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# Vertex Configurations and Their Relationship on Orthogonal Pseudo-Polyhedra

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***Abstract*—**Vertex configuration for a vertex in an orthogonal pseudo-polyhedron is an identity of a vertex that is determined by the number of edges, dihedral angles, and non-manifold properties meeting at the vertex. There are up to sixteen vertex configurations for any orthogonal pseudo-polyhedron (OPP). Understanding the relationship between these vertex configurations will give us insight into the structure of an OPP and help us design better algorithms for many 3-dimensional geometric problems. In this paper, 16 vertex configurations for OPP are described first. This is followed by a number of formulas giving insight into the relationship between different vertex configurations in an OPP. These formulas will be useful as an extension of orthogonal polyhedra usefulness on pattern analysis in 3D-digital images.

***Keywords*—**Orthogonal Pseudo Polyhedra, Vertex configuration

## I. Introduction

An Orthogonal Pseudo-Polyhedron (OPP) is a pseudo polyhedron in which every edge is parallel to one of the three orthogonal directions. In many practical applications, OPP provides a simple yet effective approximation to many important geometrical objects. The use of OPP arises frequently in practice, for instance in modeling buildings which are largely orthogonal shaped. Hence OPP deserves special attention.

Like its sub-class, orthogonal polyhedron, OPP has many applications in such area as connected component labeling[1], and pattern analysis in digital images and VLSI layout[2]. Often they are studied with respect to partitioning problem[3], and visibility problem[4].

An OPP can be described in terms of its properties including vertices, edges, faces, angles between edges and faces, and vertex configurations. Understanding the relationship between these properties will give us insight into the structure of OPP and help us design better algorithms for many 3-dimensional geometrical problems.

Each vertex in an OPP can be characterized by its vertex configuration which is defined by the number of adjacent edges, dihedral angles, and non-manifold components meeting at the vertex.

In this paper, a list of sixteen possible vertex configurations in an OPP is identified and discussed. Their quantitative relationships are conjectured. As there are large numbers of vertex configurations, to simplify the presentation, the OPPs are divided into a number of groups based on their maximum degree of vertices. The quantitative relationship for each group is discussed. We also present an algorithm to prove that any OPP can be decomposed into a number of orthogonal prisms.

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## II. Preliminaries

### *A. Terminologies*

A polygonal curve is a simple closed curve that consists of a finite and contiguous number of line segments. A polygon is a closed and bounded region of a plane whose boundary is a polygonal curve. A vertex of polygon *P* is a point on its boundary, and an edge is a line segment on the polygon's boundary that connects two vertices. Two vertices connected by an edge are adjacent, and the edge is said to be incident on the two vertices[5]. An orthogonal polygon is a polygon whose boundary sides are either parallel or perpendicular to each other. Clearly all corners of a boundary are either of $90^0$ (convex) or $270^0$ (concave) [6].A pseudo-polyhedron is a finite collection of planar faces such that (a) every edge has at least two adjacent faces, and (b) if any two faces meet, they meet at a common edge [7]. However, there is a possibility that two faces meet at a common vertex. To include this scenario, in this paper, we extend the definition of the pseudo-polyhedron by modifying condition (b) in the above definition: if two faces meet, they meet at a common edge or at a common vertex without sharing a common edge. An edge that belongs to exactly two faces is called two-manifold edge, and a vertex that is the apex of only one cone of faces is called a two manifold vertex. Conversely, a non-manifold edge belongs to more than two faces, and a non-manifold vertex is the apex of more than one cone of faces [8]. A cone is defined as a three-dimensional geometric shape that tapers smoothly from a flat base to a point called the apex. With our extended definition, a pseudo-polyhedron may have non-manifold edges as well as non-manifold vertices.Polyhedron is a sub-class of pseudo-polyhedron in which every edge is two-manifold, and every vertex is two-manifold. Hence, the boundary of a polyhedron contains two-manifold edge and two-manifold vertex only, and this kind of boundary is called two-manifold boundary. A polyhedron divides the space into two regions, one of which, called the interior, is continuous and finite [9]. A simple polyhedron is a polyhedron that can be deformed into a solid sphere; that is, a polyhedron that, unlike a torus, has no holes [10].An orthogonal pseudo-polyhedron is defined as a pseudo-polyhedron in which every edge is parallel to one of the three orthogonal directions. In an OPP, a non-manifold edge is adjacent to exactly four faces and a non-manifold vertex is the apex of exactly two cones of faces [3]. One of the most widely studied classes of pseudo-polyhedra is orthogonal polyhedra (OP) (sometimes also called isothetic polyhedra). An orthogonal polyhedron is defined a polyhedron in which every edge is parallel to one of the three orthogonal directions [7]. Fig. 1(a) and 1(b) show an instance of OP and OPP respectively. The OPP in Fig. 1(b) has two non-manifold edges and one non-manifold vertex.

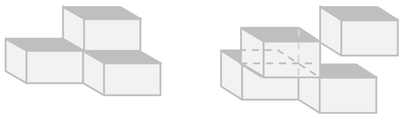



Fig. 1 (a) An OP, (b) An OPP

A simple orthogonal pseudo polyhedron, or simple OPP for short, a simple orthogonal polyhedron in which every edge is parallel to one of the three orthogonal directions. There are two kinds of angles in an OPP: facial angles and dihedral angles. Two edges incident to a common vertex may be on the same face. In such case, the angle between the two edges is referred to as a facial angle of the face. Dihedral angle is the interior angle between two faces meeting at a common edge [11].In an OP, any two adjacent faces form an interior dihedral angle and there are only two possible values for such a dihedral angle. Either the angle is $90^0$, which is called convex dihedral angle, or the angle is $270^0$, which is called concave dihedral angle. However, in an OPP, two adjacent faces may not always be capable of forming a dihedral angle due to the presence of non-manifold edge or non-manifold vertex. The vertices of an OPP can be classified into a number of groups based on the number of adjacent edges, the number of concave dihedral angles, the number of non-manifold edges, and whether the vertex is non-manifold or not. Each group can be identified by a unique label which is called as vertex configuration. For example, if a vertex in an OPP has four edges, two concave dihedral angles, one non-manifold edge, and no non-manifold vertex, then the label for the vertex is V42-10. For any manifold vertex, its label can be shortened to three digits. For instance, the aforementioned vertex can also be labeled as V42-1 instead of V42-10.

### *B. Previous Results*

Juan-Arinyo observed that, in an orthogonal polyhedron, a vertex has either three, four, or six incident edges[12]. For example, in Fig. 2a and 2b, vertex *v* has three faces, while in Fig. 2c and 2d, vertex *v* has four and six faces respectively.

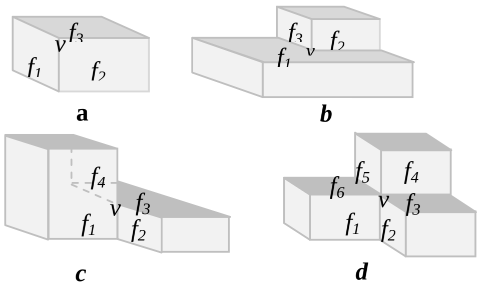



Fig. 2 Vertex *v* has three faces in ***a*** or ***b***, four faces in ***c***, and six faces in ***d***

By examining each interior dihedral angle at vertex *v* in the above four orthogonal polyhedra, it is easy to establish that, in Fig. 2(a), 2(b), 2(c), and 2(d), the number of dihedral angles at vertex *v* that are $270^0$ is 0, 2, 2 and 3 respectively.

Voss classified the orthogonal polygon boundary into the inner or outer boundary based on the relationship between concave and convex vertices in 2D digital image [13]. Yip and Klette mentioned that an OP may also have an outer boundary as well as an inner boundary [2] if the OP has a hole inside.

For simple OP where there is only outer boundary, Yip and Klette established a formula on the relationship among the different vertex configurations in an orthogonal polyhedron can be characterized by as the following formula due to Yip and Klette [2]: (HA+HG) –(HC+HE) – 2(HD1+HD2) =8, where HA, HG, HC, HE, HD1 and HD2 denote the number of V30, V33, V31, V32, V42-0, V63-0 types of vertices in the orthogonal polyhedron respectively. For an orthogonal polygon, the relationship between the number convex vertex (HC) and reflect vertices (HR) is: HC-HR = 4.

The first formula is useful for analyzing the boundaries of simple orthogonal polyhedra. The second formula can be used to analyze polygonal boundary. Yep and Klette suggested that the first formula can be used in 3D pattern analysis by providing a necessary condition for having traced a complete 3D surface of a simple OP [2].

## III. Vertex configurations for OPP

In an OPP, each edge is parallel to one of the three orthogonal directions. Therefore, for each vertex in an OPP, there are at most six distinct incident edges. At the same time, there must be at least three incident edges for each vertex to be in three-dimension. Hence the number of incident edges of any OPP vertex ranges from three to six. In this paper, an OPP vertex with *n* incident edges is denoted Vn (n = 3, 4, 5, 6), and from now on these vertices are referred to as V3, V4, V5 and V6 respectively.

For a given number of edges, an OPP's vertex may have one of several possible configurations depending on the way in which faces are formed by edges incident to the vertex. Based on this fact, many possibility shapes of OPP could be constructed for a given number of edges.

Overall, there are 255 shapes possibility (*Sp*) of OPP where each OPP's shape is composed by at most eight cubes in which each cube shares at least a vertex with others. The number of shapes possibility is achieved from the formula: *Sp* = $8^2$- 1, where 8 is the number of frames that will be occupied by cubes , 2 is the number of possibility to each frame to be occupied, and 1 is a the number of possibility for the frame having null cube. Some OPP may have a same shape; hence, we can group them as one shape of OPP. The total number of different OPP's shape is stated in following lemma.

*Lemma:* There are 16 different shapes of OPP that can be constructed by at most eight congruent cubes.

*Proof :*Let *B* a cubical frame, and it can be composed of eight smaller cubical frames in the same size *f1*, *f2*, …, *f8*. We group those smaller frames into button group and top group. The member of bottom group is *f1*, *f2*, *f3*, and *f4*; meanwhile, the member of top group is *f5*, *f6*, *f7*, *f8*.

We construct OPPs by putting a number of cubes ranges from one to eight into *B*. Each frame in the both groups can be occupied by a cube that has the same size with a smaller frame. Let *f11*, *f12*, *f21* and *f22* represent frames in first row and first column, first row and second column, second column and first row, and second row and second column, respectively in bottom and top side. A cube $c_1$ is adjacent to a cube $c_2$, if $c_1$ and $c_2$ share a face. $c_1$ is diagonally adjacent to $c_2$ if they share an edge. Two cubes are said an interstitial cubes if they share a vertex only. Three cubes $c_1$, $c_2$, $c_3$ are said a 3-consecutive cubes if a cube is adjacent to two other cubes. Four cubes $c_1$, $c_2$, $c_3$, $c_4$ are said a 4-consecutive cubes if each cube is adjacent to any two other cubes. We make several premises, *PR1*, *PR2*, *PR3*, *PR4*, and *PR5* for helping us to group OPPs as follows:

*PR1*. Two adjacent cubes are congruent with any two other adjacent cubes.
*PR2*. Two diagonally adjacent cubes are congruent with any two other diagonally adjacent cubes.
*PR3*. Two interstitial cubes are congruent with any two interstitial cubes.
*PR4*. A 3-consecutive cubes is congruent with any 3-consecutive cube.
*PR5*. A 4-consecutive cubes is congruent with any 4-consecutive cube.

Table I in the appendix shows how all possibilities of OPPs are constructed by putting at most eight cubes into *B*, and there are 256 ways to occupy a cubical frame with at most eight congruent cubes. We do not consider a frame with zero number of cubes as an OPP, hence the total possibility number of OPP's shape = 255, and they are distributed in twenty two different shapes and sizes of OPP. OPPs shape in numbers: 2, 3, 14, 23 have a similar shape; hence, we group them as a shape. OPPs shape in numbers; 4,5,15 also have a similar shape, and we put them as a shape. To conclude, there are sixteen different shape of OPP, and it means there are sixteen kinds of vertex configurations in OPP.

An OPP vertex with four edges (V3) has four possible configurations. Every vertex is two-manifold vertex and every edge is two-manifold edge. The four vertex configurations of V3 are illustrated in Fig. 3.

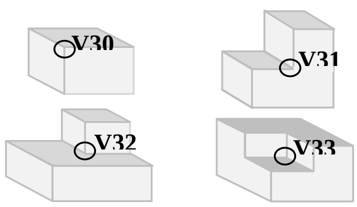



Fig. 3 Four possible vertex configuration for V3

An OPP vertex with four edges (V4) has four possible configurations, only one of which has no non-manifold edge. The four vertex configurations of V4 are illustrated in Fig. 4.

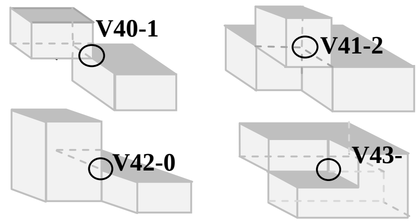



Fig. 4 Four possible vertex configurations for V4

Meanwhile, an OPP vertex with five edges (V5) has two possible vertex configurations, and both of them have non-manifold edge. The two vertex configurations for V5 are illustrated in Fig. 5.

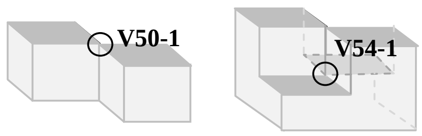



Fig. 5 Two possible vertex configurations for V5

Any OPP vertex with six edges (V6) has six possible configurations. One of them has no non-manifold edge or non-manifold vertex. Three of them have non-manifold edges but not non-manifold vertex. Two of them have non-manifold vertex but not non-manifold edge. All vertex configurations for V6 are shown in Fig. 6.

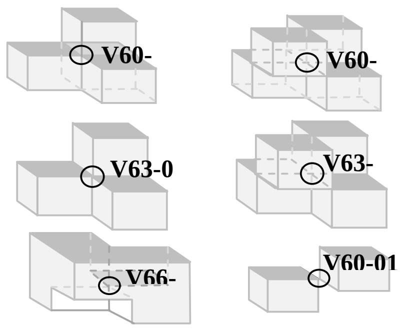



Fig. 6 Configurations a vertex having 6 edges in OPP

## IV. Vertex configurations relationship

In this section we prove a lemma saying that any OPP can be constructed by combining two or more orthogonal prisms. We also illustrate the sixteen different joining operations and use them as a tool of proof on the relationship among the vertex configurations.

### A. Constructing an OPP

**Lemma 2**: *Any OPP can be decomposed into a number of orthogonal prisms.*

*Proof:* We prove this lemma by an algorithm. OPP can be represented by their extreme vertices that are the ending vertices of all the OPP brinks. Brink is defined as the longest uninterrupted segment, built out of a sequence of collinear and contiguous two-manifold edges of an OPP [3]. Based on these definitions, the members of extreme vertices are V30,V31,V32,V33,V40-1,V43-1, V60-3, V63-3.

The following is the algorithm for decomposing an OPP *P* into rectangular prisms.

1. Calculate the value of *dim* variable of *P* by counting the number of different *X*-coordinate, *Y*-coordinate, and *Z*-coordinate. An OPP is called has multi-plane if the number of different coordinates along any axis is at least three. If each direction has multi-plane then assign *dim = 3*. If two of them has multi-planes then *dim* =2, and *dim* =1 for one multi-plane. If each axis has no multi-plane then *dim*=0, it means that *P* is orthogonal prism.
2. If *dim*=0 then add *P* as orthogonal polyhedron.
3. If *dim* > 1 then
   a. Let $m_1$, $m_2$, $m_3$ be multi-plane on *P*. Sort vertices of *P* according to the axis that has $m_1$, and then it is followed by the axis having $m_2$ (if any), and by the axis having $m_3$ (if any).
   b. Determine $SP = \{sp_1, sp_2,...sp_i)$ where *SP* is a collection of non empty splitting plane equations, and $sp_i$ is the i$^{th}$ splitting plane equation.
   c. Cut *P* at each splitting plane. *P* is represented by brinks in which each brink has two ending point, Vb and Ve respectively. Group brinks as the following rules:

```
IF Vb < s && Ve <= s THEN GroupBrink(Vb,Ve,Q) ENDIF
IF Vb >= s && Ve > s THEN GroupBrink(Vb,Ve,R) ENDIF
IF Vb < s && Ve >s THEN
        GroupBrink(Vb,s,Q)
        GroupBrink(s,Ve,R)
ENDIF
```

   d. Reduce the value *dim* by 1, and apply the steps 2 and 3
   e. If *SP* has more than one splitting plane, then continue to cut *P* at the next splitting plane (repeat the step from 3c). □

**Lemma 3:** *Every OPP can be constructed from one or more orthogonal prisms.*

*Proof:* It is proven in lemma 2 that any OPP *P* can be decomposed into a number of orthogonal prisms, so we can compose reversely all orthogonal prisms to construct *P*. □

### B. Joining Operation

We may say that V30, V31 and V5-1 vertices are basic vertex configurations because a new vertex configurations can be obtained by joining one two kind of these vertices, or by joining one kind of these vertices with another OPP's properties such as point on an edge, or point in the interior of a surface. For example, if a V30 vertex is joined with a point on an edge, then the result is a V31 vertex. In this section, we explore how to get the sixteen vertex configuration based on the basic vertex configuration and other properties of OPP.

Let *P1*, *P2* be rectangular prisms, let *P3*, *P4* be OPP only having two V31 vertex and ten V30 vertices, and let *P5*, *P6* be an OPP having two V50-1 vertices and twelve V30 vertices. We use those OPPs and other OPP's properties in joining operation, and then we show some possible joining operations to get the sixteen vertex configuration as shown in Table II (see appendix), and six of them, operation number (1) until (6) has documented by [2], and they can be used to do the task of joining operation on OPP.

### C. Relationships

Now, it is time to conjecture the vertex configurations relationship on OPP. We organize the formulation as follows (1) the formula is started with OPP having at most degree four that is every vertex has three or four edges, (2) Extend the formula to OPP having the vertex which have degrees at most degree six.

*Conjecture 1* **:** *Let P be an OPP having at most degree four, let $P_{V30}$, $P_{V31}$, $P_{V32}$, $P_{V33}$, $P_{V401}$, $P_{V412}$, $P_{V420}$, $P_{V431}$ denote the number of vertex having V30, V31, V32, V33, V40-1, V41-2, V42-0, and V43-1configurations respectively. We have*
$(P_{V30} + P_{V33} + 0P_{V412} + P_{V431}) - (P_{V31} + P_{V32} + 3P_{V401} + 2P_{V420}) = 8$

*Proof*: Let function $F(P) = (P_{V30} + P_{V33} + 0P_{V412} + P_{V431}) - (P_{V31} + P_{V32} + 3P_{V401} + 2P_{V420})$, where P is any OPP. We need to prove that F(P) = 8 for any orthogonal pseudo-polyhedron P.

It is proven in Lemma 2 that any OPP can be constructed by joining sequence of rectangular prisms one by one. In other words, any OPP can be constructed by the following steps. The process starts with marking the first rectangular prism in the sequence as the OPP. In the next step, a next rectangular prism is added to the OPP. The process finishes when the last rectangular prism is added to the OPP.

The proof starts with first rectangular prism *P*. It has exactly eight V30 vertices, and no others kind of vertices. Hence, $F(P) = P_{V30} = 8$.

The next step of arbitrary joining process is combining an OPP with an orthogonal prism. The number of possible combination is large. However, we list several of them and we are sure that there is always such a way to construct a new OPP by using them. There are at least six possibilities of the joining process. They are illustrated as operation 1, 2, 3, 9, 10, and 16 in Table II (see appendix). If both OPP contain combination V30, V31 and V50-1 vertices, line, and interior face, then there are at least 16 different possibilities of the joining process.

Let *P1* be any OPP, and *P2* be an rectangular prism, then we have F(*P1*) = 8 and F(*P2*) =8. Before joining operation, we have: F(*P1*) + F(*P2*) =R= 16. After *P1* and *P2* are joined, then they form *P*. We are going to prove that F(*P*) = 8.

So, in the joining process, some properties of *P1* and *P2* meet each other. Because of the meeting, some properties will lose, and others will change the type. For example, if V30 vertex of *P1* meets with V30 vertex of *P2* and two of incident edges of each orthogonal polyhedron then the two V30 vertices will lose. Another example, if V30 vertex meet with a point on a line and each adjacent surface of each properties coincide each other then V30 vertex will lose and instead by V31 vertex. Hence, the *R* value is changed during the combining *P1* and *P2*. Let $\Delta R$ be the value of increasing or decreasing of any joined properties.

To determine $\Delta R$, we calculate the value as the following steps:

(1) Get the relationship among vertex configurations in an OPP. It may start with an OPP that has one kind of vertex configuration, and we combine the OPP with another simpler OPP having the same vertex configuration with the current OPP, and do it until *n* times. By using the arithmetic sequence formula [14], the vertex configuration relationships for simpler OPP is achieved. We continue to look at an OPP having two kinds of vertex configurations, and so on. Table III in appendix shows the relationship among vertex configurations on simpler OPPs.

(2) Let *F*=*N* be the equation of the relationship having *N* as the value in the right side equation. Calculate $\Delta R$ of two joining properties. We have two OPP, *P1* and *P2*; the total value of both *R* is 16. Reduce *R* by ΔR until it obtains *N*. Note, we only can determine $\Delta R$ of a pair of component at the same time. It means, if we have two pairs of joining components, so $\Delta R$ of one pair must be known first.

Here is an example to calculate $\Delta R$ from the given two OPPs. For *P1*, $P_{V30} = 8 \rightarrow F(P1) = 8$, and for *P2*, $P_{V30} = 8 \rightarrow F(P2) = 8$. If the number of V30 on *P1* is added with V30 on *P2*, then we have

$F(P1) + F(P2) = 16$ ........(1)

Suppose we combine two V30 vertices on *P1* with two V30 vertices on *P2*, and two V30 vertices on *P1* with two edges on *P2*. The joining result is shown in fig. 7.

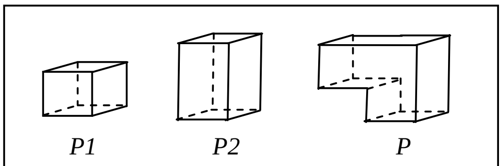



Fig. 7 *P* is a result of joining *P1* and *P2*

We join (⊕) *P1* and *P2* such that two vertices of *P1* coincide with two vertices of *P2* and the other two vertices of *P1* coincide with edges of *P2*. Relationship among vertices that contain V30 and V31 vertices is shown in Table III at number 2 as follows:

$P_{V30} - P_{V31} = 8$, so

$F(P1 \oplus P2) = 8$ ........(2)

During the joining process, the right value in the equation (1) decrease until 8 as the value in (2). We can use this value to determine the decreasing or increasing value of *R*. If a V30 vertex is joined with a V30 vertex then *R* decreases by 2 because both V30 vertices are deleted. If a V30 is joined with an edge, we calculate $\Delta R$ as follows. During the joining process, the two V30 vertices from each orthogonal prism is joined, and the other two V30 vertices from *P1* is joined with two edges of *P2*, then

$8 = 16 - 2(2) + 2\Delta R$

$\Delta R = -2$

It means that the value of *R* is decreased by 2 after joining a V30 vertex with an edge.

By using the same way, we can find the decreasing or increasing value of *R* as summarized in Table IV that also informs the new vertex configuration that gain after the joining process. As mentioned above that there are at least 16 combinations of properties on OPPs; hence, we may have other $\Delta R$ values.

TABLE IV
OPERATION NUMBER AND THEIR $\Delta R$ VALUE

| Operation Number | Properties | ΔR-value | | New vertex |
|---|---|---|---|---|
| | | Increase | Decrease | |
| 1 | V30 and V30 | 0 | 2 | - |
| 2 | V30 and line | 0 | 2 | V31 |
| 3 | V30 and interior face | 0 | 2 | V32 |
| 4 | V30 and V31 | 0 | 2 | V42 or V63 |
| 5 | V31 and V31 | 2 | 0 | - |
| 6 | V31 and interior face | 2 | 0 | V33 |
| 7 | V30, V31, edges | 0 | 0 | V41-2 |
| 8 | V31, edge | 2 | 0 | V43-1 |
| 9 | V30, edge | 0 | 4 | V40-1 |
| 10 | V30, V30, edge | 0 | 4 | V50-1 |
| 11 | V31, V31, edge | 0 | 0 | V54-1 |
| 12 | V31 and V31 | 4 | 0 | V66-0 |
| 13 | V30 and V50-1 | 4 | 0 | V60-3 |
| 14 | V31 and V50-1 | 0 | 0 | V63-3 |
| 15 | V50-1 and V50-1 | 6 | 0 | V60-6 |
| 16 | V30 and V30 | 8 | 8 | V60-01 |

By applying the suitable $\Delta R$ for each joined properties during joining *P1* and *P2* to form *P*, then *R* value changes from 16 before joining to 8 after joining. Hence, it gives us quite confidence that the equation on Conjecture 1 always remains valid. □

*Conjecture 2* **:** *Let P be an OPP having at most degree five, let* $P_{V30}$, $P_{V31}$, $P_{V32}$, $P_{V33}$ $P_{V401}$, $P_{V412}$, $P_{V420}$, $P_{V431}$, $P_{V501}$, $P_{V541}$, $P_{V600}$, $P_{V603}$, $P_{V606}$, $P_{V630}$, $P_{V633}$, *and* $P_{V660}$ *denote the number of vertex having V30, V31, V32, V33, V40-1, V41-2, V42-0,V43-1, V50 and V54-1, V60 -01, V60-3, V60-6, V63-0, V63-3, and V66-0 configurations respectively. The relationship among these vertices is:*

$$(P_{V30} + P_{V33} + 0P_{V412} + P_{V431} + 2P_{V541} + 4P_{V606} + 3P_{V633} + 2P_{V660}) - (P_{V31} + P_{V32} + 3P_{V401} + 2P_{V420} + 2P_{V501} + P_{V603} + 6P_{V600} + 2P_{V630}) = 8$$

To proof these conjecture we use similar technique as shown to prove Conjecture 1. We start with the relationship among vertex having degree five with V30 configuration or the other vertex configurations that already known their relationship, and then find the table of relationship among the V5 and V6 vertices. The later step give the increasing and decreasing of *R*-value as shown in Table IV at the operation number 10-16. By applying the suitable operation number for each step on adding a prism to the current OPP, we always find that the *R* value is always equal with 8. Hence, we are quite confident that the equation on Conjecture 2 is valid.□

## V. CONCLUSION

We have proven that there are sixteen vertex configurations of OPPs, and it also has proven that any OPP can be constructed from a sequence of orthogonal prisms. These lemmas are important to conjecture the relationship among the vertex configurations of OPPs.

For a given OPP, we always have the formula to represent the vertex configurations relationship. Conjecture 3 represents the vertex relationship for any OPP.

For the further research, the issue of inner and outer boundary is still relevant on OPP. We absolutely can find the duality of each vertex configuration on OPP, so if an OPP is a hole of another OPP then by applying the formula to the inner boundary, we can determine the vertex configurations on outer boundary.

TABLE I
CONSTRUCTING OPPS USING AT MOST EIGHT CUBES

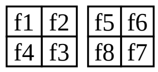
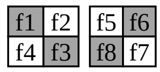
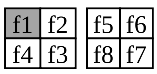
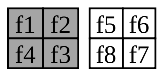
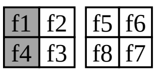
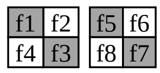
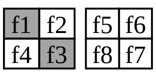
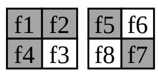
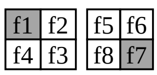
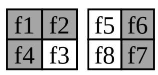
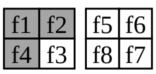
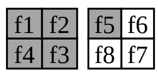
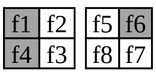
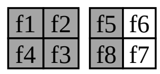
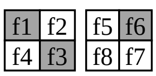
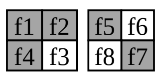
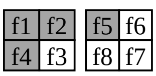
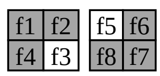
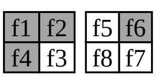
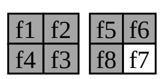
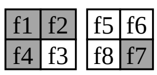
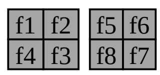
| Number | Illustrations | Description | Number of similar shapes | Number | Illustrations | Description | Number of similar shapes |
|---|---|---|---|---|---|---|---|
| 1. |  | All frames are empty | 1 | 12. |  | Two diagonally adjacent cubes share vertices and edges | 2 |
| 2. |  | A frame is occupied by a cube. | 8 | 13. |  | A 4-consecutive cubes is occupied 4 frames in B | 6 |
| 3. |  | Two frames are occupied by two adjacent cubes. | 12 | 14. |  | Two diagonally adjacent cubes share faces | 6 |
| 4. |  | Two frames are occupied by two diagonally adjacent cubes | 12 | 15. |  | A 3-consecutive cubes shares a face with a diagonally adjacent cubes | 8 |
| 5. |  | Two frames are occupied by two interstitial cubes | 4 | 16. |  | An ending face of 3-consecutive cube shares a face with an adjacent cubes | 24 |
| 6. |  | Three frames are occupied by a 3-consecutive cubes. | 24 | 17. |  | A 4-consecutive cubes shares a face with a cube | 24 |
| 7. |  | Two adjacent cubes are sharing an edge with a cube in B | 24 | 18. |  | A 4-consecutive cubes shares two face with a 2-adjacent cubes. | 12 |
| 8. |  | Two edges of two diagonally cubes meet with an edge of a cube. | 8 | 19. |  | A 3-consecutive cubes shares a face with a 2-diagonally adjacent cubes | 12 |
| 9 |  | The middle cube of a 3-consecutive cubes shares a face with a cube in B. | 8 | 20. |  | A 3-consecutive cubes shares a vertex with another 3-consecuteve cubes | 4 |
| 10 |  | An end cube of a 3-consecutive cubes shares a face with a cube in B. | 24 | 21. |  | A 3-consecutive cubes is combined with a 4-consecutive cubes. | 8 |
| 11. |  | A shared vertex of a 3-consecutive cubes shares a vertex of another cube. | 24 | 22. |  | Two 4-consecutive cubes occupy the whole frame. | 1 |

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TABLE II
. JOINING OPERATIONS ON OPP

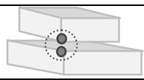
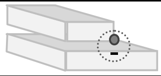
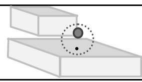
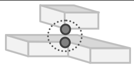
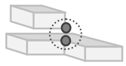
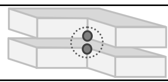
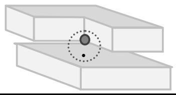
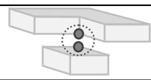
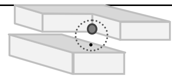
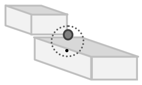
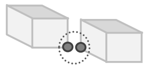
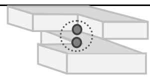
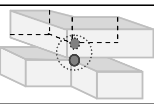
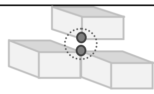
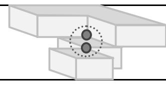
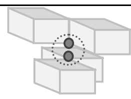
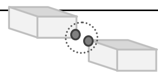
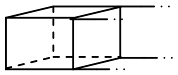
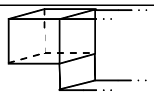
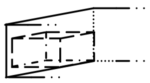
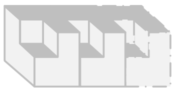
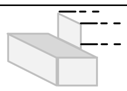
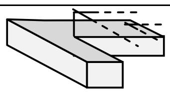
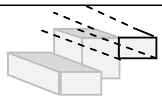
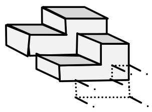
| Operation Number | Description | Illustration |
|---|---|---|
| 1. | A V30 vertex on *P1* is joined with a V30 vertex on *P2*. After joining, both V30 vertices are lost. |  |
| 2. | A V30 vertex on *P1* is joined with an edge of *P2*. After joining, the V30 vertex is lost, however, the union gains a new V31 vertex. |  |
| 3. | A V30 vertex on *P1* is joined with an interior point of *P2* surface. After joining the V30 is lost, however, the union gains a new V32 vertex. |  |
| 4. | A V30 vertex on *P1* is joined with a V31 vertex on *P3*. After joining, both V30 vertices are lost, and they either form a new V63 vertex or new V42 vertex. |   |
| 5. | A V31 vertex on *P3* is joined with a V31 vertex on *P4*. After joining, both vertices are lost without forming a new vertex. |  |
| 6. | A V31 vertex on *P1* is joined with an interior point of *P3* surface. They form the new V33 vertex. |  |
| 7. | A V30 vertex on *P1* is joined with a V31 vertex on *P3*, and two incident edges of V30 vertex are also joined with two incident edges of V31 vertex as shown in the diagram on the right. As the result, the two vertices combined to form a V43-1 vertex. |  |
| 8 | A V31 vertex on *P3* is joined with a point between the two endpoints of an edge on *P1*, and that edge is also is joined with one of the incident edges of the V31 vertex, as shown in the diagram on the right. As the result the V31 vertex is converted to a V43-1. |  |
| 9. | A V30 vertex on *P1* is joined with a point between the two endpoints of an edge on edge of *P2* and the edge is joined with one of the incident edges of the V30 vertex. As the result, the V30 vertex becomes a V40-1 vertex. |  |
| 10. | A V30 vertex on *P1* is joined with a V30 vertex *P2*, in the way shown in diagram on the right. The two vertices are combined to form one V50-1. |  |
| 11. | A V31 vertex on *P3* is joined with a V31 vertex on *P4*, and an edge incident to V31 vertex for each simpler OP is joined each other. They form V54-1. |  |
| 12. | A V31 vertex on *P3* is joined with a V31 vertex on *P4*, and two edges are incident to the vertex lie on the interior face of *P2*. They form V66-0. |  |
| 13. | A V50-1 vertex on *P5* is joined with a V30 vertex on *P1*. They gain a new V60-3 vertex. |  |
| 14. | A V50-1 vertex on *P5* is joined with a V31 vertex on *P3*. They gain a new V63-3 vertex. |  |
| 15. | A V50-1 vertex on *P5* is joined with a V50-1 vertex on *P6*, so all the incident edges coincide in pairs. They gain a new V60-6 vertex. |  |
| 16. | Only a V30 vertex on *P1* is joined with a V30 vertex on *P2*. They gain a new V60-0 vertex. |  |

TABLE III
RELATIONSHIP AMONG VERTEX CONFIGURATIONS ON SIMPLER OPP

| No. | Vertices | Illustrations | Arithmetic Sequence of vertices | | | | | | Vertices relationship |
|---|---|---|---|---|---|---|---|---|---|
| | | | Vertex | 1 | 2 | 3 | … | N | |
| 1. | V30 |  | $P_{V30}$ | 8 | 8 | 8 | … | 8 | $P_{V30}=8$ |
| 2. | V30 and V31 |  | $P_{V30}$ | 8 | 10 | 12 | … | 2n+6 | $2n+6 = (2n-2) + 8 \leftrightarrow$ |
| | | | $P_{V31}$ | 0 | 2 | 4 | … | 2n-2 | $P_{V30} - P_{V31}=8$ |
| 3. | V30, V31 and V32 |  | $P_{V30}$ | 8 | 11 | 14 | … | 3n+5 | $3n+5=(2n-2)+(n-1)=8\leftrightarrow$ |
| | | | $P_{V31}$ | 0 | 2 | 4 | … | 2n-2 | $P_{V30}-P_{V31}-2P_{V42}=8$ |
| | | | $P_{V32}$ | 0 | 1 | 2 | | n-1 | |
| 4. | V30, V31 and V33 |  | $P_{V30}$ | 10 | 12 | 14 | … | 2n+8 | $(6n+4)+(n-1)=(3n-1)+(n-1)+3(n-1) =8 \leftrightarrow$ |
| | | | $P_{V31}$ | 3 | 7 | 11 | … | 4n-1 | $P_{V30}+P_{V431} - (P_{V31} + P_{V32}+ 3P_{V401}) = 8$ |
| | | | $P_{V33}$ | 1 | 3 | 5 | … | 2n-1 | |
| 5. | V30 and V41 |  | $P_{V30}$ | 8 | 14 | 20 | … | 6n+2 | $6n+2 =3(2n-2) + 8 \leftrightarrow$ |
| | | | $P_{V41}$ | 0 | 2 | 4 | … | 2n-2 | $P_{V30}-3_{V41}=8$ |
| 6. | V30, V31, V40-1, and V41-2 |  | $P_{V30}$ | 17 | 24 | 31 | … | 2n-1 | $7n+10=n+2 + 3(2n)+8 \leftrightarrow$ |
| | | | $P_{V31}$ | 3 | 4 | 5 | … | n+2 | $P_{V30} + 0P_{V412} - P_{V31} -3P_{V401} =8$ |
| | | | $P_{V401}$ | 2 | 4 | 6 | … | 2n | |
| | | | $P_{V412}$ | 1 | 2 | 3 | … | N | |
| 7. | V30, V31, And V42-0 |  | $P_{V30}$ | 8 | 12 | 16 | … | 4n+4 | $4n+4 = (2n-2) + 2(n-1) + 8 \leftrightarrow$ |
| | | | $P_{V31}$ | 0 | 2 | 4 | … | 2n-2 | $P_{V30}-P_{V31}-2P_{V42}=8$ |
| | | | $P_{V42}$ | 0 | 1 | 2 | … | n-1 | |
| 8. | V30, V31, V32, V40-1 and V43-1 |  | $P_{V30}$ | 10 | 16 | 22 | … | 6n+4 | $(6n+4)+ (n-1)=(3n-1)+(n-1)+3(n-1)=8 \leftrightarrow$ |
| | | | $P_{V31}$ | 2 | 5 | 8 | … | 3n-1 | $P_{V30}+P_{V431} - (P_{V31} + P_{V32}+ 3P_{V401}) = 8$ |
| | | | $P_{V32}$ | 0 | 1 | 2 | … | n-1 | |
| | | | $P_{V401}$ | 0 | 1 | 2 | … | n-1 | |
| | | | $P_{V431}$ | 0 | 1 | 2 | … | n-1 | |

## REFERENCES


[1] D. Ayala and J. Rodriquez, "Connected component labeling based on the EVM Model," in The 18th spring conference on Computer graphics, 2002, pp. 63-71.

[2] B. Yip and R. Klette, "Angle Counts for Isothetic Polygons and Polyhedra," Pattern Recognition Letter, vol. 24, pp. 1275-1278, 2003.

[3] A. Aquilera and D. Ayala, "Solving point and plane vs orthogonal polyhedra using the extreme vertices model (EVM)," presented at the The Sixth International Conference in Central Europe on Computer Graphics and Visualization'98, 1998.

[4] J. Czyzowicz, et al., "Illuminating rectangles and triangles in the plane," Journal of Combinatorial Theory Series B archive, vol. 57, 1993.

[5] M. d. Berg, et al., Computational Geometry, Second ed.: Springer, 2000.

[6] F. P. Preparata and M. I. Shamos, Computational Geometry an Introduction. New York: Springer-Verlag, 1985.

[7] K. Tang and T. C. Woo, "Algorithmic aspects of alternating sum of volumes. Part 1: Data structure and difference operation," Computer-Aided Design, vol. 23, pp. 357-366, June 1991.

[8] J. R. Rossignac and A. A. G. Requicha, "Construcitve Non-Regularized Geometry," Computer - Aided Design, vol. 23, pp. 21-32, 1991.

[9] H. S. M. Coxeter, Regular polytopes. New York: Dover Publications, 1973.

[10] J. D. Foley, et al., Computer Graphics: Principles and Practice in C, 2nd edition ed.: Addison-Wesley, 1996.

[11] T. Biedl and B. Genc, "Reconstructing orthogonal polyhedra from putative vertex sets," technical reports, 2007.

[12] R. Juan-Arinyo, "Domain extension of isothetic polyhedra with minimal CSG representation," Computer Graphics Forum, vol. 5, pp. 281-293, 1995.

[13] K. Voss, Discrete Images, Objects, and Functions in Zn. Berlin: Springer, 1993.

[14] S. L. Senk, Advanced Algebra. Chicago: Scott Foresman/Addison Wesley, 1998.