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Ginzburg-Landau Model for Curved Two-Phase Shallow Mixing Layers

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Abstract**—**Method of multiple scales is used in the paper in order to derive an amplitude evolution equation for the most unstable mode from two-dimensional shallow water equations under the rigid-lid assumption. It is assumed that shallow mixing layer is slightly curved in the longitudinal direction and contains small particles. Dynamic interaction between carrier fluid and particles is neglected. It is shown that the evolution equation is the complex Ginzburg-Landau equation. Explicit formulas for the computation of the coefficients of the equation are obtained.

Keywords**—**Shallow water equations, mixing layer, weakly nonlinear analysis, Ginzburg-Landau equation

I. Introduction

SHALLOW mixing layers are widespread in nature and engineering. Examples include flows at river junctions and flows in composite and compound channels. There are three basic methods which are used to analyze the development of a mixing layer in shallow water: experimental analysis, numerical modeling and stability analysis [1]. Two major conclusions follow from experimental investigation [2]-[5]: (a) bottom friction and shallowness of water layer suppress the growth of perturbations and (b) shallow mixing layer grows at a smaller rate than free mixing layer. Several papers [5]-[9] are devoted to linear stability analysis of shallow mixing layers. Theoretical analyses in [5]-[9] confirmed that bottom friction stabilizes the flow and reduces the growth rate of a shallow mixing layer. If a carrier fluid contains solid particles one should use two-phase flow model in order to describe the development of instability. Stability of two-phase flows under several simplifying assumptions is analyzed in [10], [11]. It is shown in [10], [11] that higher particle concentration in the fluid has a stabilizing influence on the flow.

Linear stability analysis is the first step in analyzing behavior of complex flows. The evolution of the most unstable mode when the bed-friction number (introduced by Chu et al. [6]) is slightly smaller than the critical value can be analyzed by means of weakly nonlinear theories. Such models are used in the past in order to analyze spatio-temporal dynamics of complex flows [12]-[17]. It is shown in [12]-[17] that the amplitude evolution equation for the most unstable mode in both cases (Navier-Stokes equations and shallow water equations under the rigid-lid assumption) is the complex Ginzburg-Landau equation.

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In the present paper we derive the complex Ginzburg-Landau equation from the shallow water equations under the rigid-lid assumption for the case of two-phase slightly curved mixing layers. The coefficients of the equation are obtained in closed form in terms of the linear stability characteristics of the flow.

II. Mathematical Formulation of the Problem

Consider the two-dimensional shallow water equations under the rigid-lid assumption

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad (1)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}+\frac{c_f}{2h}u\sqrt{u^2+v^2}=B(u^p-u), \qquad (2)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial p}{\partial y}+\frac{c_f}{2h}v\sqrt{u^2+v^2} +\frac{1}{R}u^2=B(v^p-v), \qquad (3)$$

where u and v are the depth-averaged velocity components in the x and y directions, respectively, u^p and v^p are the components of the particle velocities, c_f is the friction coefficient, h is water depth, R is the radius of curvature $(1/R<<1)$ and B is the particle loading parameter (see [10], [11]).

System (1)-(3) can be reduced to one equation

$$(\Delta\psi)_t+\psi_y(\Delta\psi)_x-\psi_x(\Delta\psi)_y+\frac{2}{R}\psi_y\psi_{xy} +\frac{c_f}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2}+\frac{c_f}{2h\sqrt{\psi_x^2+\psi_y^2}}(\psi_y^2\psi_{yy} +2\psi_x\psi_y\psi_{xy}+\psi_x^2\psi_{xx})+B\Delta\psi=0, \qquad (4)$$

where the stream function is defined by the relations

$$u=\frac{\partial\psi}{\partial y},\ v=-\frac{\partial\psi}{\partial x}. \qquad (5)$$

A perturbed solution to (4) is sought in the form

$$\psi(x,y,t)=\psi_0(y)+\varepsilon\psi_1(x,y,t)+\varepsilon^2\psi_2(x,y,t)+\varepsilon^3\psi_3(x,y,t)+..., \tag{6}$$

where ε is a small parameter which will be defined later.

Let $\psi_{0y}=u_0(y)$ be the base flow solution. Substituting (6) into (4) and linearizing the resulting equation in the neighborhood of the base flow we obtain

$$L_1\psi_1=0, \tag{7}$$

where

$$L_1\psi\equiv\psi_{xxt}+\psi_{yyt}+\psi_{0y}\psi_{xxx}+\psi_{oy}\psi_{yyx}-\psi_{0yyy}\psi_x+\frac{c_f}{2h}(\psi_{0y}\psi_{xx}+2\psi_{oyy}\psi_y+2\psi_{0y}\psi_{yy})+\frac{2}{R}\psi_{0y}\psi_{xy}+B(\psi_{1xx}+\psi_{1yy})$$

A hyperbolic tangent velocity profile of the form

$$u_0(y)=\frac{U_1+U_2}{2}+\frac{U_2-U_1}{2}\tanh y \tag{8}$$

is often used in practice in order to represent the base flow for the case of a mixing layer. Here U_1 and U_2 are the velocities of undisturbed flow at $y=-\infty$ and $y=+\infty$, respectively.

The solution to (7) is sought in the form of a normal mode

$$\psi_1(x,y,t)=\varphi_1(y)\exp[ik(x-ct)], \tag{9}$$

where is $\varphi_1(y)$ the amplitude of the normal perturbation, k is the wave number and c is the phase speed of the perturbation. Using (7) and (9) we obtain

$$L\varphi_1=0, \tag{10}$$

where

$$L\varphi\equiv\varphi''[u_0-c-iSu_0/k-iB/k]+\varphi'(2u_0/R-iSu_{0y}/k)+\varphi(k^2c-k^2u_0-u_{0yy}+ikSu_0/2+ikB).$$

The boundary conditions are

$$\varphi_1(\pm\infty)=0. \tag{11}$$

Here $S=c_f b/h$ is the stability parameter (referred to as the bed-friction number in the literature), where b is the characteristic length scale (mixing layer width, for example).

Note that (10), (11) is an eigenvalue problem (the complex eigenvalues are $c=c_r+ic_i$). Base flow (8) is said to be stable if all $c_i<0$ and unstable if at least one $c_i>0$. Marginal stability of flow (8) is described by the relation $c_i=0$. Problem (10), (11) is usually solved numerically (details of numerical algorithm based on collocation method are given in [17]). Thus, solution of (10), (11) allows one to obtain the critical values of the parameters of the problem S_c, k_c, c_c. A typical marginal stability curve for shallow water flows is a convex curve with one maximum (the coordinates of the maximum point in the $(k,S)-$ plane are $k=k_c$ and $S=S_c$).

III. GINZBURG-LANDAU EQUATION

Assume that the bed-friction number is slightly smaller than the critical value:

$$S=S_c(1-\varepsilon^2). \tag{12}$$

Now the role of the parameter ε in (6) becomes clear: it characterizes how close is the parameter S to the critical value S_c. In addition, (12) implies that base flow (8) is unstable if the bed-friction number is equal to S. However, since ε is small, the growth rate of the most unstable perturbation is also small. Hence, one can try to characterize the development of instability analytically by means of weakly nonlinear theory.

Following [12] we introduce the following "slow" variables

$$\tau=\varepsilon^2 t, \quad \xi=\varepsilon(x-c_g t), \tag{13}$$

where c_g is the group velocity.

The stream function ψ_1 in (9) is replaced by

$$\psi_1(x,y,t,\xi,\tau)=A(\xi,\tau)\varphi_1(y)\exp[ik(x-ct)], \tag{14}$$

where $\varphi_1(y)$ is the eigenfunction of the marginally stable normal perturbation with $S=S_c, k=k_c$ and $c=c_c$. The objective is to derive equation for the evolution of the amplitude function $A(\xi,\tau)$.

Using (13) we replace the derivatives with respect to x and t in (4) by the following expressions

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial \xi},$$
$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} - \varepsilon c_g \frac{\partial}{\partial \xi} + \varepsilon^2 \frac{\partial}{\partial \tau}. \qquad (15)$$

Using (4), (6), (15) and collecting the terms that contain ε^2 we obtain

$$\begin{aligned} L_1\psi_2 &= c_g(\psi_{1xx\xi} + \psi_{1yy\xi}) - 2\psi_{1x\xi t} - 3u_0\psi_{1xx\xi} \\ &- \psi_{1y}\psi_{1xxx} - \psi_{1y}\psi_{1yyx} - u_0\psi_{1\xi yy} + \psi_{1x}\psi_{1xxy} \\ &+ \psi_{1x}\psi_{1yyy} + u_{0yy}\psi_{1\xi} - \frac{S}{2}[\psi_{1xx}\psi_{1y} + 2u_0\psi_{1x\xi} \\ &+ 2\psi_{1yy}\psi_{1y} - 2u_0u_{0y} + 2\psi_{1x}\psi_{1xy}] \\ &- \frac{2}{R}[u_0\psi_{1\xi y} + \psi_{1y}\psi_{1xy}] - 2B\psi_{1x\xi}. \end{aligned} \qquad (16)$$

Similarly, collecting the terms that contain ε^3 we obtain

$$\begin{aligned} L_1\psi_3 &= c_g(\psi_{2xx\xi} + \psi_{2yy\xi}) - \psi_{1xx\tau} - 2\psi_{2x\xi t} \\ &+ 2c_g\psi_{1x\xi\xi} - \psi_{1\xi\xi t} - \psi_{1yy\tau} - 3u_0\psi_{2xx\xi} - 3u_0\psi_{1x\xi\xi} \\ &- \psi_{1y}\psi_{2xxx} - 3\psi_{1y}\psi_{1xx\xi} - \psi_{2y}\psi_{1xxx} - \psi_{2y}\psi_{1yyx} \\ &- \psi_{1y}\psi_{2yyx} - \psi_{1y}\psi_{1\xi yy} - u_0\psi_{2\xi yy} + \psi_{2x}\psi_{1xxy} \\ &+ \psi_{1\xi}\psi_{1xxy} + \psi_{1x}\psi_{2xxy} + 2\psi_{1x}\psi_{1xy\xi} + \psi_{1x}\psi_{2yyy} \\ &+ \psi_{2x}\psi_{1yyy} + \psi_{1\xi}\psi_{1yyy} + \psi_{2\xi}u_{0yy} - \frac{S}{2}[\psi_{1xx}\psi_{2y} \\ &+ 1.5\psi_{1xx}\psi_{1x}^2/u_0 + \psi_{2xx}\psi_{1y} + 2\psi_{1x\xi}\psi_{1y} + 2u_0\psi_{2x\xi} \\ &+ u_0\psi_{1\xi\xi} + \psi_{1yy}\psi_{2y} + \psi_{2yy}\psi_{1y} - u_0\psi_{1xx} - 2u_{0y}\psi_{1y} \\ &- 2u_0\psi_{1yy} + \psi_{1yy}\psi_{2y} + \psi_{1y}\psi_{2yy} + 2\psi_{1x}\psi_{2xy} \\ &+ 2\psi_{1x}\psi_{1\xi y} + 2\psi_{2x}\psi_{1xy} + 2\psi_{1\xi}\psi_{1xy}] \\ &- \frac{2}{R}[u_0\psi_{2\xi y} + \psi_{1y}\psi_{2xy} + \psi_{1y}\psi_{1\xi y} + \psi_{2y}\psi_{1xy}] \\ &- B(2\psi_{2x\xi} + \psi_{1\xi\xi}). \end{aligned} \qquad (17)$$

Analyzing the structure of the right-hand side of (16) and using (14) we conclude that ψ_2 in (16) should be sought in the form

$$\begin{aligned} \psi_2 &= AA^*\varphi_2^{(0)}(y) + A_\xi\varphi_2^{(1)}(y)\exp[ik(x-ct)] \\ &+ A^2\varphi_2^{(2)}(y)\exp[2ik(x-ct)], \end{aligned} \qquad (18)$$

where A^* is the complex conjugate of A and $\varphi_2^{(0)}(y), \varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$ are unknown functions of y. Substituting (18) into (17) and collecting the time-independent terms we obtain the following ordinary differential equation for the function $\varphi_2^{(0)}(y)$:

$$\begin{aligned} &2S[u_{0y}(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}) + u_0(\varphi_{2yy}^{(0)} + \varphi_{2yy}^{*(0)})] \\ &+ 2B(\varphi_{2yy}^{(0)} + \varphi_{2yy}^{*(0)}) \\ &= ik(\varphi_{1y}\varphi_{1yy}^* - \varphi_{1y}^*\varphi_{1yy} + \varphi_1\varphi_{1yyy}^* - \varphi_1^*\varphi_{1yyy}) \\ &- \frac{S}{2}[k^2(\varphi_1\varphi_{1y}^* + \varphi_1^*\varphi_{1y} + +2(\varphi_{1y}^*\varphi_{1yy} + \varphi_{1yy}^*\varphi_{1y})] \\ &= 0. \end{aligned} \qquad (19)$$

The function $\varphi_2^{(0)}(y)$ satisfies the following boundary conditions:

$$\varphi_2^{(0)}(\pm\infty) = 0. \qquad (20)$$

Substituting (18) into (17) and collecting the terms containing the first harmonic we obtain the equation

$$\begin{aligned} &(u_0 - c - Su_0\frac{i}{k})\varphi_{2yy}^{(1)} + (2\frac{u_0}{R} - Su_{0y}\frac{i}{k})\varphi_{2y}^{(1)} \\ &+ (k^2c - k^2u_0 - u_{0yy} + ku_0S\frac{i}{2}) \\ &+ B(-i/k\varphi_{2yy}^{(1)} + iku_0\varphi_2^{(1)}) \\ &= -\frac{i}{k}(c_g - u_0)\varphi_{1yy} + 2\frac{iu_0}{kR}\varphi_{1y} \\ &+ (2ikc - 3iku_0 - \frac{i}{k}u_{0yy} + ikc_g - u_0S - 2B)\varphi_1 \end{aligned} \qquad (21)$$

with the boundary conditions

$$\varphi_2^{(1)}(\pm\infty) = 0. \qquad (22)$$

Finally, using (18) and (17) for the terms that contain the second harmonic, we obtain

$$\begin{aligned} &8ik^3c\varphi_2^{(2)} - 2ikc\varphi_{2yy}^{(2)} - 8ik^3u_0\varphi_2^{(2)} + 2iku_0\varphi_{2yy}^{(2)} \\ &- 2iku_{0yy}\varphi_2^{(2)} + S[-4k^2u_0\varphi_2^{(2)} + 2u_{0y}\varphi_{2y}^{(2)} \\ &+ 2u_0\varphi_{2yy}^{(2)}] + 4iku_0\varphi_{2y}^{(2)}/R + B(\varphi_{2yy}^{(2)} - 4k^2\varphi_2^{(2)}) \\ &= ik(\varphi_1\varphi_{1yyy} - \varphi_{1y}\varphi_{1yy}) \\ &- S(2\varphi_{1y}\varphi_{1yy} - 3k^2\varphi_1\varphi_{1y}) - 2ik\varphi_{1y}^2/R \end{aligned} \qquad (23)$$

The boundary conditions are

$$\varphi_2^{(2)}(\pm\infty) = 0. \tag{24}$$

Comparing (10) and (21) we see that the left-hand sides of both equations are the same. Thus, (21) has a solution if and only if the right-hand side of (21) is orthogonal to all eigenfucntions of the corresponding adjoint problem (see [18]). The adjoint operator L^a and adjoint eigenfunction φ_1^a are defined as follows

$$\int_{-\infty}^{+\infty} \varphi_1^a L\varphi_1 dy = \int_{-\infty}^{+\infty} \varphi_1 L^a \varphi_1^a dy. \tag{25}$$

The adjoint problem is

$$L^a \varphi_1^a = 0, \tag{26}$$

$$\varphi_1^a(\pm\infty) = 0. \tag{27}$$

Integrating the left-hand side of (25) by parts and using boundary conditions (11), (27) we obtain

$$\begin{aligned} L^a\varphi_1^a &\equiv \varphi_{1yy}^a\left(u_0 - c - Su_0\frac{i}{k} - B\frac{i}{k}\right) \\ &+ \varphi_{1y}^a\left(2u_{0y} - Su_{0y}\frac{i}{k} - 2\frac{u_0}{R}\right) \\ &+ \varphi_1^a\left(k^2c - k^2u_0 + \frac{ik}{2}Su_0 - 2\frac{u_{0y}}{R} + Bik\right). \end{aligned} \tag{28}$$

Solvability condition for (21) has the form

$$\begin{aligned} &\int_{-\infty}^{+\infty} \varphi_1^a[(c_g - u_0)\varphi_{1yy} - 2\frac{u_0}{R}\varphi_{1y} \\ &+ (-2k^2c + 3k^2u_0 + u_{0yy} \\ &- k^2c_g + iku_0S + 2Bik)\varphi_1]dy = 0. \end{aligned} \tag{29}$$

Hence, the group velocity c_g can be found from (29).

The evolution equation for the amplitude function $A(\xi,\tau)$ is determined from the solvability condition at the third order. Multiplying the right-hand side of (17) by φ_1^a, using (18) and the solutions of the boundary value problems (19)-(24) we obtain the complex Ginzburg-Landau equation for the amplitude $A(\xi,\tau)$ of the form

$$\frac{\partial A}{\partial \tau} = \sigma A + \delta\frac{\partial^2 A}{\partial \xi^2} - \mu |A|^2 A, \tag{30}$$

where

$$\sigma = \frac{\sigma_1}{\eta},\ \delta = \frac{\delta_1}{\eta},\ \mu = \frac{\mu_1}{\eta} \tag{31}$$

and the complex coefficients $\sigma_1, \delta_1, \mu_1$ and η are given by

$$\eta = \int_{-\infty}^{+\infty} \varphi_1^a(\varphi_{1yy} - k^2\varphi_1)dy, \tag{32}$$

$$\sigma_1 = \frac{S}{2}\int_{-\infty}^{+\infty} \varphi_1^a(-k^2u_0\varphi_1 + 2u_{0y}\varphi_{1y} + 2u_0\varphi_{1yy})dy, \tag{33}$$

$$\begin{aligned} \delta_1 &= \int_{-\infty}^{+\infty} \varphi_1^a[(c_g - u_0)\varphi_{2yy}^{(1)} - 2\frac{u_0}{R}\varphi_{2y}^{(1)} \\ &+ \varphi_2^{(1)}(-k^2c_g - 2k^2c + 3k^2u_0 + u_{0yy} - ikSu_0 - 2ikB) \\ &+ \varphi_1(2ikc_g + ikc - 3iku_0 - u_0\frac{S}{2} - B)]dy, \end{aligned} \tag{34}$$

$$\begin{aligned} \mu_1 &= \int_{-\infty}^{+\infty} \varphi_1^a\{6ik^3\varphi_2^{(2)}\varphi_{1y}^* - 2ik\varphi_{1y}^*\varphi_{2yy}^{(2)} \\ &+ 3ik^3\varphi_1^*\varphi_{2y}^{(2)} + ik^3\varphi_1(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}) \\ &- ik\varphi_{1yy}(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}) + ik\varphi_{2y}^{(2)}\varphi_{1yy}^* - ik\varphi_1^*\varphi_{2yyy}^{(2)} \\ &+ ik\varphi_1(\varphi_{2yyy}^{(0)} + \varphi_{2yyy}^{*(0)}) + 2ik\varphi_{1yyy}^*\varphi_2^{(2)} \\ &- \frac{S}{2}[-k^2\varphi_1(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}) + 3k^2\varphi_1^*\varphi_{2y}^{(2)} \\ &- \frac{3k^4}{2u_0}\varphi_1^2\varphi_1^* + 2\varphi_{1yy}(\varphi_{2y}^{(0)} + \varphi_{2y}^{*(0)}) + 2\varphi_{1yy}^*\varphi_{2y}^{(2)} \\ &+ 2\varphi_{1y}(\varphi_{2yy}^{(0)} + \varphi_{2yy}^{*(0)}) + 2\varphi_{2yy}^{(2)}\varphi_{1y}^*] \\ &- 2\frac{ik}{R}(\varphi_{2y}^{(2)}\varphi_{1y}^* + \varphi_{2y}^{(0)}\varphi_{1y})\}dy. \end{aligned} \tag{35}$$

The coefficients of the Ginzburg-Landau equation (30) can be computed using formulas (31)-(35). Note that in order to perform calculations it is necessary to solve the linear stability problem (10)-(11), the corresponding adjoint problem (26)-(28), three boundary value problems (19)-(24) and numerically evaluate integrals in (31)-(35). Computational procedure for such type of problems is described in detail in [17].

IV. Conclusions

Method of multiple scales is used in the paper in order to derive an amplitude evolution equation for the most unstable mode. The equation is obtained for the case of a shallow mixing layer which is slightly curved in the longitudinal direction and contains small particles. It is shown that the amplitude equation in this case is the complex Ginzburg-Landau equation. Explicit formulas for the calculation of the coefficients of the equation are derived.

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