International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:5, No:2, 2011

An eighth order Backward Differentiation Formula with Continuous Coefficients for Stiff Ordinary Differential Equations

Olusheye Akinfenwa, *Member, IEEE,* Samuel Jator, and Nianmin Yoa

Abstract—A block backward differentiation formula of uniform order eight is proposed for solving first order stiff initial value problems (IVPs). The conventional 8-step Backward Differentiation Formula (BDF) and additional methods are obtained from the same continuous scheme and assembled into a block matrix equation which is applied to provide the solutions of IVPs on non-overlapping intervals. The stability analysis of the method indicates that the method is L_0-stable. Numerical results obtained using the proposed new block form show that it is attractive for solutions of stiff problems and compares favourably with existing ones.

Keywords—Stiff IVPs, System of ODEs,Backward differentiation formulas, Block methods, Stability.

I. INTRODUCTION

NUMERICAL solutions for ordinary differential equations (ODEs) are very important in scientific computation, as they are widely used to model real world problems.Stiff systems are considered difficult because explicit numerical methods designed for non-stiff problems are used with very small step sizes.In the quest for better methods for solving these systems, Curtiss and Hirschfelder [1] discovered the backward differentiation formulae (BDF).

Since then, a great effort has been made in order to obtain new numerical integration methods with strong stability properties desirable for solving stiff systems. For a survey on methods for stiff systems (see [2]). Since we are concerned with the 8-step BDF which is an example of a linear multistep method, we review briefly the k-step linear multistep methods (LMMs) for the solution of the differential equations of the form

$$y' = f(t, y), \; y(t_0) = y_0 \; , \; x \; \epsilon \; [t_0, T_n] \tag{1}$$

where f satisfies the Lipschitz condition as given in Henrici [3]). The k-step LMM is conventionally written as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{2}$$

Which has $2k+1$ unknown α's and β's and therefore can be of order $2k$, where k is is the step number, however, according to Dahlquist[4], the order of (2) cannot exceed $k+1$ (k is odd) or $k+2$ (k is even) for the method to be stable. Several authors such as Lambert [5], Gear[6], Gragg and Stetter[7], Butcher[8], Akinfenwa et-al[9] proposed modified forms of (2) known as hybrid methods which were shown to overcome the Dahlquist barrier theorem. Several other methods have been proposed for efficiently solving (1) (see Keiper and Gear [10], Enright([11], [12]), Hairer and Wanner[2], Cash[13] and Brugnano and Trigiante[14]).

In this paper, the conventional 8-step BDF and additional methods are obtained from the same continuous scheme and assembled into a block matrix equation which is applied to provide the solutions for (1). We note that block methods were first introduced by Milne[15] for use only as a means of obtaining starting values for predictor-corrector algorithms and has since then been developed by several researchers (see [16], [17], [18]), for general use. The advantage of a block method is that in each application, the solution is approximated at more than one point. The number of points depends on the structure of the block method. Therefore, applying these methods can give faster solutions to the problem which can be managed to produce a desired accuracy.

The paper is presented as follows: In section 2, we discuss the basic idea behind the algorithm and obtain a continuous representation $Y(t)$ for the exact solution $y(t)$ which is used to generate members of the block method for solving (1). In section 3, we present the stability analysis of our block implicit algorithm. In section 4, we briefly discuss the implementation of the method. In section 5, we show the accuracy of our method. Finally, in section 6 we present some concluding remarks.

II. DERIVATION OF THE METHOD

We proceed by assuming that the exact solution $y(t)$ is locally represented in the range $[t_0, t_0 + 8h]$ by the continuous solution $Y(t)$ of the form

$$Y(t) = \sum_{j=0}^{8} b_i \phi_j(t) \tag{3}$$

where b_j are unknown coefficients to be determined and $\phi_j t$ are polynomial basis function of degree 8. We thus construct the 8-point BDF method with $\phi_j t = t^j, j = 0, \ldots, 8$ by imposing the following conditions

O. Akinfenwa is with the College of Computer Science and Technology, Harbin Engineering University, Harbin, 15001 P.R China e-mail: (akinolu35@yahoo.com).

S. Jator is with the Department of Mathematics, Austin Peay State University Clarksville,TN 37044 USA e-mail:(Jators@apsu.edu).

N. Yao is with the College of Computer Science and Technology, Harbin Engineering University, Harbin, 15001 P.R China e-mail: (yaonianmin@hrbeu.edu.cn).

Manuscript received April 04, 2011; revised April 19, 2011.

$$Y\left(t_{n+i}\right)=y_{n+j}, \quad j=0,\ldots,7 \quad Y'\left(t_{n+8}\right)=f_{n+8}, \quad (4)$$

where y_{n+j} is the approximation for the exact solution $y(t_{n+j}), f_{n+8} = f(t_{n+8}, y_{n+8})$ and n is the grid index. It should be noted that equation (4) leads to a system of equations which must be solved to obtain the coefficients $b_j, j = 0, \ldots, 8$ which are substituted into (3) and after some algebraic computation, our continuous representation yields the form

$$Y(t) = -\sum_{j=0}^{7} \alpha_j(t) y_{n+j} + h\beta_8(t) f_{n+8} \quad (5)$$

where $\alpha_j(t) and \beta_8(t)$ are continuous coefficients. The method (5) is then used to generate the $8-step$ standard BDF (6) at point $t = t_{n+8}$.

The additional methods are obtained by evaluating the first derivative of (5) given by (7) at the points $t = t_{n+j}, j = 1, \ldots, 7$. Thus we have the additional methods as (8).

The integrators (8) together with (6) are combined as a one block 8 point block BDF methods of order $(8,8,8,8,8,8,8,8)^T$ with error constants:

$$\mathrm{C}_9 = (\tfrac{89}{6088}, -\tfrac{2423}{575316}, \tfrac{817}{383544}, -\tfrac{277}{159810}, \tfrac{2563}{1150632}, -\tfrac{901}{191772}, \tfrac{347}{18264}, \tfrac{280}{6849})^T$$

III. STABILITY ANALYSIS

In what follows, (6) and (8) can be rearranged and rewritten as a matrix finite difference equation of the form

$$A^{(1)} Y_{\omega+1} = A^{(0)} Y_\omega + hB^{(1)} F_\omega \quad (9)$$

where

$$\begin{aligned} Y_{\omega+1} &= (y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5}, y_{n+6}, y_{n+7}, y_{n+8})^T \\ Y_\omega &= (y_{n-7}, y_{n-6}, y_{n-5}, y_{n-4}, y_{n-3}, y_{n-2}, y_{n-1}, y_n)^T \\ F_\omega &= (f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, f_{n+5}, f_{n+6}, f_{n+7}, f_{n+8})^T \end{aligned}$$

for $\omega = 0, \ldots$ and $n = 0, 8, \ldots, N-8$, and the matrices $A^{(1)}$, $A^{(0)}$, $B^{(1)}$ are 8 by 8 matrices whose entries are given by the coefficients of (6) and (8). In particular, the matrices are defined as equation (10).

A. Zero-stability

It is worth noting that zero-stability is concerned with the stability of the difference system in the limit as h tends to zero. Thus, as $h \to 0$, the method (9) tends to the difference system

$$A^{(1)} Y_{\omega+1} - A^{(0)} Y_\omega = 0$$

whose first characteristic polynomial $\rho(R)$ is given by

$$\rho(R) = \det(RA^{(0)} + A^{(1)}) = \frac{280}{761} R^7 (1-R) \quad (11)$$

Following Fatunla[19], the block method (9) is zero-stable, since from (11), $\rho(R) = 0$ satisfies $|R_j| \le 1$, $j = 1, \ldots, \nu$, and for those roots with $|R_j| = 1$, the multiplicity does not exceed 1.

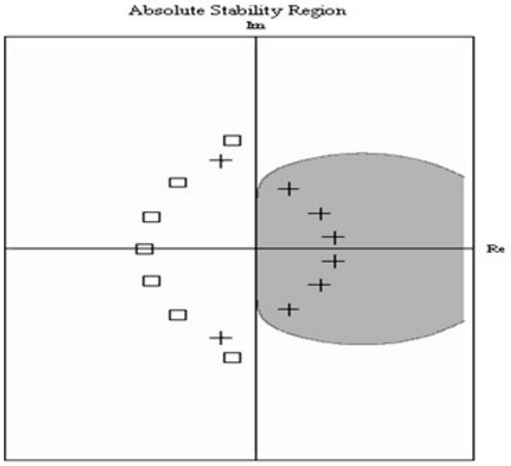

Fig. 1. Stability Region

1) Consistency: The block method (9) is consistent as it has order $p > 1$. According to Henrici[3],convergent, since $convergence = zerostability + consistency.$

B. Linear stability

The linear stability properties of the eight point block BDF methods are determined by expressing them in the form (9) and applying them to the test equation

$$y' = \lambda y \quad , \quad \lambda < 0$$

which is applied to (9) to yield

$$Y_{\omega+1} = D(z) Y_\omega \ , \ z = \lambda h, \quad (12)$$

where the matrix $D(z)$ is given by

$$D(z) = -(A^{(1)} - zB^{(1)})^{-1} A^{(0)}$$

From (12) we obtain the stability function $R(z) : C \to C$ which is a rational function with real coefficients given by (13).

The stability domain of the method (or region of absolute stability), S, is defined as

$$S = [z \in C : R(z) \le 1] \quad (14)$$

Specifically, when the left-half complex plane is contained in S, the method is said to be A-stable. Below in Fig. 1, we show the plot with rectangle representing the zeros and plus sign representing the poles of (13). The plot in white represents the stability region which corresponds to the stability function (13). Clearly, from the figure, it is obvious that our method is not A- stable since according to Hairer and Wanner [2] it has at least a pole of the stability function (13) in the left half complex plane.

However , the method is L_0-stable as in Cash [13] since it satisfies the requirement that:

$Max_{z \le 0} |R(z)| \le 1$, z real and $\lim_{z \to -\infty} R(z) = 0$

$$y_{n+8}=\frac{280h}{761}f_{n+8}-\frac{35}{761}y_n+\frac{320}{761}y_{n+1}-\frac{3920}{2283}y_{n+2}+\frac{3136}{761}y_{n+3}-\frac{4900}{761}y_{n+4}+\frac{15680}{2283}y_{n+5}-\frac{3920}{761}y_{n+6}+\frac{3920}{761}y_{n+7} \tag{6}$$

$$Y'(t)=\frac{1}{h}(\sum_{i=0}^{7}\alpha_j'(t)y_{n+j}+h\beta_8'(t)f_{n+8})=f(t_{n+j},y_{n+j}) \tag{7}$$

$$\left.\begin{aligned}
hf_{n+1}-\tfrac{5h}{761}f_{n+8}&=-\tfrac{383}{3040}y_n-\tfrac{24129}{15220}y_{n+1}+\tfrac{15841}{4566}y_{n+2}-\tfrac{5215}{1522}y_{n+3}+\tfrac{25585}{9132}y_{n+4}-\tfrac{14861}{9132}y_{n+5}+\tfrac{4627}{7610}y_{n+6}-\tfrac{521}{4566}y_{n+7}\\
hf_{n+2}+\tfrac{5h}{2283}f_{n+8}&=\tfrac{1159}{63924}y_n-\tfrac{658}{2283}y_{n+1}-\tfrac{128731}{136980}y_{n+2}+\tfrac{4510}{2283}y_{n+3}-\tfrac{11065}{9132}y_{n+4}+\tfrac{4286}{6849}y_{n+5}-\tfrac{2003}{9132}y_{n+6}+\tfrac{3166}{79905}y_{n+7}\\
hf_{n+3}-\tfrac{h}{761}f_{n+8}&=-\tfrac{391}{63924}y_n+\tfrac{111}{1522}y_{n+1}-\tfrac{2311}{4566}y_{n+2}-\tfrac{1325}{3044}y_{n+3}+\tfrac{3735}{3044}y_{n+4}-\tfrac{2171}{4566}y_{n+5}+\tfrac{677}{4566}y_{n+6}-\tfrac{537}{21308}y_{n+7}\\
hf_{n+4}+\tfrac{h}{761}f_{n+8}&=\tfrac{199}{53270}y_n-\tfrac{425}{11415}y_{n+1}+\tfrac{2353}{11415}y_{n+2}-\tfrac{620}{761}y_{n+3}+\tfrac{35}{1522}y_{n+4}+\tfrac{8852}{11415}y_{n+5}-\tfrac{691}{3805}y_{n+6}+\tfrac{2204}{79905}y_{n+7}\\
hf_{n+5}-\tfrac{5h}{2283}f_{n+8}&=-\tfrac{1229}{319620}y_n+\tfrac{349}{9132}y_{n+1}-\tfrac{2423}{13698}y_{n+2}+\tfrac{2395}{4566}y_{n+3}-\tfrac{11765}{9132}y_{n+4}+\tfrac{67241}{136980}y_{n+5}+\tfrac{2143}{4566}y_{n+6}-\tfrac{1723}{31962}y_{n+7}\\
hf_{n+6}+\tfrac{5h}{761}f_{n+8}&=\tfrac{433}{63924}y_n-\tfrac{246}{3805}y_{n+1}+\tfrac{2563}{9132}y_{n+2}-\tfrac{1690}{2283}y_{n+3}+\tfrac{4155}{3044}y_{n+4}-\tfrac{4846}{2283}y_{n+5}+\tfrac{15859}{15220}y_{n+6}+\tfrac{1242}{5327}y_{n+7}\\
hf_{n+7}-\tfrac{35h}{761}f_{n+8}&=-\tfrac{503}{21308}y_n+\tfrac{1001}{4566}y_{n+1}-\tfrac{20881}{22830}y_{n+2}+\tfrac{6895}{3044}y_{n+3}-\tfrac{33985}{9132}y_{n+4}+\tfrac{19901}{4566}y_{n+5}-\tfrac{6307}{1522}y_{n+6}+\tfrac{208903}{106540}y_{n+7}
\end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned}
A^{(1)}&=\begin{pmatrix}
\frac{24129}{15220} & -\frac{15841}{4566} & \frac{5215}{1522} & -\frac{25585}{9132} & \frac{14861}{9132} & -\frac{4627}{7610} & \frac{521}{4566} & 0\\
\frac{658}{2283} & \frac{128731}{136980} & -\frac{4510}{2283} & \frac{11065}{9132} & -\frac{4286}{6849} & \frac{2003}{9132} & -\frac{3166}{79905} & 0\\
-\frac{111}{1522} & \frac{2311}{4566} & \frac{1325}{3044} & -\frac{35}{1522} & \frac{2171}{4566} & -\frac{677}{4566} & \frac{537}{21308} & 0\\
\frac{425}{11415} & -\frac{2353}{11415} & \frac{620}{761} & -\frac{35}{1522} & -\frac{8852}{11415} & \frac{691}{3805} & -\frac{2204}{79905} & 0\\
-\frac{349}{9132} & \frac{2423}{13698} & -\frac{2395}{4566} & \frac{11765}{9132} & -\frac{67241}{136980} & -\frac{2143}{4566} & \frac{1723}{31962} & 0\\
\frac{246}{3805} & -\frac{2563}{9132} & \frac{1690}{2283} & -\frac{4155}{3044} & \frac{4846}{2283} & -\frac{15859}{15220} & -\frac{1242}{5327} & 0\\
-\frac{1001}{4566} & \frac{20881}{22830} & -\frac{6895}{3044} & \frac{33985}{9132} & -\frac{19901}{4566} & \frac{6307}{1522} & -\frac{208903}{106540} & 0\\
-\frac{320}{761} & \frac{3920}{2283} & -\frac{3136}{761} & \frac{4900}{761} & -\frac{15680}{2283} & \frac{3920}{761} & -\frac{3920}{761} & 1
\end{pmatrix}\\
A^{(0)}&=\begin{pmatrix}
0&0&0&0&0&0&0&-\frac{383}{3044}\\
0&0&0&0&0&0&0&\frac{1159}{63924}\\
0&0&0&0&0&0&0&-\frac{391}{63924}\\
0&0&0&0&0&0&0&\frac{199}{53270}\\
0&0&0&0&0&0&0&-\frac{1229}{319620}\\
0&0&0&0&0&0&0&\frac{433}{63924}\\
0&0&0&0&0&0&0&-\frac{503}{21308}\\
0&0&0&0&0&0&0&-\frac{35}{761}
\end{pmatrix}\\
B^{(1)}&=\begin{pmatrix}
-1&0&0&0&0&0&0&\frac{5}{761}\\
0&-1&0&0&0&0&0&-\frac{5}{761}\\
0&0&-1&0&0&0&0&\frac{1}{761}\\
0&0&0&-1&0&0&0&-\frac{1}{761}\\
0&0&0&0&-1&0&0&\frac{5}{761}\\
0&0&0&0&0&-1&0&-\frac{5}{761}\\
0&0&0&0&0&0&-1&\frac{35}{761}\\
0&0&0&0&0&0&0&\frac{280}{761}
\end{pmatrix}
\end{aligned}\right\} \tag{10}$$

$$R(z)=\frac{3(1680+5880z+9660z^2+9800z^3+6769z^4+3283z^5+1089z^6+210z^7)}{5040-22680z+49140z^2-68040z^3+67347z^4-50463z^5+29531z^6-13698z^7+5040z^8} \tag{13}$$

IV. IMPLEMENTATION

The implementation of the above block methods is summarized as follows:

A. Summary

On the partition $I_N:\{a=t_0<t_1<\ldots<t_{N-1}<t_N=b$,$n=0,1,2,\ldots,N-1$.

Step 1. Choose N for $k=8$, $h=\frac{b-a}{N}$ the number of blocks $\pi=\frac{N}{8}$ using (9) $n=0,\omega=0$ the values $(y_1,y_2,\ldots,y_8)^T$ are generated simultaneously over the subinterval $[t_0,t_8]$ as y_0 are known from the IVP (1).

Step 2. for $n=8,\omega=1$, $(y_9,y_{10},\ldots,y_{16})^T$ are obtained over the subinterval $[t_8,t_{16}]$ since y_8 is known from the first block.

Step 3. The process is continued for $n=2k,\ldots,N-k$ and $\omega=2,\ldots,\pi$ to obtain approximate solutions to (1) on subintervals $[t_0, t_k], \dots, [t_{N-k}, t_N]$ N is a positive integer and n the grid index.

We explain briefly the implementation of the block methods. For linear problem we use the Gaussian elimination to solve the resulting k x k matrix in each block with our written Matlab code. While for non- linear problem the code uses the Newton iteration. The following notation is used to specify the iteration y^{j+1}_{n+i} denotes the $(j+1)th$ iterative value of y_{n+i} and $\delta^{j+1}_{n+i} = y^{j+1}_{n+i} - y^{j}_{n+i}$ for $i = 1, 2, \dots, k$ and $i = 1, 2, \dots$ Thus the Newton iteration of the 8 point block BDF method for (15) takes the form

$$y^{(j+1)}_{n+i} = y^{(j)}_{n+i} - \frac{f^{(j)}_{n+i}}{f'^{(j)}_{n+i}} \tag{15}$$

$$y^{(j+1)}_{n+1} - y^{(j)}_{n+1} = \frac{a_1 y^{(j)}_{n+1} + a_2 y^{(j)}_{n+2} + \dots + a_7 y^{(j)}_{n+7} + h f^{(j)}_{n+1} + h\beta_8 f^{(j)}_{n+8}}{1 + h\frac{\delta f_{n+1}}{\delta y_{n+1}} + h\beta_8 \frac{\delta f_{n+8}}{\delta y_{n+8}}} + D_1$$

$$y^{(j+1)}_{n+2} - y^{(j)}_{n+2} = \frac{c_1 y^{(j)}_{n+1} + c_2 y^{(j)}_{n+2} + \dots + c_7 y^{(j)}_{n+7} + h f^{(j)}_{n+2} + h V_8 f^{(j)}_{n+8}}{1 + h\frac{\delta f_{n+2}}{\delta y_{n+2}} + h V_8 \frac{\delta f_{n+8}}{\delta y_{n+8}}} + D_2$$

$$y^{(j+1)}_{n+3} - y^{(j)}_{n+3} = \frac{d_1 y^{(j)}_{n+1} + d_2 y^{(j)}_{n+2} + \dots + c_7 y^{(j)}_{n+7} + h f^{(j)}_{n+3} + h\nu_8 f^{(j)}_{n+8}}{1 + h\frac{\delta f_{n+3}}{\delta y_{n+2}} + h\nu_8 \frac{\delta f_{n+8}}{\delta y_{n+8}}} + D_3$$

$$\dots$$
$$\dots$$
$$\dots$$

$$y^{(j+1)}_{n+8} - y^{(j)}_{n+8} = \frac{g_1 y^{(j)}_{n+1} + g_2 y^{(j)}_{n+2} + \dots + g_7 y^{(j)}_{n+7} + y^{(j)}_{n+8} + h\psi_8 f^{(j)}_{n+8}}{1 + h\psi_8 \frac{\delta f_{n+8}}{\delta y_{n+8}}} + D_8$$

Put in matrix form then becomes:

$$J^{(1)}\delta^{(1)} = \alpha^{(0)} Y^{(1)} + h\beta^{(0)} F^{(1)} + D \tag{16}$$

Where

$D_1, D_2, \dots, D_8$ are known from the initial value of the problem. Thus we obtain the approximated values of $y_{n+1}, y_{n+2}, \dots, y_{n+8}$ as

$$y^{(j+1)}_{n+1} = y^{(j)}_{n+1} + \delta^{(j+1)}_{n+1}$$
$$y^{(j+1)}_{n+2} = y^{(j)}_{n+2} + \delta^{(j+1)}_{n+2}$$
$$\vdots$$
$$y^{(j+1)}_{n+8} = y^{(j)}_{n+8} + \delta^{(j+1)}_{n+8}$$

V. NUMERICAL EXAMPLES

A. Example 1

Example 1: Our first example is the problem whose Jacobian matrix J has purely imaginary eigenvalues on the range $0 \leq t \leq T$

$$y'_1 = -\alpha y_2 + (1+\eta)cos(t), \; y_1(0) = 0$$
$$y'_2 = \alpha y_2 - (1+\eta)sin(t), \; y_2(0) = 1$$

With exact solution of the system given by

$$y_1 = sin(t) \; , \; y_2 = cos(t)$$

For any value of the parameter η.Thus, the jacobian J has the following expression

$$J = \left(\begin{array}{cc} 0 & -\eta \\ \eta & 0 \end{array}) \right)$$

the eigenvalues $-i\eta$, $i\eta$.

We compare our method with that of [20] for the correct digit $\Delta = -\log_{10}(\frac{||y_i(T) - y_{n,i}||_\infty}{||y_{n,i}||_\infty})$ at the end of the interval for various values of h as shown in Table I.

TABLE I
A COMPARISON OF METHODS FOR NUMBER OF CORRECT DIGITS Δ ,$T = 100$, AND $\eta = 10$ FOR EXAMPLE 1

h	$M(8, r8)$	Our $BBDF_8$
4/5	3.43	3.97
2/5	5.67	6.38
1/5	8.23	8.28
1/10	9.29	10.72
1/20	11.24	12.45
1/40	12.57	14.23

B. Example 2

Example 2: Next, we consider a well known classical system see ([21], [22], [23]) in the range $0 \leq t \leq 10$

$$y'_1 = 998y_1 + 1998y_2, \; y_1(0) = 1$$
$$y'_2 = -999y_1 - 1999y_2, \; y_2(0) = 1$$

Its exact solution is given by the sum of two decaying exponentials components.

$$y_1 = 4e^{-t} - 3e^{-1000t} \; , \; y_2 = -2e^{-t} + 3e^{-1000t}$$

The stiffness ratio is 1:1000. In Table II, we present result for that BGH stiff solver in Hall and Watt [24] and the version of the Gear method in Stabrowski [22], along with that of $VSCRK_8$ in Vigo-Aguiar and Ramos [23]. For our method we use the step $h = 0.1$. The parameters considered are the number of function evaluations, feval, and the total number of integration steps, nstep. The exact solution, our numerical solution and the absolute error at the end of the last 10 time step $(9.1, 10)$ are presented in Table III.

TABLE II
A COMPARISON OF METHODS FOR EXAMPLE 2

method	feval	nstep
BGH stiff	256	214
Gear type	317	248
$VSCRK_8$	99	8
Our $BBDF_8$	100	13

Remark: Although the $VSCRK_8$ has fewer functions evaluations the method was evaluated with an initial step $h = 10^{-4}$ but our method uses relatively large step size at $h = 10^{-4}$ which shows its' efficiency and good accuracy.

$$J^{(1)} = \begin{pmatrix} 1 + h\frac{\delta f_{n+1}}{\delta y_{n+1}} + h\beta_8 \frac{\delta f_{n+8}}{\delta y_{n+8}} & a_2 & \dots & a_7 & 0 \\ c_1 & 1 + h\frac{\delta f_{n+2}}{\delta y_{n+2}} + hV_8 \frac{\delta f_{n+8}}{\delta y_{n+8}} & \dots & c_7 & 0 \\ . & . & . & & \\ . & . & . & & \\ . & . & . & & \\ e_1 & \dots & 1 + h\frac{\delta f_{n+7}}{\delta y_{n+7}} + h\nu_8 \frac{\delta f_{n+8}}{\delta y_{n+8}} & 0 & \\ g_1 & g_2 & \dots & g_7 & 1 + h\psi_8 \frac{\delta f_{n+8}}{\delta y_{n+8}} \end{pmatrix}$$

$$\delta^{(1)} = \begin{pmatrix} \delta_{n+1}^{(j+1)} \\ \delta_{n+2}^{(j+1)} \\ \delta_{n+3}^{(j+1)} \\ \vdots \\ \delta_{n+7}^{(j+1)} \\ \delta_{n+8}^{(j+1)} \end{pmatrix}, \qquad \alpha^{(0)} = \begin{pmatrix} -1 & -a_1 & -a_2 & \dots & -a_7 & 0 \\ -c_1 & -1 & -c_3 & \dots & -c_7 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -e_1 & -e_2 & \dots & -e_6 & -1 & 0 \\ -g_1 & -g_2 & -g_3 & \dots & -g_7 & -1 \end{pmatrix}, \qquad Y_{(1)} = \begin{pmatrix} y_{n+1}^{(j)} \\ y_{n+2}^{(j)} \\ y_{n+3}^{(j)} \\ \vdots \\ y_{n+7}^{(j)} \\ y_{n+8}^{(j)} \end{pmatrix}$$

$$\beta^{(0)} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & -\beta \\ 0 & 1 & 0 & \dots & 0 & -V \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & \dots & 1 & -nu \\ 0 & 0 & 0 & \dots & 0 & -\psi \end{pmatrix}, \qquad F^{(1)} = \begin{pmatrix} f_{n+1}^{(j)} \\ f_{n+2}^{(j)} \\ f_{n+3}^{(j)} \\ \vdots \\ f_{n+7}^{(j)} \\ f_{n+8}^{(j)} \end{pmatrix}, \qquad D = \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_7 \\ D_8 \end{pmatrix}$$

TABLE III
RESULT FOR $BBDF_8$ AT $h = 0.1$ FOR EXAMPLE 2

t	Exact $y_1(t) \times 10^{-3}$ $y_2(t) \times 10^{-3}$	$BBDF_8$ $y_1 \times 10^{-3}$ $y_2 \times 10^{-3}$	Absolute error $(\lvert y_1(t) - y_1\rvert)$ $(\lvert y_2(t) - y_2\rvert)$
9.1	0.44666323396046 -0.22333161698023	0.44666323491111 -0.22333161745555	9.506×10^{-13} 4.753×10^{-13}
9.2	0.40415760734837 -0.20207880367419	0.40415760820822 -0.20207880410411	8.598×10^{-13} 4.299×10^{-13}
9.3	0.36569692591269 -0.18284846295635	0.36569692669097 -0.18284846334548	7.782×10^{-13} 3.891×10^{-13}
9.4	0.33089626222653 -0.16544813111326	0.33089626293039 -0.16544813146520	7.038×10^{-13} 3.519×10^{-13}
9.5	0.29940731955080 -0.14970365977540	0.29940732018840 -0.14970366009420	6.376×10^{-13} 3.188×10^{-13}
9.6	0.27091494596342 -0.13545747298171	0.27091494653686 -0.13545747326843	5.734×10^{-13} 2.867×10^{-13}
9.7	0.24513398021289 -0.12256699010644	0.24513398077913 -0.12256699038957	5.662×10^{-13} 2.831×10^{-13}
9.8	0.22180639772871 -0.11090319886435	0.22180639823954 -0.11090319911977	5.108×10^{-13} 2.554×10^{-13}
9.9	0.20069872822470 -0.10034936411235	0.20069872868725 -0.10034936434363	4.625×10^{-13} 2.312×10^{-13}
10.0	0.18159971904994 -0.09079985952497	0.18159971946833 -0.09079985973416	4.183×10^{-13} 2.092×10^{-13}

C. *Example 3*

Example 3: Consider the Stiffly nonlinear problem which was proposed by Kaps [25] in the range $0 \leq t \leq 10$

$$y_1' = (\epsilon^{-1} + 2)y_1 + \epsilon^{-1}y_2,\ y_1(0) = 1$$
$$y_2' = y_1 - y_2 - y_2^2,\ y_2(0) = 1$$

The smaller ϵ is, the more serious the stiffness of the system. Its exact solution is given by

$$y_1 = y_2^2\ ,\ y_2 = e^{-t}$$

We compare our method with that of Wu and Xia [26], PRM of Li rong and and de-gui liu [27] and $M(8, r8)$ in Chartier [20] taking values ϵ^{-3}, ϵ^{-6} and ϵ^{-8} respectively.

TABLE IV
A COMPARISON OF METHODS FOR EXAMPLE 3
$\epsilon = 10^{-3}$,$Erri = |y_i(t) - y_i|$

method	t	h	N	err1	err2
$WuandXia$	1 10	0.002 0.001	500 10000	2.5606×10^{-7} 5.5468×10^{-16}	8.0150×10^{-8} 6.0936×10^{-12}
$BBDF_8$	1 10	0.05 0.01	20 1000	4.5602×10^{-13} 6.6466×10^{-20}	6.2638×10^{-13} 2.3988×10^{-17}

TABLE V
A COMPARISON OF METHODS FOR EXAMPLE 3 USING $\epsilon = 10^{-6}$,
$Erri = \frac{|y_i(t) - y_i|}{|-y_i|}$

method	h	err1	err2
$PRM(3stage4^{th}order$	0.1 0.01	7.283×10^{-2} 4.076×10^{-5}	1.259×10^{-8} 2.349×10^{-6}
$BBDF_8$	0.5 0.05	4.780×10^{-11} 4.034×10^{-18}	69.268×10^{-11} 1.078×10^{-19}

The table below shows the result of our method compapared with that of [27].

Lastly, for this example the result of our method compared with that of [20]. It can be seen that for this example our method show superiority over the all the three methods for the different values of ϵ compared especially when the step size h is relatively high.

For our last example we present without comparison the result for different choices of the constant stepsize h, the absolute error for h at the end of the interval $T = 10$.

D. *Example 4*

Example 4: Consider the weakly damped oscillatory problem in the range $0 \leq t \leq 10$

TABLE VI
A comparison of methods for the number of correct digits $\Delta$for Example 3 using $\epsilon = 10^{-8}$

method	$M(8, r8)$	$BBDF_8$
$h = 1/4$	4.66	5.80
$h = 1/8$	5.67	7.93
$h = 1/16$	6.26	10.21
$h = 1/32$	8.47	12.52
$h = 1/64$	10.83	12.87
$h = 1/128$	15.63	12.58

TABLE VII
Result for our method $BBDF_8$ for Example 4 $erri = |y_i(t) - y_i|$

h	err1	err2	err3
0.1	6.565×10^{-7}	2.302×10^{-6}	2.302×10^{-6}
0.05	5.849×10^{-9}	6.767×10^{-9}	6.767×10^{-9}
0.01	2.237×10^{-14}	1.747×10^{-13}	1.747×10^{-13}

$$y' = Ay, \; y(0) = y_0$$

Where

$$A = \begin{pmatrix} 0.01 & -1 & 1 \\ 2 & -100.005 & 99.995 \\ 2 & 99.995 & -100.005 \end{pmatrix}$$

The exact solution is

$$y_1(t) = e^{-0.01t}(\cos(2t) - \sin(2t))$$

$$y_2(t) = e^{-0.01t}(\cos(2t) + \sin(2t)) + e^{-200t}$$

$$y_3(t) = e^{-0.01t}(\cos(2t) + \sin(2t)) - e^{-200t}$$

VI. Conclusion

A 8-step BDF with continuous coefficients has been proposed and implemented as a self-starting method for solution of stiff systems of ODEs. The method avoids complicated subroutines needed for existing methods requiring starting values or predictors. The good stability and consistency property of our method makes it attractive for numerical solution of stiff problems. We have demonstrated the accuracy of the methods for both linear and non linear problems. Our future research will be focused on the implementation of the method to parabolic partial differential equations, since it is L_0 -stability.

Acknowledgment

The authors would like to thank the State Key Laboratory of High-end Server and Storage Technology. Number: 009HSSA08 and Fundamental Research Funds for the Central Universities. Numbers: HEUCFT1007, HEUCF100607 China, for the their financial support in this research work. Also, we thank the referees whose useful suggestions greatly improve the quality of this paper.

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