

# Traveling Wave Solutions For The Sawada-Kotera-Kadomtsev-Petviashvili Equation And The Bogoyavlensky-Konoplechenko Equation By $\left(\frac{G'}{G}\right)$ -Expansion Method

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**Abstract**—This paper presents a new function expansion method for finding traveling wave solutions of a nonlinear equations and calls it the  $\left(\frac{G'}{G}\right)$ -expansion method, given by Wang et al recently. As an application of this new method, we study the well-known Sawada-Kotera-Kadomtsev-Petviashvili equation and Bogoyavlensky-Konoplechenko equation. With two new expansions, general types of soliton solutions and periodic solutions for these two equations are obtained.

**Keywords**—Sawada-Kotera-Kadomtsev-Petviashvili equation, Bogoyavlensky-Konoplechenko equation,  $\left(\frac{G'}{G}\right)$ -expansion method, Exact solutions.

## I. INTRODUCTION

**T**HE nonlinear phenomena exist in all the fields including either the scientific work or engineering fields, such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, chemical physics, and so on. It is well known that many non-linear evolution equations (NLEEs) are widely used to describe these complex phenomena. Research on solutions of NLEEs is popular. So, the powerful and efficient methods to find analytic solutions and numerical solutions of nonlinear equations have drawn a lot of interest by a diverse group of scientists. Some of these approaches are the homogeneous balance method [1,2], the hyperbolic tangent expansion method [3,4], the tanh-method [5], the inverse scattering transform [6], the Bäcklund transform [7], the Hirota bilinear method [8,9], the generalized Riccati equation [10,11], the Weierstrass elliptic function method [12,13], the sine-cosine method [14,15], the Jacobi elliptic function expansion [16,17], the truncated Painleve expansion [18], Lie Classical method [19] and so on.

Among the possible exact solutions of NLEEs, certain solutions for special form may depend only on a single combination of variables such as traveling wave variables. Also there is a wide variety of approaches to nonlinear problems for constructing traveling wave solutions. Recently a so-called  $\left(\frac{G'}{G}\right)$ -expansion method has drawn a lot of attention. The method was presented by Mingliang Wang in [20] at first. The

main merits of the  $\left(\frac{G'}{G}\right)$ -expansion method over the other methods are that it gives more general solutions with some free parameters and it handles NLEEs in a direct manner with no requirement for initial/boundary condition or initial trial function at the outset. The method was soon been applied to other non-linear problems by several authors [21,22,23].

In this paper, we pay attention to the analytical method for getting the exact solution of some NLEEs. Among the possible exact solutions of NLEEs, certain solutions for special form may depend only on a single combination of variables such as traveling wave variables. Our main goal in this study is to present the  $\left(\frac{G'}{G}\right)$ -expansion method for constructing the traveling wave solutions.

In section II, we describe the  $\left(\frac{G'}{G}\right)$ -expansion method. In section III, in order to illustrate the method we apply the method to two physically important nonlinear evolution equations, namely, the Sawada-Kotera-Kadomtsev-Petviashvili equation and the Bogoyavlensky-Konoplechenko equation and abundant exact solutions are obtained which included the hyperbolic functions, the trigonometric functions and rational functions. Finally, we record some concluding remarks.

## II. $\left(\frac{G'}{G}\right)$ -EXPANSION METHOD

We assume the given nonlinear partial differential equation for  $u(x, y, t)$  to be in the form

$$P(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, u_{xt}, \dots) = 0, \quad (1)$$

where P is a polynomial in its arguments. The essence of the  $\left(\frac{G'}{G}\right)$ -expansion method can be presented in the following steps:

**Step 1.** Find traveling wave solutions of equation (1) by taking  $u(x, y, t) = u(\xi)$ ,  $\xi = x + y - kt$  and transform equation (1) to the ordinary differential equation

$$Q(u, u', u'', \dots) = 0, \quad (2)$$

where prime denotes the derivative with respect to  $\xi$ .

**Step 2.** If possible, integrate equation (2) term by term one or more times. This yields constants of integration. For

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simplicity, the integration constants can be set to zero.

**Step 3.** Introduce the solution  $u(\xi)$  of equation (2) in the finite series form

$$u(\xi) = \sum_{i=0}^N a_i \left( \frac{G'(\xi)}{G(\xi)} \right), \quad (3)$$

where  $a_i$  are real constants with  $a_N \neq 0$  to be determined,  $N$  is a positive integer to be determined. The function  $G(\xi)$  is the solution of the auxiliary linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \quad (4)$$

where  $\lambda$  and  $\mu$  are real constants to be determined.

**Step 4.** Determine  $N$ . This, usually, can be accomplished by balancing the linear terms of highest order with the highest order nonlinear terms in equation (2).

**Step 5.** Substituting (3) together with (4) into equation (2) yields an algebraic equation involving powers of  $\left(\frac{G'}{G}\right)$ .

Equating the coefficients of each power of  $\left(\frac{G'}{G}\right)$  to zero gives a system of algebraic equations for  $a_i, \lambda, \mu$  and  $k$ . Then, we solve the system with the aid of a computer algebra system, such as Maple, to determine these constants. On the other hand, depending on the sign of the discriminant  $D = \lambda^2 - 4\mu$  the solutions of equation (4) are well known for us. So, we can obtain exact solutions of equation (1).

### III. APPLICATIONS OF $\left(\frac{G'}{G}\right)$ METHOD

In this section, we apply the  $\left(\frac{G'}{G}\right)$ -expansion method to solve the Sawada-Kotera-Kadomtsev-Petviashvili equation and Bogoyavlensky-Konoplechenko equation.

#### III.1 Sawada-Kotera-Kadomtsev-Petviashvili (SKKP) Equation

The Sawada-Kotera-Kadomtsev-Petviashvili (SKKP) equation is

$$(u_t + 15uu_{xxx} + 15u_x u_{xx} + 45u^2 u_x + u_{xxxxx})_x + u_{yy} = 0, \quad (5)$$

where  $u$  is function of  $x, y$  and  $t$ .

Equation (5) can be written as

$$u_{tx} + 15u_x u_{xxx} + 15u u_{xxx} + 15u_{xx}^2 + 15u_x u_{xxx} + 45u^2 u_{xx} + 90u u_x^2 + u_{xxxxx} + u_{yy} = 0. \quad (6)$$

According to the method described above in section 2, we make the transformation  $u(x, y, t) = u(\xi), \xi = x + y - kt$ . Then we get

$$-ku'' + 15u'u''' + 15uu'''' + 15u''^2 + 15u'u''' + 45u^2u'' + 90uu'^2 + u'''''' + u'' = 0, \quad (7)$$

where prime denotes the derivative with respect to  $\xi$ .

Now, balancing  $uu''''$  with  $u''''''$  gives  $N = 2$ . Therefore, we can write the solution of equation (7) in the form

$$u(\xi) = a_0 + a_1 \left( \frac{G'}{G} \right) + a_2 \left( \frac{G'}{G} \right)^2, \quad (8)$$

where  $a_2 \neq 0$  and  $G = G(\xi)$ . From equations (4) and (8), we derive

$$u'(\xi) = -2a_2 \left( \frac{G'}{G} \right)^3 - (a_1 + 2a_2\lambda) \left( \frac{G'}{G} \right)^2 - (a_1\lambda + 2a_2\mu) \left( \frac{G'}{G} \right) - a_1\mu, \quad (9)$$

$$u''(\xi) = 6a_2 \left( \frac{G'}{G} \right)^4 + (10a_2\lambda + 2a_1) \left( \frac{G'}{G} \right)^3 + (4a_2\lambda^2 + 8a_2\mu + 3a_1\lambda) \left( \frac{G'}{G} \right)^2 + (a_1\lambda^2 + 2a_1\mu + 6a_2\lambda\mu) \left( \frac{G'}{G} \right) + a_1\lambda\mu, \quad (10)$$

$$u'''(\xi) = -24a_2 \left( \frac{G'}{G} \right)^5 - (6a_1 + 54a_2\lambda) \left( \frac{G'}{G} \right)^4 - (40a_2\mu + 38a_2\lambda^2 + 12a_1\lambda) \left( \frac{G'}{G} \right)^3 - (52a_2\lambda\mu + 8a_2\lambda^3 + 7a_1\lambda^2 + 8a_1\mu) \left( \frac{G'}{G} \right)^2 - (8a_1\lambda\mu + 14a_2\lambda^2\mu + 16a_2\mu^2 + a_1\lambda^3) \left( \frac{G'}{G} \right) - 6a_2\lambda\mu^2 - 2a_1\mu^2 - a_1\lambda^2\mu, \quad (11)$$

$$u''''(\xi) = 120a_2 \left( \frac{G'}{G} \right)^6 + (336a_2\lambda + 24a_1) \left( \frac{G'}{G} \right)^5 + (330a_2\lambda^2 + 240a_2\mu + 60a_1\lambda) \left( \frac{G'}{G} \right)^4 + (50a_1\lambda^2 + 130a_2\lambda^3 + 40a_1\mu + 440a_2\lambda\mu) \left( \frac{G'}{G} \right)^3 + (15a_1\lambda^3 + 16a_2\lambda^4 + 60a_1\lambda\mu + 232a_2\lambda^2\mu + 136a_2\mu^2) \left( \frac{G'}{G} \right)^2 + (a_1\lambda^4 + 22a_1\lambda^2\mu + 120a_2\lambda\mu^2 + 16a_1\mu^2 + 30a_2\lambda^3\mu) \left( \frac{G'}{G} \right) + 16a_2\mu^3 + 14a_2\lambda^2\mu^2 + a_1\lambda^3\mu + 8a_1\lambda\mu^2, \quad (12)$$

$$u''''''(\xi) = 5040a_2 \left( \frac{G'}{G} \right)^8 + (19440a_2\lambda + 720a_1) \left( \frac{G'}{G} \right)^7 + (2520a_1\lambda + 29400a_2\lambda^2 + 13440a_2\mu) \left( \frac{G'}{G} \right)^6 + (21840a_2\lambda^3 + 38640a_2\lambda\mu + 1680a_1\mu) \left( \frac{G'}{G} \right)^5 + (3360a_1\lambda^2) \left( \frac{G'}{G} \right)^4 + (40152a_2\lambda^2\mu + 8106a_2\lambda^4 + 4200a_1\lambda\mu) \left( \frac{G'}{G} \right)^3 + (12096a_2\mu^2 + 2100a_1\lambda^3) \left( \frac{G'}{G} \right)^2 + (1232a_1\mu^2 + 17920a_2\lambda^3\mu) \left( \frac{G'}{G} \right) + 22960a_2\lambda\mu^2 + 602a_1\lambda^4 + 1330a_2\lambda^5 + 3584a_1\lambda^2\mu \left( \frac{G'}{G} \right)^3 + (3968a_2\mu^2 + 63a_1\lambda^5 + 3096a_2\lambda^4\mu + 1176a_1\lambda^3\mu + 13320a_2\lambda^2\mu^2) \left( \frac{G'}{G} \right)^2 + (64a_2\lambda^6 + 1848a_1\lambda\mu^2) \left( \frac{G'}{G} \right) + (2352a_2\lambda^3\mu^2 + a_1\lambda^6 + 720a_1\lambda^2\mu^2) \left( \frac{G'}{G} \right) + 3696a_2\lambda\mu^3 + 272a_1\mu^3 + 114a_1\lambda^4\mu + 126\lambda^5\mu \left( \frac{G'}{G} \right) + 272a_2\mu^4 + 52a_1\lambda^3\mu^2 + 62a_2\lambda^4\mu^2 + a_1\lambda^5\mu + 584a_2\lambda^2\mu^3 + 136a_1\lambda\mu^3, \quad (13)$$

Substituting equations (9-13) into equation (7), setting the coefficients of  $\left(\frac{G'}{G}\right)^i, (i = 0, 1, 2, 3, 4, 5, 6, 7, 8)$  to zero, we obtain a system of algebraic equations for  $a_0, a_1, a_2, k, \lambda$  and

$\mu$  as follows:

$$\begin{aligned} \left(\frac{G'}{G}\right)^8 &: -3780 a_2^2 - 630 a_2^3 - 5040 a_2 = 0, \\ \left(\frac{G'}{G}\right)^7 &: -720 a_1 - 1170 a_2^3 \lambda - 3600 a_1 a_2 - 19440 a_2 \lambda \\ &- 11520 a_2^2 \lambda - 1350 a_1 a_2^2 = 0, \\ \left(\frac{G'}{G}\right)^6 &: -13440 a_2 \mu - 29400 a_2 \lambda^2 - 10500 a_1 a_2 \lambda \\ &- 540 a_2^3 \lambda^2 - 12690 a_2^2 \lambda^2 - 2520 a_1 \lambda - 900 a_0 a_2^2 \\ &- 2475 a_1 a_2^2 \lambda - 8880 a_2^2 \mu - 1800 a_0 a_2 - 1080 a_2^3 \mu \\ &- 600 a_1^2 - 900 a_1^2 a_2 = 0, \\ \left(\frac{G'}{G}\right)^5 &: -180 a_1^3 - 1080 a_0 a_1 a_2 - 360 a_0 a_1 - 990 a_2^3 \lambda \mu \\ &- 3360 a_1 \lambda^2 - 1620 a_1^2 \lambda - 2250 a_1 a_2^2 \mu - 10920 a_1 \lambda^2 a_2 \\ &- 1680 a_1 \mu - 21840 a_2 \lambda^3 - 1620 a_1^2 a_2 \lambda - 1620 a_0 a_2^2 \lambda \\ &- 18840 a_2^2 \lambda \mu - 5040 a_0 a_2 \lambda - 38640 a_2 \lambda \mu - 7800 a_1 \mu a_2 \\ &- 5910 a_2^2 \lambda^3 - 1125 a_1 a_2^2 \lambda^2 = 0, \\ \left(\frac{G'}{G}\right)^4 &: -6 a_2 - 1890 a_0 a_1 a_2 \lambda - 2025 a_1 a_2^2 \lambda \mu \\ &- 15420 a_1 \lambda \mu a_2 - 4695 a_1 \lambda^3 a_2 - 4950 a_0 a_2 \lambda^2 \\ &- 1440 a_1^2 a_2 \mu - 1440 a_0 a_2^2 \mu - 4200 a_1 \lambda \mu - 270 a_0^2 a_2 \\ &- 960 a_2^2 \lambda^4 - 12096 a_2 \mu^2 - 2100 a_1 \lambda^3 - 1515 a_1^2 \lambda^2 \\ &- 450 a_2^3 \mu^2 - 6720 a_2^2 \mu^2 + 6 k a_2 - 315 a_1^3 \lambda - 1140 a_1^2 \mu \\ &- 270 a_0 a_1^2 - 8106 a_2 \lambda^4 - 900 a_0 a_1 \lambda - 720 a_0 a_2^2 \lambda^2 \\ &- 40152 a_2 \lambda^2 \mu - 720 a_1^2 a_2 \lambda^2 - 3600 a_0 a_2 \mu \\ &- 12480 a_2^2 \lambda^2 \mu = 0, \\ \left(\frac{G'}{G}\right)^3 &: -10 a_2 \lambda - 2 a_1 - 1980 a_1^2 \lambda \mu - 1950 a_0 a_2 \lambda^3 \mu \\ &- 8280 a_2^2 \lambda \mu^2 - 750 a_0 a_1 \lambda^2 - 3584 a_1 \lambda^2 \mu - 2490 a_2^2 \lambda^3 \mu \\ &- 900 a_1 a_2^2 \mu^2 - 450 a_0 a_1^2 \lambda - 675 a_1 \lambda^4 a_2 - 600 a_0 a_1 \mu \\ &- 22960 a_2 \lambda \mu^2 - 450 a_0^2 a_2 \lambda - 1232 a_1 \mu^2 - 602 a_1 \lambda^4 \\ &- 9210 a_1 \lambda^2 a_2 \mu - 6600 a_0 a_2 \lambda \mu - 810 a_0 a_1 a_2 \lambda^2 \\ &- 1620 a_0 a_1 a_2 \mu - 1260 a_0 a_2^2 \lambda \mu - 1260 a_1^2 a_2 \lambda \mu \\ &- 17920 a_2 \lambda^3 \mu - 5160 a_1 \mu^2 a_2 + 10 k a_2 \lambda - 135 a_1^3 \lambda^2 \\ &- 1330 a_2 \lambda^5 - 90 a_0^2 a_1 - 555 a_1^2 \lambda^3 + 2 k a_1 - \\ &270 a_1^3 \mu = 0, \\ \left(\frac{G'}{G}\right)^2 &: -1848 a_1 \lambda \mu^2 - 135 a_0^2 a_1 \lambda - 180 a_0 a_1^2 \lambda^2 \\ &- 240 a_0 a_2 \lambda^4 - 13320 a_2 \lambda^2 \mu^2 - 1176 a_1 \lambda^3 \mu - 540 a_1^2 a_2 \mu^2 \\ &+ 8 k a_2 \mu - 360 a_0 a_1^2 \mu - 2040 a_0 a_2 \mu^2 - 360 a_0^2 a_2 \mu \\ &- 180 a_0^2 a_2 \lambda^2 + 3 k a_1 \lambda - 3096 a_2 \lambda^4 \mu - 960 a_1^2 \lambda^2 \mu \\ &- 540 a_0 a_2^2 \mu^2 - 2190 a_2^2 \lambda^2 \mu^2 + 4 k a_2 \lambda^2 - 225 a_1^3 \lambda \mu \\ &- 1350 a_0 a_1 a_2 \lambda \mu - 225 a_0 a_1 \lambda^3 - 60 a_1^2 \lambda^4 - 64 a_2 \lambda^6 \\ &- 3968 a_2 \mu^3 - 600 a_1^2 \mu^2 - 1680 a_2^2 \mu^3 - 63 a_1 \lambda^5 \\ &- 3480 a_0 a_2 \lambda^2 \mu - 900 a_0 a_1 \lambda \mu - 5520 a_1 \lambda \mu^2 a_2 \\ &- 1545 a_1 \lambda^3 a_2 \mu - 3 a_1 \lambda - 8 a_2 \mu - 4 a_2 \lambda^2 = 0, \\ \left(\frac{G'}{G}\right) &: -450 a_0 a_2 \lambda^3 \mu - 15 a_0 a_1 \lambda^4 - 240 a_0 a_1 \mu^2 \\ &- 90 a_0^2 a_1 \mu - 720 a_1 \lambda^2 \mu^2 - 272 a_1 \mu^3 - 960 a_1 \mu^3 a_2 \\ &- 3696 a_2 \lambda \mu^3 + 2 k a_1 \mu - 114 a_1 \lambda^4 \mu - 126 a_2 \lambda^5 \mu - a_1 \lambda^2 \\ &- 6 a_2 \lambda \mu - 2 a_1 \mu - a_1 \lambda^6 - 90 a_1^3 \mu^2 - 105 a_1^2 \lambda^3 \mu \\ &+ k a_1 \lambda^2 - 720 a_2^2 \lambda \mu^3 - 2352 a_2 \lambda^3 \mu^2 - 480 a_1^2 \lambda \mu^2 \\ &- 270 a_0^2 a_2 \lambda \mu - 270 a_0 a_1^2 \lambda \mu - 540 a_0 a_1 a_2 \mu^2 + 6 k a_2 \lambda \mu \\ &- 1110 a_1 \lambda^2 a_2 \mu^2 - 330 a_0 a_1 \lambda^2 \mu - 1800 a_0 a_2 \lambda \mu^2 \\ &- 45 a_0^2 a_1 \lambda^2 = 0, \\ \left(\frac{G'}{G}\right)^0 &: -62 a_2 \lambda^4 \mu^2 - a_1 \lambda \mu - 210 a_0 a_2 \lambda^2 \mu^2 - 60 a_2^2 \mu^4 \\ &- 60 a_1^2 \mu^3 - 15 a_0 a_1 \lambda^3 \mu - 240 a_1 \lambda \mu^3 a_2 + k a_1 \lambda \mu \\ &+ 2 k a_2 \mu^2 - 45 a_1^2 \lambda^2 \mu^2 - 120 a_0 a_1 \lambda \mu^2 - 90 a_0 a_1^2 \mu^2 \\ &- 90 a_0^2 a_2 \mu^2 - 272 a_2 \mu^4 - 52 a_1 \lambda^3 \mu^2 - 45 a_0^2 a_1 \lambda \mu \\ &- 2 a_2 \mu^2 - 240 a_0 a_2 \mu^3 - a_1 \lambda^5 \mu - 136 a_1 \lambda \mu^3 \\ &- 584 a_2 \lambda^2 \mu^3 = 0, \end{aligned} \tag{14}$$

Solving these systems of algebraic equations by Maple gives

**Case 1.**

$$\begin{aligned} k &= 1 + \lambda^4 + 76\mu^2 + 22\lambda^2\mu + 120a_0\mu + 15a_0\lambda^2 + 45a_0^2, \\ a_1 &= -2\lambda, a_2 = -2, \end{aligned} \tag{15}$$

and  $\mu, \lambda$  and  $a_0$  arbitrary constants.

**Case 2.**

$$\begin{aligned} k &= 1 + \frac{15}{4} a_0 a_1^2 + \frac{1}{16} a_1^4 + 45 a_0^2, \\ \lambda &= -\frac{1}{2} a_1, a_2 = -2, \mu = 0, \end{aligned} \tag{16}$$

and  $a_0$  and  $a_1$  are arbitrary constants.

For Case 1, Substituting the solution set (15) and the corresponding solutions of (4) into (8), we have the solutions of equation (7) as follows:

When  $\lambda^2 - 4\mu > 0$ , we obtain the hyperbolic function traveling wave solutions

$$\begin{aligned} u_{11}(\xi) &= \\ a_0 - 2\lambda &\left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)} - \frac{\lambda}{2} \right) \right) \\ &- 2 \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)} - \frac{\lambda}{2} \right) \right)^2. \end{aligned} \tag{17}$$

When  $\lambda^2 - 4\mu < 0$ , we obtain the trigonometric function traveling wave solutions

$$\begin{aligned} u_{12}(\xi) &= \\ a_0 - 2\lambda &\left( \frac{\sqrt{4\mu - \lambda^2}}{2} \left( \frac{-C_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + C_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)}{C_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)} - \frac{\lambda}{2} \right) \right) \\ &- 2 \left( \frac{\sqrt{4\mu - \lambda^2}}{2} \left( \frac{-C_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + C_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)}{C_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)} - \frac{\lambda}{2} \right) \right)^2. \end{aligned} \tag{18}$$

When  $\lambda^2 - 4\mu = 0$ , we obtain the rational function solutions

$$u_{13}(\xi) = \frac{C_2}{C_1 + C_2 \xi} - \frac{\lambda}{2}, \tag{19}$$

where  $\xi = x + y - (1 + \lambda^4 + 76\mu^2 + 22\lambda^2\mu + 120a_0\mu + 15a_0\lambda^2 + 45a_0^2)t$ .

For Case 2, Substituting the solution set (16) and the corresponding solutions of (4) into (8), we have the solutions of equation (7) as follows:

When  $\lambda^2 - 4\mu > 0$ , we obtain the hyperbolic function traveling wave solutions

$$\begin{aligned} u_{21}(\xi) &= a_0 \\ &+ a_1 \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)} - \frac{\lambda}{2} \right) \right) \\ &- 2 \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)} - \frac{\lambda}{2} \right) \right)^2. \end{aligned} \tag{20}$$

When  $\lambda^2 - 4\mu < 0$ , we obtain the trigonometric function traveling wave solutions

$$u_{22}(\xi) = a_0 + a_1 \left( \frac{\sqrt{4\mu - \lambda^2}}{2} \left( \frac{-C_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + C_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)}{C_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)} - \frac{\lambda}{2} \right) \right) - 2 \left( \frac{\sqrt{4\mu - \lambda^2}}{2} \left( \frac{-C_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + C_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)}{C_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)} - \frac{\lambda}{2} \right) \right)^2 \quad (21)$$

When  $\lambda^2 - 4\mu = 0$ , we obtain the rational function solutions

$$u_{23}(\xi) = \frac{C_2}{C_1 + C_2 \xi} - \frac{\lambda}{2}, \quad (22)$$

where  $\lambda = -\frac{1}{2}a_1, \mu = 0$  and  $\xi = x + y - (1 + \frac{15}{4}a_0a_1^2 + \frac{1}{16}a_1^4 + 45a_0^2)t$ .

### III.2 Bogoyavlensky-Konoplechenko Equation

Now, let us consider the following Bogoyavlensky-Konoplechenko equation in the form

$$u_{xt} + \alpha u_{xxxx} + \beta u_{xxy} + 6\alpha u_{xx}u_x + 4\beta u_{xy}u_x + 4\beta u_{xx}u_y = 0, \quad (23)$$

where  $u$  is function of  $x, y$  and  $t$ .

We make the transformation  $u(x, y, t) = u(\xi), \xi = x + y - kt$ . Then we get

$$-ku'' + \alpha u'''' + \beta u'''' + 6\alpha u''u' + 8\beta u'u'' = 0, \quad (24)$$

where prime denotes differentiation w.r.t  $\xi$ .

Balancing  $u'u''$  with  $u''''$  gives  $N = 1$ .

Therefore, we can write the solution of equation (24) in the form

$$u(\xi) = a_0 + a_1 \left( \frac{G'}{G} \right), a_1 \neq 0 \quad (25)$$

By using equations (4) and (25) we have

$$\begin{aligned} u'(\xi) &= -a_1 \left( \frac{G'}{G} \right)^2 - a_1 \lambda \left( \frac{G'}{G} \right) - a_1 \mu, \\ u''(\xi) &= 2a_1 \left( \frac{G'}{G} \right)^3 + 3a_1 \left( \frac{G'}{G} \right)^2 + (a_1 \lambda^2 + 2a_1 \mu) \left( \frac{G'}{G} \right) + a_1 \lambda \mu, \\ u'''(\xi) &= -6a_1 \left( \frac{G'}{G} \right)^4 - 12a_1 \lambda \left( \frac{G'}{G} \right)^3 - a_1 (7\lambda^2 + 8\mu) \left( \frac{G'}{G} \right)^2 - a_1 (\lambda^3 + 8\lambda \mu) \left( \frac{G'}{G} \right) - a_1 (\lambda^2 \mu + 2\mu^2), \\ u''''(\xi) &= 24a_1 \left( \frac{G'}{G} \right)^5 + 60a_1 \lambda \left( \frac{G'}{G} \right)^4 + a_1 (50\lambda^2 + 40\mu) \left( \frac{G'}{G} \right)^3 + a_1 (15\lambda^3 + 60\lambda \mu) \left( \frac{G'}{G} \right)^2 + a_1 (\lambda^4 + 22\lambda^2 \mu + 16\mu^2) \left( \frac{G'}{G} \right) + a_1 (\lambda^3 \mu + 8\lambda \mu^2), \end{aligned} \quad (26)$$

Substituting equations (26) into (24), setting coefficients of  $\left( \frac{G'}{G} \right)^i, (i = 0, 1, 2, 3, 4, 5)$  to zero, we obtain a system of

nonlinear algebraic equations  $a_0, a_1, k, \lambda$  and  $\mu$  as follows:

$$\begin{aligned} \left( \frac{G'}{G} \right)^5 &: 12aa_1 - 24a - 24b + 16ba_1 = 0, \\ \left( \frac{G'}{G} \right)^4 &: -60a\lambda - 60b\lambda + 30aa_1\lambda + 40ba_1\lambda = 0, \\ \left( \frac{G'}{G} \right)^3 &: 2k + 32ba_1\lambda^2 - 40b\mu - 40a\mu + 24aa_1\mu + 32ba_1\mu + 24aa_1\lambda^2 - 50b\lambda^2 - 50a\lambda^2 = 0, \\ \left( \frac{G'}{G} \right)^2 &: -60a\lambda\mu + 6aa_1\lambda^3 + 8ba_1\lambda^3 + 48ba_1\lambda\mu - 15a\lambda^3 - 60b\lambda\mu + 3k\lambda - 15b\lambda^3 + 36aa_1\lambda\mu = 0, \\ \left( \frac{G'}{G} \right) &: 16ba_1\lambda^2\mu - 22b\lambda^2\mu - a\lambda^4 + 12aa_1\mu^2 + 16ba_1\mu^2 - 22a\lambda^2\mu + 12aa_1\lambda^2\mu - 16a\mu^2 - b\lambda^4 + k\lambda^2 + 2k\mu - 16b\mu^2 = 0, \\ \left( \frac{G'}{G} \right)^0 &: -8a\lambda\mu^2 + 8ba_1\lambda\mu^2 - b\lambda^3\mu + k\lambda\mu + 6aa_1\lambda\mu^2 - a\lambda^3\mu - 8b\lambda\mu^2 = 0. \end{aligned} \quad (27)$$

Solving this system by Maple gives

$$\begin{aligned} \alpha &= -\frac{2\beta(-3+2a_1)}{3(a_1-2)}, a_1 = 1, \\ k &= \frac{\beta a_1(-\lambda^2+4\mu)}{3(a_1-2)}. \end{aligned} \quad (28)$$

Substituting the solution set (28) and the corresponding solutions of (4) into (25), we have the solutions of equation (24) as follows:

When  $\lambda^2 - 4\mu > 0$ , we obtain the hyperbolic function traveling wave solutions

$$u_{11}(\xi) = a_0 + a_1 \lambda \left( \frac{\sqrt{\lambda^2 - 4\mu}}{2} \left( \frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \xi\right)} - \frac{\lambda}{2} \right) \right) \quad (29)$$

When  $\lambda^2 - 4\mu < 0$ , we obtain the trigonometric function traveling wave solutions

$$u_{12}(\xi) = a_0 + a_1 \lambda \left( \frac{\sqrt{4\mu - \lambda^2}}{2} \left( \frac{-C_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + C_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)}{C_1 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \xi\right)} - \frac{\lambda}{2} \right) \right) \quad (30)$$

When  $\lambda^2 - 4\mu = 0$ , we obtain the rational function solutions

$$u_{13}(\xi) = \frac{C_2}{C_1 + C_2 \xi} - \frac{\lambda}{2}, \quad (31)$$

where  $\xi = x + y - \left( \frac{\beta a_1(-\lambda^2+4\mu)}{3(a_1-2)} \right) t$ .

### IV. DISCUSSION AND CONCLUDING REMARKS

In this paper, an implementation of the  $\left( \frac{G'}{G} \right)$ -expansion method is given by applying it to three nonlinear equations to illustrate the validity and advantages of the method. As a result, hyperbolic function solutions, trigonometric function solutions and rational function solutions with parameters are obtained. The  $\left( \frac{G'}{G} \right)$ -expansion method is direct, concise and effective. The performance of this method is reliable, simple and gives many new exact solutions. The obtained solutions with free parameters may be important to explain

some physical phenomena. The paper shows that the devised algorithm is effective and can be used for many other NPDEs in mathematical physics.

#### ACKNOWLEDGMENT

Nisha Goyal, wants to thank for finical support from Human Resource Development Group Council of Scientific Industrial Research (CSIR), India [09/677(0014)2009 – EMR – 1].

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