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Periodic solutions for a higher order nonlinear neutral functional differential equation

Yanling Zhu

Abstract—In this paper, a higher order nonlinear neutral functional differential equation with distributed delay is studied by using the continuation theorem of coincidence degree theory. Some new results on the existence of periodic solutions are obtained.

Keywords—Neutral functional differential equation, higher order, periodic solution, coincidence degree theory

I. Introduction

In the last several decades, by applying the continuation theorem of coincidence degree theory, some researchers have studied some kinds of second order delay functional differential equations, see [3-12] and the references therein. The conditions imposed on $g(x)$ are: there are two positive constants A, M such that $xg(x) > 0, |x| > A$, and $g(x) > -M$, for $x < M$, which are required in [5,6]; or $g(x) > 0, \forall x \in R$, $\lim_{x\to+\infty} g(x) = +\infty$ and $\lim_{x\to-\infty} g(x) = -\infty$ which are required in [10]. Furthermore, the delays of these equations are discrete. But the work to get the existence of periodic solutions of neutral distributed delay functional differential equations(NFDE), especially higher order nonlinear neutral distributed functional differential equations rarely appeared.

In present paper, we discuss the existence of periodic solutions to a kind of higher order nonlinear neutral functional differential equation with distributed delay as follows,

$$(Ax)^{(m)}(t) = f(x(t))x'(t) + g(t, \int_{-r}^{0} x(t+s)d\alpha(s)) + e(t), \tag{1.1}$$

where $(Ax)(t) = x(t) - kx(t-\tau)$, $f : R \to R$ is continuous functions, $g : R^2 \to R$ is contiunous function which is periodic to the first argumtent with positive period ω, $e : R \to R$ is a continuous periodic function with period ω, $r > 0, m$ is a positive integer, $k, \tau \in R$ are two constants. $\alpha : [-r, 0] \to R$ is a bounded variation function. It is well known that such a kind of distributed delay NFDE has been used for studying many problems in some fields, such as physics, mechanics and ecology.

By employing the continuation of coincidence degree theory developed by Mawhin, we obtain some new results on the existence of periodic solutions of Eq.(1.1). The significance is that even if for the case of $m = 2$, the conditions imposed on $g(x)$ and $f(x)$, and the methods to estimate *a priori bounds* are different from the corresponding ones of [3-12].

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II. Lemmas

We firstly give some useful notations: $\bigvee_{-r}^{0}(\alpha) = 1$, where $\bigvee_{-r}^{0}(\alpha)$ is the total variation of $\alpha(s)$ over $[-r, 0]$. $C_\omega = \{x | x \in C(R, R), x(t+\omega) \equiv x(t)\}$, with the norm $||x||_0 = \max_{t\in[0,\omega]} |x(t)|$. $C_\omega^1 = \{x | x \in C^1(R, R), x(t+\omega) \equiv x(t)\}$, with the norm $||x|| = \max\{||x||_0, ||x'||_0\}$. Clearly, C_ω and C_ω^1 are two $Banach$ spaces. We also define operators A and L in the following form respectively,

$$A : X \to X,\ L : Dom(L) \subset Y \to X,\ Lx = (Ax)^{(m)},$$

where $Dom(L) = \{x \in C^m(R, R) : x(t+\omega) \equiv x(t)\}$.

Lemma 2.1[6] If $|k| < 1$, then A has continuous bounded inverse on X, and

[1] $||A^{-1}x|| \le \frac{||x||_0}{||k|-1|},\ \forall x \in X$,

[2] $\int_0^\omega |(A^{-1}f)(t)|dt \le \frac{1}{|1-|k||} \int_0^\omega |f(s)|ds, \forall f \in X$.

By Hale's terminology[2], a solution $x(t)$ of Eq.(1.1) is that $x(t) \in C^1(R, R)$ such that $Ax \in C^m(R, R)$ and Eq.(1.1) is satisfied on R. In general, $x(t)$ does not belong to $C^m(R, R)$. But under the condition $|k| \ne 1$, we can see from Lemma 2.1 that $(Ax)'(t) = Ax'(t)$, $(Ax)''(t) = Ax''(t), \cdots, (Ax)^{(m)}(t) = Ax^{(m)}(t)$. So a solution $x(t)$ of Eq.(1.1) must belong to $C^{(m)}(R, R)$. According to the first part of Lemma 2.1, we can easily obtain that

$$KerL = R,\ ImL = \{x | x \in X : \int_0^\omega x(s)ds = 0\}.$$

L is a Fredholm operator with index zero. Now we project operators P and Q as follows, respectively,

$$P : Y \to KerL,\ Px = (Ax)(0),$$

$$Q : X \to X/ImL,\ Qy = \frac{1}{\omega}\int_0^\omega y(s)ds.$$

Then $ImP = KerL$, $KerQ = ImL$. Let $L_p^{-1} : ImL \to DomL \cap KerP$ denotes the inverse of L.

Lemma 2.2[1] Let X and Y be two $Banach$ spaces, $L : Dom(L) \subset Y \to X$ be a $Fredholm$ operator with index zero, $\Omega \subset Y$ be an open bounded set, and $N : \overline{\Omega} \to X$ be $L-$ compact on $\overline{\Omega}$. If all the following conditions hold,

$[A_1]$ $Lx \ne \lambda Nx, \forall x \in \partial\Omega \cap Dom(L), \forall \lambda \in (0, 1)$,
$[A_2]$ $Nx \notin ImL, \forall x \in \partial\Omega \cap KerL$,
$[A_3]$ $deg\{JQN, \Omega \cap KerL, 0\} \ne 0$, $J : ImQ \to KerL$ is an isomorphism. Then equation $Lx = Nx$ has at least one solution on $\overline{\Omega} \bigcap Dom(L)$.

III. MAIN RESULTS

For the sake of convenience, we denote: Z^+ is a set of all positive integers, $X := C_\omega$, $Y := C^1_\omega$ and $\overline{e} = \frac{1}{\omega}\int_0^\omega e(t)dt$.

Theorem 3.1 If there exist constants $M > 0$ and $W \geq 0$ such that

$[B_1]$ $x(g(t,x) + \overline{e}\,) > 0$ (or $x(g(t,x) + \overline{e}\,) < 0$), for $t \in R$, $|x| > M$,

$[B_2]$ $\lim\limits_{|x|\to+\infty} \sup |\frac{F(x)}{x}| = W$, where $F(x) = \int_0^x f(s)ds$,

$[B_3]$ $\overline{e} < 0$, $g(t,x) > 0$, for $t, x \in R$.

Then Eq.(1.1) has at least one ω-periodic solution, if $|k| - 1 > W\omega^{m-1}$.

Corollary 3.1 If there exist constants $M > 0$ and $W \geq 0$ such that

$[B_1^*]$ $x(g(t,x) + \overline{e}\,) > 0$ (or $x(g(t,x) + \overline{e}\,) < 0$), for $t \in R$, $|x| > M$,

$[B_2^*]$ $\lim\limits_{|x|\to+\infty} \sup |\frac{F(x)}{x}| = W$, where $F(x) = \int_0^x f(s)ds$,

$[B_3^*]$ $\overline{e} > 0$, $g(t,x) < 0$, for $t, x \in R$.

Then Eq.(1.1) has at least one ω-periodic solution, if $|k| - 1 > W\omega^{m-1}$. .

Theorem 3.2 Assume n is an even integer, and if there exist constants $C \geq 0$ and $M > 0$ such that

$[C_1]$ $xg(t,x) > 0$ (or $xg(t,x) < 0$), for $t \in R, |x| > M$,

$[C_2]$ $\lim\limits_{x\to+\infty} \sup\limits_{t\in R} |\frac{g(t,x)}{x}| \leq C$.

Then Eq.(1.1) has at least one ω-periodic solution, if $1 - |k| > 2C\omega^m$.

Corollary 3.2 Assume n is an even integer, and if there exist constants $C \geq 0$ and $M > 0$ such that

$[C_1^*]$ $xg(t,x) > 0$ (or $xg(t,x) < 0$), for $t \in R, |x| > M$,

$[C_2^*]$ $\lim\limits_{x\to-\infty} \sup\limits_{t\in R} |\frac{g(t,x)}{x}| \leq C$.

Then Eq.(1.1) has at least one ω-periodic solution, if $1 - |k| > 2C\omega^m$.

Theorem 3.3 Assume n is an odd integer, and if there exist constants $C \geq 0$ and $M > 0$ such that

$[H_1]$ $xg(t,x) > 0$ (or $xg(t,x) < 0$), for $t \in R, |x| > M$,

$[H_2]$ $\lim\limits_{x\to+\infty} \sup\limits_{t\in R} |\frac{g(t,x)}{x}| \leq C$.

$[H_3]$ $\sup f(y) \leq 0$, for $y \in R$.

Then Eq.(1.1) has at least one ω-periodic solution, if $1 - |k| > 2C\omega^m$.

Corollary 3.3 Assume n is an odd integer, and if there exist constants $C \geq 0$ and $M > 0$ such that

$[H_1^*]$ $xg(t,x) > 0$ (or $xg(t,x) < 0$), for $t \in R, |x| > M$,

$[H_2^*]$ $\lim\limits_{x\to-\infty} \sup\limits_{t\in R} |\frac{g(t,x)}{x}| \leq C$.

$[H_3^*]$ $\sup f(y) \leq 0$, for $y \in R$.

Then Eq.(1.1) has at least one ω-periodic solution, if $1 - |k| > 2C\omega^m$.

Remark Conditions of Theorem 3.1 and it's corollary are different from all those in papers[3-12]. Furthermore Conditions $[C_2]$ and $[C_2^*]$ imposed on $g(x)$ in this paper are the type of one sided linear growth, which are weaker than the corresponding ones of [5,6], and also different from the corresponding ones of [10].

IV. PROOF

As the proof of corollary is similar to the corresponding theorem, we only prove the theorems.

Firstly, we show the proof for Theorem 3.1.

Proof It is clear that Eq.(1.1) has an ω-periodic solution if and only if the operator equation $Lx = Nx$ has an ω-periodic solution, where $N : Y \to X$,

$$(Nx)(t) = f(x(t))x'(t) + g(t, \int_{-r}^0 x(t+s)d\alpha(s)) + e(t).$$

Then N is $L-$compact on $\overline{\Omega}$, where Ω is any open and bounded subset of Y, see paper [9] for more details. Take

$$\Omega_1 = \{x | x \in Dom(L),\ Lx = \lambda Nx,\ \lambda \in (0,1)\}.$$

$\forall\, x \in \Omega_1$, then x must satisfy the following equation,

$$(Ax)^{(m)}(t) = \lambda f(x(t))x'(t) + \lambda g(t, \int_{-r}^0 x(t+s)d\alpha(s)) + \lambda e(t). \tag{4.1}$$

Integrating both sides of Eq.(4.1) over $[0,\omega]$, we have

$$\int_0^\omega \left(g(t, \int_{-r}^0 x(t+s)d\alpha(s)) + \overline{e} \right) dt = 0. \tag{4.2}$$

The integral mean value theorem yields there exists a constant $\xi \in (0,\omega)$ such that

$$g(\xi, \int_{-r}^0 x(\xi+s)d\alpha(s)) + \overline{e} = 0. \tag{4.3}$$

So from assumption $[B_1]$ we get $|\int_{-r}^0 x(\xi+s)d\alpha(s)| \leq M$. By the properties of Riemann-Stieltes integral, we know that there must exist a constant $\zeta \in (-r, 0)$ such that $|x(\xi+\zeta)| \leq M$. Because $\xi + \zeta \in R$, there is an integer k_0 such that $\xi + \zeta = k_0\omega + t^*, t^* \in (0,\omega]$, then $|x(t^*)| \leq M$. Hence we have

$$|x(t)| \leq M + \int_0^\omega |x'(s)|ds$$

for all $t \in [0,\omega]$, i.e.,

$$||x||_0 \leq M + \int_0^\omega |x'(t)|dt,\ \forall t \in [0,\omega]. \tag{4.4}$$

On the other hand, multiplying both sides of Eq.(4.1) by $x^{(m-2)}(t-\tau)$ and integrating them on $[0,\omega]$, we obtain

$$\begin{aligned} & k\int_0^\omega |x^{(m-1)}(t-\tau)|^2 dt \\ = & \int_0^\omega x^{(m-1)}(t)x^{(m-1)}(t-\tau)dt \\ & + \lambda \int_0^\omega f(x(t))x'(t)x^{(m-2)}(t-\tau)dt \\ & + \lambda \int_0^\omega x^{(m-2)}(t-\tau)g(t, \int_{-r}^0 x(t+s)d\alpha(s))dt \\ & + \lambda \int_0^\omega x^{(m-2)}(t-\tau)e(t)dt. \end{aligned}$$

By Cauchy inequality, we have

$$\begin{aligned}
&|k|\int_0^\omega |x^{(m-1)}(t-\tau)|^2dt\\
\le&\Big(\int_0^\omega |x^{(m-1)}(t)|^2dt\Big)^{1/2}\Big(\int_0^\omega |x^{(m-1)}(t-\tau)|^2dt\Big)^{1/2}\\
&+|\int_0^\omega f(x(t))x'(t)x^{(m-2)}(t-\tau)dt|\\
&+\int_0^\omega |x^{(m-2)}(t-\tau)||g(t,\int_{-r}^0 x(t+s)d\alpha(s))|dt\\
&+\int_0^\omega |x^{(m-2)}(t-\tau)||e(t)|dt\\
=&\Big(\int_0^\omega |x^{(m-1)}(t)|^2dt\Big)^{1/2}\Big(\int_{-\tau}^{\omega-\tau} |x^{(m-1)}(t)|^2dt\Big)^{1/2}\\
&+|\int_0^\omega f(x(t))x'(t)x^{(m-2)}(t-\tau)dt|\\
&+||e||_0\int_0^\omega |x^{(m-2)}(t)|dt\\
&+\int_0^\omega |x^{(m-2)}(t-\tau)||g(t,\int_{-r}^0 x(t+s)d\alpha(s))|dt\\
=&\int_0^\omega |x^{(m-1)}(t)|^2dt+|\int_0^\omega f(x(t))x'(t)x^{(m-2)}(t-\tau)dt|\\
&+\int_0^\omega |x^{(m-2)}(t-\tau)||g(t,\int_{-r}^0 x(t+s)d\alpha(s))|dt\\
&+||e||_0\int_0^\omega |x^{(m-2)}(t)|dt,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&(|k|-1)\int_0^\omega |x^{(m-1)}(t)|^2dt\\
\le&|\int_0^\omega f(x(t))x'(t)x^{(m-2)}(t-\tau)dt|+||e||_0\int_0^\omega |x^{(m-2)}(t)|dt\\
&+\int_0^\omega |x^{(m-2)}(t-\tau)||g(t,\int_{-r}^0 x(t+s)d\alpha(s))|dt.
\end{aligned}\tag{4.5}$$

In view of $|k|-1>W\omega^{m-1}$, there exists a small constant $\varepsilon>0$ such that

$$|k|-1>(W+\varepsilon)\omega^{m-1}.\tag{4.6}$$

For the small ε, condition implies that there is a constant $\rho>0$(independent of λ) such that

$$|F(x)|\le (W+\varepsilon)|x|\le (W+\varepsilon)||x||_0,\ \text{for } |x|>\rho.\tag{4.7}$$

Let $D_1=\{t\in[0,\omega]:|x(t)|>\rho\}$, $D_2=\{t\in[0,\omega]:|x(t)|\le\rho\}$. Because that

$$\begin{aligned}
&|\int_0^\omega f(x(t))x'(t)x^{(m-2)}(t-\tau)dt|\\
\le&\int_0^\omega |F(x(t))x^{(m-1)}(t-\tau)|dt\\
=&\int_{D_1} |F(x(t))x^{(m-1)}(t-\tau)|dt\\
&+\int_{D_2} |F(x(t))x^{(m-1)}(t-\tau)|dt.
\end{aligned}\tag{4.8}$$

Taking (4.8) into (4.5), we obtain

$$\begin{aligned}
&(|k|-1)\int_0^\omega |x^{(m-1)}(t)|^2dt\\
\le&\int_{D_1} |F(x(t))x^{(m-1)}(t-\tau)|dt\\
&+\int_{D_2} |F(x(t))x^{(m-1)}(t-\tau)|dt\\
&+\int_0^\omega |x^{(m-2)}(t-\tau)||g(t,\int_{-r}^0 x(t+s)d\alpha(s))|dt\\
&+||e||_0\int_0^\omega |x^{(m-2)}(t)|dt.
\end{aligned}\tag{4.9}$$

From assumption $[B_3]$ and (4.3), we know that

$$\begin{aligned}
&\int_0^\omega |g(t,\int_{-r}^0 x(t+s)d\alpha(s))|dt\\
&=\int_0^\omega g(t,\int_{-r}^0 x(t+s)d\alpha(s))dt=-\bar{e}\omega.
\end{aligned}\tag{4.10}$$

Submitting (4.7) and (4.10) into (4.9), we get

$$\begin{aligned}
&(|k|-1)\int_0^\omega |x^{(m-1)}(t)|^2dt\\
\le&(W+\varepsilon)||x||_0\int_0^\omega |x^{(m-1)}(t-\tau)|dt\\
&+(F_\rho+\omega||e||_0)\int_0^\omega |x^{(m-1)}(t-\tau)|dt\\
&+\int_0^\omega |x^{(m-2)}(t)||g(t,\int_{-r}^0 x(t+s)d\alpha(s))|dt\\
\le&(W+\varepsilon)||x||_0\int_0^\omega |x^{(m-1)}(t)|dt\\
&+(F_\rho+\omega||e||_0-\omega^2\bar{e})\int_0^\omega |x^{(m-1)}(t)|dt,
\end{aligned}\tag{4.11}$$

where $F_\rho=\max_{x\in D_2}|F(x)|$. As $x(0)=x(\omega),x'(0)=x'(\omega),\cdots,x^{(m)}(0)=x^{(m)}(\omega)$, there exist constants $t_i\in(0,\omega)$ such that $x^{(i)}(t_i)=0$, $i=1,2,\cdots,m$, it follows from (4.4) that

$$||x||_0\le M+\omega^{m-2}\int_0^\omega |x^{(m-1)}(t)|dt.\tag{4.12}$$

Taking (4.12) into (4.11), we get

$$\begin{aligned}
&(|k|-1)\int_0^\omega |x^{(m-1)}(t)|^2dt\\
\le&(W+\varepsilon)\Big(M+\omega^{m-2}\int_0^\omega |x^{(m-1)}(t)|dt\Big)\int_0^\omega |x^{(m-1)}(t)|dt\\
&+(F_\rho+\omega||e||_0-\bar{e}\omega^2)\int_0^\omega |x^{(m-1)}(t)|dt\\
\le&(W+\varepsilon)\omega^{m-1}\int_0^\omega |x^{(m-1)}(t)|^2dt\\
&+\Big((W+\varepsilon)M+F_\rho+\omega||e||_0-\bar{e}\omega^2\Big)\int_0^\omega |x^{(m-1)}(t)|dt,
\end{aligned}$$

which together with (4.6) gives

$$\begin{aligned}
&\int_0^\omega |x^{(m-1)}(t)|^2dt\\
&\le\frac{(W+\varepsilon)M+F_\rho+\omega||e||_0-\bar{e}\omega^2}{|k|-1-(W+\varepsilon)\omega^{(m-1)}}\int_0^\omega |x^{(m-1)}(t)|dt.
\end{aligned}$$

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So there exists a constant M_1(which independent of λ and x), such that $\int_0^\omega |x^{(m-1)}(t)|^2 dt \le M_1$, which together with (4.12) yields there exist positive constants M_2 and M_3, such that

$$||x||_0 \le M + \omega^{m-2}\sqrt{\omega M_1} := M_2$$

and

$$||x'||_0 \le \omega^{m-3}\sqrt{\omega M_1} := M_3.$$

Let $\widetilde{M} = \max\{M_2, M_3\} + 1$, $\Omega = \{x : ||x|| < \widetilde{M}\}$ and $\Omega_2 = \{x \in \partial(\Omega \bigcap KerL)\}$. Then

$$QNx = \frac{1}{\omega}\int_0^\omega \left(g(t, \int_{-r}^0 x(t+s)d\alpha(s)) + \overline{e}\right)dt.$$

If $x = \widetilde{M}\ or\ -\widetilde{M}$, then

$$g(t, \int_{-r}^0 x(t+s)d\alpha(s)) + \overline{e} > 0,$$

which yields $QNx \neq 0$ for all $x \in \Omega_2$. Thus condition $[A_1]$ and $[A_2]$ of Lemma 2.2 are both satisfied. Next, we show that condition $[A_3]$ of Lemma 2.2 is also satisfied. In order to do it, define the isomorphism $J : ImQ \to KerL, J(x) \equiv x$, and the operator $H(x, \mu)$ as follows,

$$H(x, \mu) = -\mu x - \frac{1-\mu}{T}JQNx,\ \forall (x, \mu) \in \Omega \times [0, 1].$$

Then we have, for all $(x, \mu) \in \Omega_2 \times [0, 1]$,

$$H(x, \mu) = -\mu x - \frac{1-\mu}{\omega}\int_0^\omega \left(g(t, \int_{-r}^0 x(t+s)d\alpha(s)) + \overline{e}\right)dt.$$

Similar to the above proof, we can prove $H(x, \mu) \neq 0$. Hence

$$deg\{JQN, \Omega \cap KerL, 0\} = deg\{H(x, 1), \Omega \cap KerL, 0\} \neq 0.$$

So condition $[A_3]$ of Lemma 2.2 is also satisfied. By applying Lemma 2.2, we know that the operator equation $Lx = Nx$ has at least one solution $x(t)$ in $\overline{\Omega} \cap D(L)$, i.e., Eq.(1.1) has at least one $\omega-$periodic solution $x(t)$.

Secondly, we prove Theorem 3.2.

proof As m is even, there must be an integer $z(z \in Z^+)$ such that $m = 2z$, then multiplying both sides of Eq.(3.2) by $x(t)$ and integrating them on interval $[0, \omega]$, we obtain

$$\begin{aligned}
&\int_0^\omega |x^{(z)}(t)|^2 dt \\
=&k\int_0^\omega x^{(z)}(t)x^{(z)}(t-\tau)dt \\
&+ (-1)^z\lambda\int_0^\omega x(t)g(t, \int_{-r}^0 x(t+s)d\alpha(s))dt \\
&+ (-1)^z\lambda\int_0^\omega x(t)e(t)dt \\
\le&|k|\int_0^\omega |x^{(z)}(t)||x^{(z)}(t-\tau)|dt \\
&+ \int_0^\omega |x(t)|\,|g(t, \int_{-r}^0 x(t+s)d\alpha(s))|dt \\
&+ \int_0^\omega |x(t)||e(t)|dt.
\end{aligned}$$

By Cauchy inequality, we have

$$\begin{aligned}
&\int_0^\omega |x^{(z)}(t)|^2 dt \\
\le&|k|\left(\int_0^\omega |x^{(z)}(t)|^2 dt\right)^{1/2}\left(\int_0^\omega |x^{(z)}(t-\tau)|^2 dt\right)^{1/2} \\
&+ \int_0^\omega |x(t)||g(\int_{-r}^0 x(t+s)d\alpha(s))dt + \int_0^\omega |x(t)||e(t)|dt \\
\le&\frac{1}{1-|k|}||x||_0(\int_0^\omega |g(t, \int_{-r}^0 x(t+s)d\alpha(s))|dt + \omega||e||_0).
\end{aligned} \tag{4.13}$$

Note that $1 - |k| > 2C\omega^m$, so there exists a small constant $\varepsilon > 0$ such that $1 - |k| > 2(C + \varepsilon)\omega^m$. From condition $[C_2]$, and the properties of bounded variation function, we get that there exists a constant $\rho > M$ such that

$$\begin{aligned}
&|g(t, \int_{-r}^0 x(t+s)d\alpha(s))| \\
&\le (C+\varepsilon)|\int_{-r}^0 x(t+s)d\alpha(s)| \\
&\le (C+\varepsilon)||x||_0,\ \forall\ t \in R, \int_{-r}^0 x(t+s)d\alpha(s) > \rho.
\end{aligned} \tag{4.14}$$

Let $X(t) = \int_{-r}^0 x(t+s)d\alpha(s)$, we set $E_1 = \{t \in [0, \omega] : X(t) > \rho\}$, $E_2 = \{t \in [0, \omega] : |X(t)| \le \rho\}$, $E_3 = \{t \in [0, \omega] : X(t) < -\rho\}$. It is easy to see from (4.2) that

$$\left(\int_{E_1} + \int_{E_2} + \int_{E_3}\right)g(t, \int_{-r}^0 x(t+s)d\alpha(s))dt = -\omega\overline{e},$$

which together with assumption $[C_1]$ leads to that

$$\begin{aligned}
&\int_{E_3} |g(t, \int_{-r}^0 (x(t+s)d\alpha(s))|dt \\
&\le \left(\int_{E_1} + \int_{E_2}\right)|g(t, \int_{-r}^0 x(t+s)d\alpha(s))|dt + \omega\overline{e}.
\end{aligned} \tag{4.15}$$

Combination of (4.14) and (4.15) gives

$$\begin{aligned}
&\int_0^\omega |g(t, \int_{-r}^0 x(t+s)d\alpha(s))|dt \\
&\le 2\left(\int_{E_1} + \int_{E_2}\right)|g(t, \int_{-r}^0 x(t+s)d\alpha(s))|dt + \omega\overline{e} \\
&\le 2\omega(C+\varepsilon)||x||_0 + 2\tilde{g}_\rho\omega + \omega\overline{e},
\end{aligned} \tag{4.16}$$

where $\tilde{g}_\rho = \max_{t \in E_2}|g(\int_{-r}^0 x(t+s)d\alpha(s))|$. From (4.12) and (4.16), we have

$$\int_0^\omega |x^{(z)}(t)|^2 dt \le \frac{2\omega(C+\varepsilon)}{1-|k|}||x||_0^2 + \frac{(2\tilde{g}_\rho + \overline{e} + ||e||_0)\omega}{1-|k|}||x||_0.$$

From $x(0) = x(\omega), x'(0) = x'(\omega), \cdots, x^{(z-1)}(0) = x^{(z-1)}(\omega)$, we know that there exist $\xi_i \in (0, \omega)$, $i = 1, 2, \cdots, z$, such that $x'(\xi_1) = x''(\xi_2) = \cdots = x^{(z)}(\xi_z) = 0$. Hence we get

$$||x||_0 \le M + \omega^{z-1}\int_0^\omega |x^{(z)}(t)|dt, \tag{4.17}$$

and

$$||x'||_0 \le \omega^{z-2}\int_0^\omega |x^{(z)}(t)|dt. \tag{4.18}$$

So it follows from (4.17) and (4.18) that

$$\begin{aligned}
&\int_0^\omega |x^{(z)}(t)|^2 dt \\
\leq &\frac{2\omega(C+\varepsilon)}{1-|k|}||x||_0^2 + \frac{(2\tilde{g}_\rho + \overline{e} + ||e||_0)\omega}{1-|k|}||x||_0 \\
\leq &\frac{2\omega(C+\varepsilon)}{1-|k|}\left(M + \omega^{z-1}\int_0^\omega |x^{(z)}(t)|dt\right)^2 \\
&+ \frac{(2\tilde{g}_\rho + \overline{e} + ||e||_0)\omega}{1-|k|}\left(M + \omega^{z-1}\int_0^\omega |x^{(z)}(t)|dt\right) \\
\leq &\frac{2\omega^m(C+\varepsilon)}{1-|k|}\int_0^\omega |x^{(z)}(t)|^2 dt + d_1\int_0^\omega |x^{(z)}(t)|dt + d_2,
\end{aligned}$$

where $d_1 = (4\omega M(C+\varepsilon) + 2\tilde{g}_\rho + \overline{e} + ||e||_0)\omega^{z-1}/1-|k|$, $d_2 = (2\omega M(C+\varepsilon) + (2\tilde{g}_\rho + \overline{e} + ||e||_0))M/1-|k|$. As $1-|k| > 2(C+\varepsilon)\omega^m$, there is a constant $M_2 > 0$ such that $\int_0^\omega |x^{(z)}(t)|^2 dt < M_2$. The remainder can be proved in the same way as that in theorem 3.1.

Now, we give the proof of Theorem 3.3 briefly.

Proof Note that m is an odd number, so there exists a constant $z (z \in Z^+)$ such that $m = 2z-1$. Multiplying both sides of Eq.(4.2) by $x'(t)$, and integrating them on interval $[0,\omega]$, from assumption $[H_3]$, we have

$$\begin{aligned}
&\int_0^\omega |x^{(z)}(t)|^2 dt \\
=&k\int_0^\omega x^{(z)}(t-\tau)\, x^{(z)}(t)dt \\
&+ (-1)^{z-1}\lambda\int_0^\omega f(x(t))[x'(t)]^2 dt \\
&+ (-1)^{z-1}\lambda\int_0^\omega x'(t)g(t,\int_{-r}^0 x(t+s)d\alpha(s))dt \\
&+ (-1)^{z-1}\lambda\int_0^\omega x'(t)e(t)dt \\
\leq&|k|\int_0^\omega |x^{(z)}(t-\tau)|\,|x^{(z)}(t)|dt \\
&+ \int_0^\omega |x'(t)|\,|g(t,\int_{-r}^0 x(t+s)d\alpha(s))|dt \\
&+ \omega||x'||_0||e||_0.
\end{aligned} \tag{4.19}$$

Since $||x||_0 \leq M + \int_0^\omega |x'(t)|dt \leq M + \omega||x'||_0$, then by Cauchy inequality and (4.19), we obtain

$$\begin{aligned}
&\int_0^\omega |x^{(z)}(t)|^2 dt \\
\leq &|k|\left(\int_0^\omega |x^{(z)}(t)|^2 dt\right)^{1/2}\left(\int_{-\tau}^{\omega-\tau} |x^{(z)}(t)|^2 dt\right)^{1/2} \\
&+ \int_0^\omega |x'(t)||g(t,\int_{-r}^0 x(t+s)d\alpha(s))|dt + \omega||x'||_0||e||_0 \\
\leq &\frac{2\omega^2(C+\varepsilon)}{1-|k|}||x'||_0^2 + d_1||x'||_0,
\end{aligned} \tag{4.20}$$

where $d_1 = (2M(C+\varepsilon) + 2\tilde{g}_\rho + \overline{e} + ||e||_0)\omega/1-|k|$.

Thus, from (4.18) and (4.20), we get

$$\begin{aligned}
\int_0^\omega |x^{(z)}(t)|^2 dt \leq &\frac{2\omega^{2z-2}(C+\varepsilon)}{1-|k|}\left(\int_0^\omega |x^{(z)}(t)|dt\right)^2 \\
&+ d_1\omega^{z-2}\int_0^\omega |x^{(z)}(t)|dt. \\
\leq &\frac{2\omega^m(C+\varepsilon)}{1-|k|}\int_0^\omega |x^{(z)}(t)|^2 dt \\
&+ d_1\omega^{z-2}\int_0^\omega |x^{(z)}(t)|dt.
\end{aligned} \tag{4.21}$$

Assumption $1-|k| > 2C\omega^m$ implies that there exists a constant $\varepsilon > 0$ such that $1-|k| > 2(C+\varepsilon)\omega^m$. Hence (4.21) yields that there is a constant M_2(independent of λ and x) such that $\int_0^\omega |x^{(z)}(t)|^2 dt \leq M_2$. In the same way as that in theorem 3.1, we can easily prove Theorem 3.3.

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