$ is a ternary relation). $(x\,y) \equiv x + y$
5) Let "( )" be an identity relation — I (I is a binary relation). $(x\,y) \equiv xIy$ iff $x = y$
6) Let "( )" be a concatenation relation — "$\frown$" ($\frown$ is a binary operation, i.e. a ternary relation). $(x\,y) \equiv x \frown y$
7) Let "( )" be an equivalence relation — "$\cong$" ($\cong$ is a binary relation). $(x\,y) \equiv x \cong y$.
8) Finally when x is considered a quality and y stands for any object, $(x\,y)$ is read "an object x has a quality q".

The presented notation allows us to introduce a primitive relation "( )" of any arity and of any nature. Without the lost of generality, we can assume that we do not know what is the internal mechanism which puts together objects. We can only assume that $(x\,y)$ "creates" a new object, in a sense that $(x\,y)$ is a whole which turns out to consist of two related entities x and y.

In this paper we will focus on the case when "( )" can act as an operation; we can perform computations and obtain results. We propose two simple equations which will stand at the basis of our system in a way to describe interrelationship between relations. In the first part, we will explore a model composed of three particular relations and show that an extensional property of any operation interpreted as "( )" is derivable from axioms of that model. In the second part, we will adopt a system in a way to interpret other standard well-known theories, such as Robinson's arithmetic *Q* and a piece of a theory of concatenation. Finally, we will discuss obtained results.

II. Calculus on Relations

A. Basic Definitions

We define a language of Calculus of Relations *CR* in which well formed formulas are those formed from atomic constants, i. e., the logical operators $\forall$, $\exists$, $\wedge$, $\vee$, $\neg$, $\Longrightarrow$, $\Longleftrightarrow$ and $=$, parentheses [ ], { } and variables, possibly connected by means of application ( ).

The nature of the objects remains explicitly open, as long as these objects are capable of being in relation with other objects in accordance with the Association Rule to be defined in Axiom 2.1. The objects themselves can even be relations.

Definition 2.1: The language of calculus of relations [*CR*] terms is built from an infinite number of variables: x, y, z,... using the application operator ( ) as follows:

1) If x is a variable, then x is a *CR* term,
2) if x, y are variables, then $(x\,y)$ is a *CR* term,
3) if x is a variable and *M* is a *CR* term, then $(M\,x)$ and $(x\,M)$ are *CR* terms.

The application operator obeys the law of associativity:

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Axiom 2.1: Law of Associativity:
$$\forall\, a, b, c : (a\,b\,c) \equiv ((a\,b)\,c) = (a\,(b\,c))$$

As stated above, the application operator allows us to consider $(a\,b\,c)$ as a single object without constraining the view of its inner structure. Hence, one can either find $(a\,b)$ in relation with c or a in relation with $(b\,c)$.

Furthermore, we will use the classical *Deduction Rules* and substitution property for equality "$=$":

Axiom 2.2: $\forall\, p, q, r : [p = p]$,
$$\begin{aligned} &[p = q] \Longrightarrow [q = p], \\ &\{[p = q] \wedge [q = r]\} \Longrightarrow [p = r] \\ &[p = q] \Longrightarrow [(p\,r) = (q\,r)] \\ &[p = q] \Longrightarrow [(r\,p) = (r\,q)] \end{aligned}$$

*B. A Ternary Model (*TM*)*

We start with a set of three axioms for specific variables $\overline{x}$, $\overline{y}$, $\overline{z}$, which will act as constants of our system:

Axiom 2.3: $\overline{x} \neq \overline{y}$, $\overline{y} \neq \overline{z}$, $\overline{z} \neq \overline{x}$.

Axiom 2.4: $(\overline{x}\,\overline{y}) = \overline{y}$

Axiom 2.5: $(\overline{z}\,\overline{y}) = \overline{x}$

Axiom 2.3 assures the existence of three distinct relations, Axioms 2.4 and 2.5 describe the relationship among $\overline{x}, \overline{y}$ and $\overline{z}$. Axiom 2.4 can be understood as a Distinction Rule (object $\overline{y}$ is separated from object $\overline{x}$) or as a kind of relation under which $\overline{y}$ remains invariant. Axiom 2.5 describes the process of returning to $\overline{x}$: $\overline{y}$ returns to $\overline{x}$ via $\overline{z}$.

Definition 2.2: A Ternary Model *TM* is a model composed of three specific relations *TM* $= [\overline{x}, \overline{y}, \overline{z}]$ satisfying Axioms 2.1, 2.2, 2.3, 2.4 and 2.5

Adding a symmetry condition on $\overline{z}$: $[(\overline{y}\,\overline{z}) = (\overline{z}\,\overline{y})]$ leads to symmetric behavior of the entire *TM* model, as the following lemmas show:

Lemma 2.1: $[(\overline{y}\,\overline{z}) = (\overline{z}\,\overline{y})] \Longrightarrow [(\overline{x}\,\overline{z}) = (\overline{z}\,\overline{x})]$

Proof: $(\overline{x}\,\overline{z}) =_{\text{(Ax 2.5)}} ((\overline{z}\,\overline{y})\,\overline{z}) =_{\text{(Ax 2.1)}} (\overline{z}\,(\overline{y}\,\overline{z})) = (\overline{z}\,(\overline{z}\,\overline{y})) =_{\text{(Ax 2.5)}} (\overline{z}\,\overline{x})$ ∎

Lemma 2.2: $[(\overline{y}\,\overline{z}) = (\overline{z}\,\overline{y})] \Longrightarrow [(\overline{x}\,\overline{y}) = (\overline{y}\,\overline{x})]$

Proof: $(\overline{x}\,\overline{y}) =_{\text{(Ax 2.5)}} ((\overline{z}\,\overline{y})\,\overline{y}) = ((\overline{y}\,\overline{z})\,\overline{y}) =_{\text{(Ax 2.1)}} (\overline{y}\,(\overline{z}\,\overline{y})) =_{\text{(Ax 2.5)}} (\overline{y}\,\overline{x})$ ∎

Lemma 2.3: $[(\overline{y}\,\overline{z}) = (\overline{z}\,\overline{y})] \Longrightarrow [(\overline{x}\,\overline{x}) = \overline{x}]$

Proof: $(\overline{x}\,\overline{x}) =_{\text{(Ax 2.5)}} (\overline{x}\,(\overline{z}\,\overline{y})) =_{\text{(Ax 2.1)}} ((\overline{x}\,\overline{z})\,\overline{y}) =_{\text{(Lm 2.1)}} ((\overline{z}\,\overline{x})\,\overline{y}) =_{\text{(Ax 2.1)}} (\overline{z}\,(\overline{x}\,\overline{y})) =_{\text{(Ax 2.4)}} (\overline{z}\,\overline{y}) =_{\text{(Ax 2.5)}} \overline{x}$ ∎

Lemma 2.3 provides a useful extension to Axiom 2.4.

Because the symmetry condition $[(\overline{y}\,\overline{z}) = (\overline{z}\,\overline{y})]$ is not a theorem of *TM*, we add it in another axiom:

Axiom 2.6: $(\overline{y}\,\overline{z}) = (\overline{z}\,\overline{y})$

Definition 2.3: A Ternary Model with Symmetry *TMS* is a *TM* model with Axiom 2.6.

*C. A Model of Dynamic Generative System (*DGS*)*

We are now going to subject *TM* to certain modifications, introducing other variables. Let us assume that Axioms 2.4 and 2.5 are true for any variable y.

Definition 2.4: A Dynamic Generative System *DGS*$=[\overline{x}, y, z]$ is a model satisfying Axioms 2.1, 2.2, 2.7 and 2.8.

Axiom 2.7: $\forall\, y : (\overline{x}\,y) = y$

Axiom 2.8: $\forall\, y\, \exists\, z : (z\,y) = \overline{x}$

Note that Axiom 2.7 and Lemma 2.4 do not exclude the possibility of $r = \overline{x}$. Hence, we get $(\overline{x}\,\overline{x}) = \overline{x}$ by Axiom 2.7.

As in the case of *TM*, we can add a similar symmetry condition to *DGS*.

Axiom 2.9: $\forall\, y\, \exists\, z : (z\,y) = (y\,z) = \overline{x}$

As a result:

Lemma 2.4: $\forall\, r : (r\,\overline{x}) = r$

Proof: $\forall\, y : (y\,\overline{x}) =_{\text{(Ax 2.8)}} (y\,(z\,y)) =_{\text{(Ax 2.1)}} ((y\,z)\,y) =_{\text{(Ax 2.9)}} (\overline{x}\,y) =_{\text{(Ax 2.7)}} y$. ∎

Definition 2.5: A Dynamic Generative System with Symmetry (*DGSS*) is a *DGS* model with Axiom 2.9.

Now two lemmas will show that the symmetry condition on y causes the uniqueness of z in Axiom 2.9.

Lemma 2.5: .
$$\forall\, x, y : [x = y] \Longrightarrow \exists\, s, t : [(s\,x) = (t\,y) = \overline{x}] \wedge [s = t]$$

Proof: The existence of s and t is assured by Axiom 2.8:
$\Longrightarrow_{\text{(Ax 2.8)}} \exists\, s : \overline{x} = (s\,x)$ and $\overline{x} = (s\,y)$
$\Longrightarrow_{\text{(Ax 2.8)}} \exists\, t : \overline{x} = (t\,y)$ and $\overline{x} = (t\,x)$

Furthermore, Axiom 2.8 guarantees the equivalence of $(s\,x)$ and $(t\,y)$:
$(s\,y) = (s\,x) =_{\text{(Ax 2.8)}} \overline{x} =_{\text{(Ax 2.8)}} (t\,y) = (t\,x)$

Finally, we prove the equivalence of s and t:
$s =_{\text{(Lm 2.4)}} (s\,\overline{x}) = (s\,(t\,y)) =_{\text{(Ax 2.9)}} (s\,(y\,t)) =_{\text{(Ax 2.1)}} ((s\,y)\,t) = (\overline{x}\,t) =_{\text{(Ax 2.7)}} t$ ∎

Lemma 2.6: .
$$\forall\, x, y : [x = y] \Longrightarrow \exists\, s, t : [(x\,s) = (y\,t) = \overline{x}] \wedge [s = t]$$

Proof: The existence of s and t is assured by Axiom 2.8:
$\Longrightarrow_{\text{(Ax 2.8)}} \exists\, s : \overline{x} = (s\,x)$ and $\overline{x} = (s\,y)$
$\Longrightarrow_{\text{(Ax 2.8)}} \exists\, t : \overline{x} = (t\,y)$ and $\overline{x} = (t\,x)$

Furthermore, Axiom 2.8 guarantees the equivalence of $(s\,x)$ and $(t\,y)$:
$(x\,s) =_{\text{(Ax 2.9)}} (s\,x) =_{\text{(Ax 2.8)}} \overline{x} =_{\text{(Ax 2.8)}} (t\,y) =_{\text{(Ax 2.9)}} (y\,t)$

$s =_{\text{(Lm 2.4)}} (s\,\overline{x}) = (s\,(y\,t)) =_{\text{(Ax 2.1)}} ((s\,y)\,t) = (\overline{x}\,t) =_{\text{(Ax 2.7)}} t$ ∎

Corollary 2.1: Axiom 2.9 assures the uniqueness of z in Axiom 2.8: $\forall\, y\, \dot{\exists}\, z : [(z\,y) = \overline{x}]$.

Now, we will verify that for arbitrary *DGSS* terms, extensional property of equality holds. We start with two lemmas on extensionality for special *DGSS* terms:

Lemma 2.7: $\forall\, x, y, z : [(z\, x) = (z\, y) = \overline{x}] \Longrightarrow [x = y]$

Proof: $x =_{\text{(Lm 2.4)}} (x\, \overline{x}) = (x\, (z\, y)) =_{\text{(Ax 2.1)}}$
$((x\, z)\, y) =_{\text{(Ax 2.9)}} ((z\, x)\, y) = (\overline{x}\, y) =_{\text{(Ax 2.7)}} y$ ■

Analogously:

Lemma 2.8: $\forall\, x, y, z : [(x\, z) = (y\, z) = \overline{x}] \Longrightarrow [x = y]$

Proof: $[(x\, z) = (y\, z) = \overline{x}] \Longleftrightarrow_{\text{(Ax 2.9)}} [(z\, x) = (z\, y) = \overline{x}]$
$\Longrightarrow_{\text{(Lm 2.7)}} [x = y]$ ■

Finally:

Theorem 2.1: Left Extensional Property of Equality: $\forall\, x, y, z : [(z\, x) = (z\, y)] \Longrightarrow [x = y]$

Proof: $\forall\, x, y, z : [(z\, x) = (z\, y)] \Longrightarrow [x = y]$:
$[(z\, x) = (z\, y)] \Longrightarrow_{\text{(Lm 2.5)}}$
$\exists\, s : [(s\, (z\, x)) = (s\, (z\, y)) = \overline{x}]$
$[(z\, x) = (z\, y)] \Longrightarrow$
$[(s\, (z\, x)) = (s\, (z\, y)) = \overline{x}] \Longleftrightarrow_{\text{(Ax 2.1)}}$
$[((s\, z)\, x) = ((s\, z)\, y) = \overline{x}] \Longrightarrow_{\text{(Lm 2.7)}} [x = y]$ ■

Theorem 2.2: Right Extensional Property of Equality: $\forall\, x, y, z : [(x\, z) = (y\, z)] \Longrightarrow [x = y]$

Proof: $\forall\, x, y, z : [(x\, z) = (y\, z)] \Longrightarrow [x = y]$:
$[(x\, z) = (y\, z)] \Longrightarrow_{\text{(Lm 2.6)}}$
$\exists\, s : [((x\, z)\, s) = ((y\, z)\, s) = \overline{x}]$
$[(x\, z) = (y\, z)] \Longrightarrow$
$[((x\, z)\, s) = ((y\, z)\, s) = \overline{x}] \Longleftrightarrow_{\text{(Ax 2.1)}}$
$[(x\, (z\, s)) = (y\, (z\, s)) = \overline{x}] \Longrightarrow_{\text{(Lm 2.8)}} [x = y]$ ■

Summary:
A *Dynamic Generative System with symmetry* $DGSS =< R, (\ ), \overline{x} >$, where R is a set of relations, is a model satisfying the following axioms:

(G1) : $\forall\, x, y, z \in R\ [x = x]$
(G2) : $[x = y] \Longrightarrow [y = x]$
(G3) : $\{[x = y] \wedge [y = z]\} \Longrightarrow [x = z]$
(G4) : $[x = y] \Longrightarrow [(x\, z) = (y\, z)]$
(G5) : $[x = y] \Longrightarrow [(z\, x) = (z\, y)]$
(G6) : $(x\, y\, z) \equiv ((x\, y)\, z) = (x\, (y\, z))$
(G7) : $\forall\, y : (\overline{x}\, y) = (y\, \overline{x}) = y$
(G8) : $\forall\, y\, \exists\, z : (z\, y) = (y\, z) = \overline{x}$

The following statements hold for *DGSS*:

(T1) : $\forall\, y\, \dot{\exists}\, z : [(z\, y) = \overline{x}]$.
(T2) : $\forall\, x, y, z : [(z\, x) = (z\, y)] \Longrightarrow [x = y]$
(T3) : $\forall\, x, y, z : [(x\, z) = (y\, z)] \Longrightarrow [x = y]$

III. Calculus on Relations and Formalized Arithmetic of Natural Numbers

The formalized arithmetic of natural numbers in which we are interested here will be referred to as Theory *N*. The set of all constants of *N* is assumed to consist of an individual constant called *0*, a unary operation symbol **S** and two binary operation symbols "$+, \cdot$". We shall be interested in an axiomatic subtheory of *N* referred to as Theory *Q* to denote Robinson's arithmetic.

It is known that *Q* is very weak, but all its recursively axiomatizable consistent extensions are both incomplete and undecidable [11]. In *Q* the individuals are numbers that can be added and multiplied.

Now we will show that within the language of *DGSS* we can interpret natural numbers as strings of irreducible relations $\overline{y}$. Let us interpret the primitive relation "()" over any two relations x, y as a Boolean operation of addition "$+$":

Definition 3.1: Addition: $\forall\, x, y : x + y = (x\, y)$.

The axiom system of *DGSS*, which we will call a theory *M*, adopted for "$+$" will consist of the following eight axioms:

(M1) : $\forall\, x, y, z :\ [x = x]$
(M2) : $[x = y] \Longrightarrow [y = x]$
(M3) : $\{[x = y] \wedge [y = z]\} \Longrightarrow [x = z]$
(M4) : $[x = y] \Longrightarrow [x + z = y + z]$
(M5) : $[x = y] \Longrightarrow [z + x = z + y]$
(M6) : $x + y + z = [x + y] + z = x + [y + z]$
(M7) : $\forall\, y : \overline{x} + y = y + \overline{x} = y$
(M8) : $\forall\, y\, \dot{\exists}\, z : z + y = y + z = \overline{x}$

Axioms (M6) — (M8) are the axioms of a group where (M7) defines the number $\overline{x} = 0$.

Let us define natural numbers as terms of the form: $\overline{y}, (\overline{y}\, \overline{y}), (\overline{y}\, \overline{y}\, \overline{y})$. Since such terms are irreducible, they can naturally be interpreted as numbers [8]. Moreover let us interpret a unary operation **S** over any number y as an expression of the form $(\overline{y}\, y)$. Observe that $\mathbf{S}y \equiv (\overline{y}\, y) = \overline{y} + y$. Let us set $\overline{x} = 0$ and $\overline{y} = 1$. $(\overline{y}\, \overline{y}) = \overline{y} + \overline{y} = 1 + 1 = 2$. As a result any natural number $n = \underbrace{(\overline{y}\, \overline{y} \ldots \overline{y})}_{n}$.

Definition 3.2: We will call a natural number any term of the form: $\overline{y}, (\overline{y}\, \overline{y}), (\overline{y}\, \overline{y}\, \overline{y}), ..., \underbrace{(\overline{y}\, \overline{y} \ldots \overline{y})}_{n-times} \equiv n$.

The axiom system of *Q* with the usual meaning of "$+, \cdot$" and 1 adopted at the place of **S** and x, y interpreted as natural numbers, consist of the following seven sentences.

(Q1) : $[(1\, x) = (1\, y)] \Longrightarrow [x = y]$
(Q2) : $0 \neq (1\, y)$
(Q3) : $x \neq 0 \Longrightarrow \exists\, y : x = (1\, y)$
(Q4) : $x + 0 = x$
(Q5) : $x + (1\, y) = (1\, [x + y])$
(Q6) : $x \cdot 0 = 0$
(Q7) : $x \cdot (1\, y) = [x \cdot y] + x$

We state that:

Theorem 3.1: The axioms of *Q* are derivable from *M*.

The proof of this statement will be done in several steps. We shall derive all the sentences of (Q1) — (Q7) from the axioms of *M*.

(Q1) is valid in *M* by Theorem 2.1.

Proposition 3.1: $\forall\, y : 0 \neq (1\, y)$

Proof: Assume $\exists\, y : [(1\, y) = 0]$.
$[(1\, y) = 0] \equiv [(\overline{y}\, y) = \overline{x}]$.
(M8) $\Longrightarrow \exists\, z : (\overline{y}\, \overline{z}) = \overline{x}$. This implies $[y = \overline{z}]$.
Since $[\overline{z} \neq \overline{y}]$, by Definition 3.2
y is not a natural number.
The result follows.

■

Proposition 3.2: $[x \neq 0] \Longrightarrow \exists\, y : [x = (1\, y)]$

Proof: Assume $\exists\, x \neq 0 : \;\forall\, y \;[x \neq (1\, y)]$.

(Q2) $\Longrightarrow [x = 0]$. This is a contradiction.

■

To show (Q4), notice that by Definition 3.1 and by (M7) a stronger statement holds:

Corollary 3.1: $\forall\, y : y + 0 = 0 + y$.

Additionally, by Definition 3.2, the following corollary results:

Corollary 3.2: $\forall\, y : y + 1 = 1 + y$.

Proof: Since $y = \underbrace{(\overline{y}\,\overline{y}\ldots\overline{y})}_{y}$ then

$$y + 1 = \underbrace{(\overline{y}\,\overline{y}\ldots\overline{y})}_{y+1} =_{\text{(Df 3.1)},\text{(Df 3.2)}}$$

$$(\underbrace{(\overline{y}\,\overline{y}\ldots\overline{y})}_{y}\;\overline{y}) =_{\text{(Ax 2.1)}} (\overline{y}\;\underbrace{(\overline{y}\,\overline{y}\ldots\overline{y})}_{y}) =_{\text{(Df 3.1)},\text{(Df 3.2)}}$$

$$\overline{y} + \underbrace{(\overline{y}\,\overline{y}\ldots\overline{y})}_{y} = 1 + y$$

■

In [13], Tarski showed that (Q3) is logically equivalent with the particular instance of the induction scheme $\{\phi(0) \wedge \forall\, u\,[\phi(u) \Longrightarrow \phi(\mathbf{S}u)] \Longrightarrow \forall\, u \;:\; \phi(u)\}$, by taking for ϕ the formula $\{[u \neq 0] \Longrightarrow \exists\, k \;:\; [u = (1\, k)]\}$. In fact, by (Q3) $[(1\, u) \neq 0] \Longrightarrow \exists\, k \;:\; [(1\, u) = (1\, k)]$. By Theorem 2.1 this implies $[k = u]$. In this way we are allowed to use inductive arguments in the sense of metamathematical inductions, and not inductions within Theory *Q*.

Observe that if we set $x = 0$ then $0 + 0 = (0\, 0) =_{(M7)} 0$. $(M7) \Longrightarrow x = 0{+}x =_{\text{(Df 3.1)}} (0\, x) = (0\,(0\,0)) =_{\text{(Df 3.1)}} 0{+}0{+}0$. The following Corollary results:

Corollary 3.3: $\forall\, k$, which is a natural number: $\underbrace{0 + 0 + 0 + 0 + \ldots + 0}_{k} = 0$.

Proof: We will use inductive argument:

1) $0 + 0 = (0\, 0) =_{(M7)} 0$
2) Assume $\underbrace{0 + 0 + 0 + 0 + \ldots + 0}_{k} = 0$
3) We have to show that $\underbrace{0 + 0 + 0 + 0 + \ldots + 0}_{k+1} = 0$

$$\underbrace{0 + 0 + 0 + 0 + \ldots + 0}_{k+1} =_{\text{(Df 3.1)},\text{(Df 3.2)}}$$

$$(\underbrace{0 + 0 + 0 + 0 + \ldots + 0}_{k}\; 0) =_{by\ assumption}$$

$$(0\, 0) =_{(1)} 0.$$

■

Finally:

Corollary 3.4: $\forall\, x, y : x + y = y + x$.

Proof: Again, we use inductive argument:

1) Corollary 3.2 $\Longrightarrow 1 + y = y + 1$
2) Assume: $\forall\, k : k + y = y + k$
3) We have to show that
$\forall\, k : [1 + k] + y = y + [1 + k]$
$[1 + k] + y =_{\text{(Df 3.1)}} ((1\, k)\, y) =_{\text{(Ax 2.1)}}$
$(1\,(k\, y)) =_{by\ assumption}$
$(1\,(y\, k)) =_{\text{(Ax 2.1)}} ((1\, y)\, k) =_{(1)} ((y\, 1)\, k) =_{\text{(Ax 2.1)}}$
$(y\,(1\, k)) =_{\text{(Df 3.1)}} y + [1 + k]$

■

Now we can show (Q5):

Proposition 3.3: $\forall\, x, y : x + (1\, y) = (1\, [x + y])$

Proof: $\forall\, x, y : (1\, x) + y =_{\text{(Df 3.1)}}$
$((1\, x)\, y) =_{\text{(Ax 2.1)}} (1\,(x\, y)) =_{\text{(Df 3.1)}}$
$1 + (x\, y) =_{\text{(Df 3.1)}} 1 + [x + y]$
and $(1\,(x\, y)) =_{\text{(Ax 2.1)}} ((1\, x)\, y) =_{\text{(Cor 3.4)}}$
$((x\, 1)\, y) =_{\text{(Ax 2.1)}} (x\,(1\, y)) =_{\text{(Df 3.1)}} x + (1\, y)$.
Thus $x + (1\, y) = 1 + [x + y] = (1\, [x + y])$.

■

In order to prove (Q6) and (Q7) we need define the multiplication operation "$\cdot$". Let us consider a primitive symbol denoted by "$*$" related to $\overline{x} = 0$ and $\overline{y} = 1$ due to Axiom (M8) as follows: $[(*\, 1) = (1\, *) = 0] \equiv [* + 1 = 1 + * = 0]$. In this context, it seems natural to interpret "$*$" as a negative natural number "-1". In fact, (M6) — (M8) are the axioms of a group. Observe first, some interesting properties of "$*$", a kind of a distributive property:

Lemma 3.1: $\forall\, x, y :$

1) $(1\, *\, x) = (x\, *\, 1) = x$
2) $(1\, *\, (x\, y)) = ((1\, *\, x)\,(1\, *\, y))$

Proof: .

1) $(1\, *\, x) =_{\text{(Ax 2.1)}} ((1\, *)\, x) =_{(M8)} (0\, x) =_{(M7)} x$
$(x\, *\, 1) =_{\text{(Ax 2.1)}} (x\,(*\, 1)) =_{(M8)}$
$(x\, 0) =_{(M7)} x$
2) $(1\, *\, (x\, y)) =_{\text{(Ax 2.1)}} (1\,(*\,(x\, y))) =_{\text{(Ax 2.1)}}$
$((1\, *)\,(x\, y)) =_{(M8)}$
$(0\,(x\, y)) =_{(M7)} (x\, y) =_{(1)} ((1\, *\, x)\,(1\, *\, y))$

■

Additionally:

$0 + x =_{\text{(Df 3.1)}} (0\, x) =_{(M7)} x =_{\text{(Lm 3.1)}} (1\, *\, x)$
$x + x = (x\, x) =_{\text{(Lm 3.1)}} ((1\, *\, x)\,(1\, *\, x)) =_{\text{(Lm 3.1)}} (1\, *\, (x\, x))$
$x + x + x = (1\, *\, (x\, x)) + x = ((1\, *\, (x\, x))\,(1\, *\, x)) = (1\, *\, ((x\, x)\, x)) =_{\text{(Ax 2.1)}} (1\, *\, (x\, x\, x))$

Continuing, it seems natural to introduce a classical recursive definition of "$\cdot$", as follows:

Definition 3.3: Multiplication:
$\forall\, x, y : x \cdot y \equiv (1\, *\, \underbrace{(y\, y \ldots y)}_{x})$

As a result $(Q6)$ follows:

Proposition 3.4: $\forall\, x : x \cdot 0 = 0$.

Proof: $x \cdot 0 =_{\text{(Df 3.3)}} (1 * \underbrace{(0\, 0 \ldots 0)}_{x}) =_{\text{(Cor 3.3)}} (1 * 0) =_{\text{(Lm 3.1)}} 0$

■

Additionally:

Proposition 3.5: The operation of multiplication "$\cdot$" has the classical properties:

1) $\forall\, y : 1 \cdot y = y$
2) $\forall\, x, y : (1\, x) \cdot y = x \cdot y + y$

Proof: .

1) $1 \cdot y =_{\text{(Df 3.3)}} (1\, *\, y) =_{\text{(Lm 3.1)}} y$

2) $x \cdot y =_{\text{(Df 3.3)}} (1 * \underbrace{(y\, y \ldots y)}_{x})$

$(M4) \Longrightarrow x \cdot y + y =_{\text{(Df 3.3)}}$
$(1 * \underbrace{(y\, y \ldots y)}_{x}) + y =_{\text{(Lm 3.1)}}$
$(1 * \underbrace{(y \ldots y)}_{x}) + (1 * y) =_{\text{(Df 3.1)}}$
$((1 * \underbrace{(y\, y \ldots y)}_{x}) (1 * y)) =_{\text{(Lm 3.1)}}$
$(1 * \underbrace{(y\, y \ldots y)}_{1+x}) =_{\text{(Df 3.3)}} (1\, x) \cdot y$

■

By Proposition 3.5 follows:

Corollary 3.5: $\forall\, y : 0 \cdot y = 0$

Proof: $[1 \cdot y =_{\text{(Pr 3.5)}} y] \Longrightarrow_{(Q4)} [[1+0] \cdot y = y] \equiv_{\text{(Df 3.1)}}$
$[(1\, 0) \cdot y = y] \equiv_{\text{(Pr 3.5)}} [0 \cdot y + y = y]$
$\Longrightarrow_{(M7)} [0 \cdot y + y = 0 + y] \equiv_{\text{(Df 3.1)}}$
$[((0 \cdot y)\, y) = (0\, y)] \Longrightarrow_{\text{(Thm 2.2)}} [0 \cdot y = y]$

■

To show (Q7), we will prove first the following lemma:

Lemma 3.2: $\forall\, y : y \cdot 1 = y$

Proof: Let us verify two cases:

1) $y \neq 0$
(Q3) $\Longrightarrow \exists\, k : y = (1\, k)$.
Thus we have to verify whether
$[y \cdot 1 = y] \Longrightarrow [(1\, k) \cdot 1 = (1\, k)]$
$(1\, k) \cdot 1 =_{\text{(Pr 3.5)}} k \cdot 1 + 1 =_{by\ assumption}$
$k + 1 =_{\text{(Cor 3.4)}} 1 + k =_{\text{(Df 3.1)}} (1\, k)$

2) $y = 0$
Observe that $1 \cdot 1 =_{\text{(Pr 3.5)}} 1$. (Q4) $\Longrightarrow 1 = (1\, 0)$.
$[1 = (1\, 0) \cdot 1 =_{\text{(Pr 3.5)}} 0 \cdot 1 + 1] \Longrightarrow_{(M7),\text{(Df 3.1)}}$
$[(1\, 0) = (0\, 1) = ((0 \cdot 1)\, 1)]$
$\Longrightarrow_{\text{(Thm 2.2)}} [0 = 0 \cdot 1]$

■

Axioms of *M* implies a stronger statement than (Q7), i.e. a distributive law:

Proposition 3.6: $\forall\, x, y, z : x \cdot [y + z] = x \cdot y + x \cdot z$

Proof: .

$[x = 1] \Longrightarrow \{1 \cdot [y + z]\} =_{\text{(Pr 3.5)}} [y + z]$ and
$\{1 \cdot y + 1 \cdot z\} =_{\text{(Pr 3.5)}} [y + z]$
Applying inductive argument,
if $k \cdot [y + z] = k \cdot y + k \cdot z$ then
$(1\, k) \cdot [y + z] =_{\text{(Pr 3.5)}} k \cdot [y + z] + [y + z] =$
$k \cdot y + k \cdot z + [y + z] =_{\text{(Cor 3.4)}}$
$k \cdot y + y + k \cdot z + z =_{\text{(Pr 3.5)}}$
$(1\, k) \cdot y + (1\, k) \cdot z$

■

(Q7) is a particular case of Proposition 3.6 for $y = 1$, applying Corollary 3.4 and Lemma 3.2: $x \cdot [1 + z] = x \cdot z + x$.

In this way all the axioms of *Q* turn out to be derivable from the axioms of *M*, and the proof is complete.

IV. Calculus on relations and theory of concatenation

Besides the theory *Q* we will consider another weak theory, a theory of concatenation — *TC* [1], [3]. Its language is composed of a binary function symbol, three constants: $\{{}^\frown, \epsilon,\ \alpha,\ \beta\}$ and the following six axioms:

(C1) $\forall\, x : x^\frown \epsilon = \epsilon^\frown x = x$
(C2) $\forall\, x \forall\, y \forall\, z : x^\frown [y^\frown z] = [x^\frown y]^\frown z$
(C3) $\forall\, x \forall\, y \forall\, u \forall\, v : [x^\frown y = u^\frown v] \Longrightarrow \{[x = u\ and\ y = v] \vee \exists\, w\{[u = x^\frown w\ and\ w^\frown v = y] \vee [x = u^\frown w\ and\ w^\frown y = v]\}\}$
(C4) $\alpha \neq \epsilon\ and\ \forall\, x \forall\, y : \neg[\alpha = x^\frown y]$
(C5) $\beta \neq \epsilon\ and\ \forall\, x \forall\, y : \neg[\beta = x^\frown y]$
(C6) $\alpha \neq \beta$

The objects of the theory *TC* are called texts or strings. The axioms (C4) — (C6) say that α, β are irreducible; they are one-letter strings that are mutually different. The axiom (C3) is called an editor axiom and describes what happens if two editors independently suggest splitting a large text into two volumes. It was proved [5] the undecidability of the theory *TC*.

In the previous Section we showed that *Q* is interpretable within *DGSS* when multiplication is defined in classical recursive way and "( )" is interpretable as a unary operation of addition. Now, we will show that the axioms of *TC* are also derivable from the axioms of *DGSS*. Hence, let us interpret "( )" as a binary function of concatenation "$^\frown$" and irreducible symbols α, β as particular relations: $\alpha = \overline{y},\ \beta = \overline{z}$ according to *TM* such that: $(\overline{z}\, \overline{y}) = (\overline{y}\, \overline{z}) = \overline{x}$ where $\overline{x} = \epsilon$ can be considered as a neutral string, which can act not as a null string but as an identity element for the operation of concatenation.

We state that:

Theorem 4.1: The axioms of *TC* are provable within the *DGSS* model.

Proof: .

(C1) is equivalent to (G7).
(C2) is equivalent to (G6).
(C3) is valid for any x, y, v, w, thus, in particular let $v = y$.
$[(x\, y) = (u\, v)] \equiv [(x\, y) = (u\, y)] \Longrightarrow_{(T3)} [x = u]$.
Let $u = x$. $[(x\, y) = (u\, v)] \equiv$
$[(u\, y) = (u\, v)] \Longrightarrow_{(T2)} [y = v]$.
Assume $x \neq u\ and\ y \neq v$.
$(G8) \Longrightarrow \exists\, z : (z\, y) = (y\, z) = \epsilon$
$[(x\, y) = (u\, v)] \Longrightarrow_{(G4)} [((x\, y)\, z) = ((u\, v)\, z)]$
$((x\, y)\, z) =_{(G6)} (x\, (y\, z)) =_{(G8)} (x\, \epsilon) =_{(G7)} x$.
Thus $x = ((u\, v)\, z) =_{(G6)} (u\, (v\, z))$
Let $[w = (v\, z)] \Longrightarrow [x = (u\, w)]$ and
$(w\, y) = ((v\, z)\, y) =_{(G6)} (v\, (z\, y)) =_{(G8)} (v\, \epsilon) =_{(G7)} v$.
Analogously for v: $(G8) \Longrightarrow \exists\, k : (v\, k) = (k\, v) = \epsilon$
$(G4) \Longrightarrow ((x\, y)\, k) = ((u\, v)\, k) =_{(G6)} (u\, (v\, k)) =_{(G8)} (u\, \epsilon) =_{(G7)} u$.
Thus $u = ((x\, y)\, k) =_{(G6)} (x\, (y\, k))$
Let $[w = (y\, k)] \Longrightarrow [u = (x\, w)]$ and
$(w\, v) = ((y\, k)\, v) =_{(G6)} (y\, (k\, v)) =_{(G8)} (y\, \epsilon) =_{(G7)} y$.

(C4) means that α cannot be defined as a composition of two different one-letter terms $x, y \neq \epsilon$. Assume there exists

two different one-letter terms $x, y \neq \epsilon$: $\alpha = (x\, y)$.
$(G7) \Longrightarrow [\overline{y} = (\epsilon\, \overline{y})]$. Thus $(\epsilon\, \overline{y}) = (x\, y)$.
$(TC3) \Longrightarrow [x = \epsilon \ and \ y = \overline{y}]$. This is a contradiction. The results follows.
The case of the existence of w is not considered because w is a composed term.

(C5) The proof is analogously to (C4).

(C6) It holds by Axiom 2.3.

■

V. Discussion of Results and Conclusions

We have shown that for any interpretation of a primitive relation "()", the presented *DGSS* model implies the uniqueness of an "inverse" element and satisfies cancellation laws. In particular, when addition is interpreted as "()" and the number "0" is associated to a specific unary relation $\overline{x}$, the axioms (M6) — (M8) are the axioms of a group under addition. In this context it is easy to define an operation of multiplication in terms of addition following usual approach. We derive all the axioms of Robinson's arithmetic from the axioms of *M* and even a distributive law.

In the case of theory of concatenation we showed that axioms of *TC* can be derived from the axioms of *DGSS* if we assume the existence of some kind of "inverse strings". This means that a complete model suggests that every letter or word ought to have some kind of inverse even if it is not so obvious. (c.f. [11], [12])

In the introduction we made reference to the insights of Husserl [10] regarding the possibility and need for a radical revision of logic. In fact his notion of parthood and of reciprocal dependence set forth in the 3^{rd} Logical Investigation. In reviewing Husserl's proposed remedy to the impasse of modern science, Rota [9] spoke of the constructing ideal objects to be subjected to yet-to-be-discovered ideal laws and relations. Also Gödel, with an explicit reference to Husserl, expressed the desire to reach a new state of awareness enabling us to describe the fundamental concepts of thought with precision, and even to grasp new, yet unknown concepts. What we are suggesting is that the calculus on unary relations outlined in [6] and developed in this paper may be a step in this direction, in offering a "language" with which new models and phenomena may be expressed.

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