International Journal of Electrical, Electronic and Communication Sciences
ISSN: 2517-9438
Vol:2, No:8, 2008

New Approach to Spectral Analysis of High Bit Rate PCM Signals

J. P. Dubois

Abstract**—**Pulse code modulation is a widespread technique in digital communication with significant impact on existing modern and proposed future communication technologies. Its widespread utilization is due to its simplicity and attractive spectral characteristics. In this paper, we present a new approach to the spectral analysis of PCM signals using Riemann-Stieltjes integrals, which is very accurate for high bit rates. This approach can serve as a model for similar spectral analysis of other competing modulation schemes.

Keywords**—**Coding, discrete Fourier, power spectral density, pulse code modulation, Riemann-Stieltjes integrals.

I. Introduction

PULSE code modulation (PCM) has undoubtedly revolutionalised digital communication. In retrospect, PCM appears to be simple, trivial, and as old as telephony itself. Historically, PCM was invented by the British engineer Alec Reeves while working for the ITT (International Telephone and Telegraph) in France in 1937. Ever since that time, never has a modulation technique been as ubiquitous as PCM. Nowadays, PCM is employed by advanced digital technologies such as GSM, GPRS, and Ethernet.

While PCM is basically a baseband modulation technique, it serves as a model for the phase diagram of passband modulation schemes such as the equally popular binary phase shift keying (BPSK).

PCM has traditionally found widespread applications in digital telephony, combining signal processing with coding. The conventional standard audio signal for a single voice path is sampled at a rate of 8 KHz (the Nyquist rate), each sample is coded with 8 bits, yielding 64 Kbits/s digital signal known as DS0. DS0 serves as a building block for an even more popular standard G.711. In such a system, the encoding on a DS0 is a logarithmic compression law, known as μ-law in North American and Japan and A-law in Europe and the rest of the world [1]. Fig. 1 illustrates sampling and quantization of a 4-bit (or 2^4 levels) PCM signal.

As research in compression techniques advanced, further compression was possible and additional standards were published. We note here the powerful ADPCM scheme, which is widely used in VoIP (voice over IP) communication.

Manuscript received June 30, 2006.

J. P. Dubois is with the University of Balamand, Deir El Balamand, Koura, North District, Lebanon (phone: 961-3-841472; fax: 961-6930250; e-mail: jeanpierre_dubois@hotmail.com).

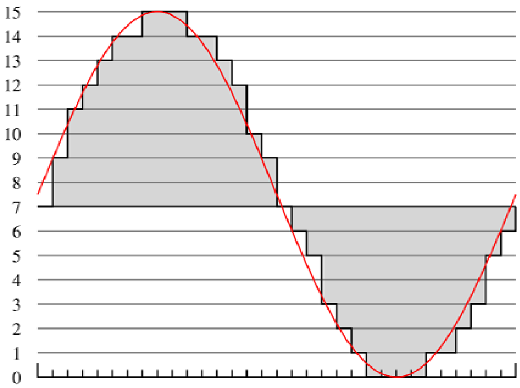

Fig. 1 A 4-bit PCM signal

Encoding the PCM bitstream as a signal (ready for transmission through a channel) can be done using a variety of line codes. Most notably are the following formats: BNRZ (bipolar non-return to zero), UNRZ (unipolar non-return-to-zero), BRZ (bipolar return-to-zero), URZ (unipolar return-to-zero), AMI (alternate mark inversion), Manchester (or bi-phase). The latter is used in Ethernet local area computer networks (LAN).

The choice of a given PCM line code or format depends on a number of factors: (1) power efficiency; (2) bandwidth efficiency; (3) synchronization; (4) error detection capabilities; (5) ability of carrier recovery.

Key to determining most of these characteristics, is an accurate spectral analysis of the PCM signal. This paper presents a new approach to the spectral analysis of PCM signals using Riemann-Stieltjes integrals, which is very accurate for high bit rates.

II. Background on Riemann-Stieltjes Integrals

A. Definition

The Riemann-Stieltjes integral, developed by Bernhard Riemann and Thomas Joannes Stieltjes, is a generalization of the Riemann integral.

The Riemann-Stieltjes integral of a real-valued function ψ (termed the integrand) of a real variable with respect to a real function Φ (termed the integrator) is denoted by

$$\mathcal{RS}\langle \psi, \Phi \rangle = \int_{x_0}^{x_1} \psi(x)\, d\Phi(x) . \qquad (1)$$

Strictly speaking, the Riemann-Stieltjes integral is mathematically defined to be the limit as the mesh of the partition $\mathcal{P}$ of the interval $[x_0, x_1]$ approaches 0, of the approximating sum

$$\mathcal{RS}\langle\psi,\Phi\rangle \approx \sum_{x_i \in \mathcal{P}} \psi(\zeta_i)\left(\Psi(x_{i+1}) - \Psi(x_i)\right), \tag{2}$$

where $\zeta_i \in [x_i, x_{i+1}]$.

B. Existence

Perhaps the best simple existence theorem states: If ψ is continuous and Φ is of *bounded variation* on the interval $[x_0, x_1]$, then the integral exists.

Simply stated, the Riemann-Stieltjes integral of (1) exists if the integrand ψ and the integrator Φ do not share any points of discontinuities.

C. Properties

The Riemann integral is defined as

$$\mathcal{R}\langle\psi,\Phi\rangle = \int_{x_0}^{x_1} \psi(x)\Phi^{'}(x)dx\ . \tag{3}$$

If Φ is the *Lebesgue integral* of its derivative, then the Riemann-Stieltjes integral of (1) is equal to the Riemann integral of (3). In this case, Φ is said to be *absolutely continuous*. Below is a few mathematical highlights on some of the mentioned concepts.

Lebesgue integral: The Lebesgue integral is defined using *Lebesgue measure* which is used throughout real analysis. (On a historical note, Henri Lesbesgue described his measure in 1901 and described his integral the next year).

Absolute continuity: A real function Φ of a real variable is said to be absolutely continuous if $\forall\,\varepsilon > 0$ (arbitrarily small), $\exists\,\delta > 0$ small enough so that if a sequence of pairwise disjoint intervals $[y_k, z_k], k = 1, 2, \cdots n$, satisfies $\sum_{k=1}^{n}(z_k - y_k) < \delta$, then $\sum_{k=1}^{n}\left|\Phi(z_k) - \Phi(y_k)\right| < \varepsilon$.

The Riemann-Stieltjes integral is not captured by any expression involving derivatives of the integrator Φ, that is, the Riemann-Stieltjes and Riemann integrals are different if: (1) Φ has jump discontinuities and (2) Φ has derivative 0 almost everywhere while still being continuous and increasing. The latter condition may arise if Φ is, for example, a *Cantor function* or a *Minkowski*'s question mark function. In Fig. 2, the Cantor function is depicted as the standard example of what is sometimes called a devil's staircase.

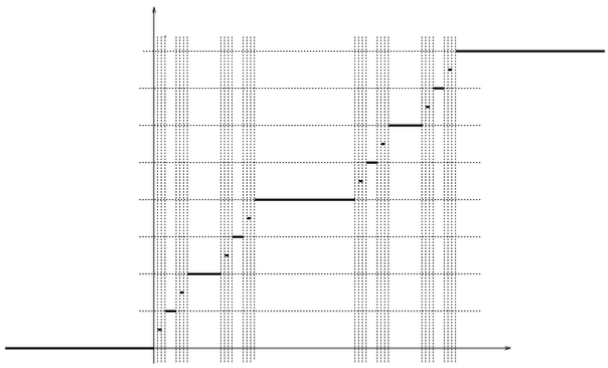

Fig. 2 The Cantor function as an example of a devil's staircase

III. Power Spectral Density of PCM

In this section, we present a new approach to deriving the power spectral density (PSD) of PCM signals, which can serve as a model for similar spectral analysis of other competing modulation schemes.

Throughout the communication literature [2-6], the canonical form of the PSD of PCM is derived using discrete Fourier transforms (DFT). The derivation is complicated, lengthy, and involves DFT. The drawback of such analysis is that if DFT is used as a guide for spectral analysis of other modulation schemes, experimental validation of such analysis can only be accomplished using fast Fourier transform (FFT) chips and ultimately highly expensive digital spectrum analyzers (which process signals using FFT at a cost of around $25,000). On the other hand, the approach we follow only invokes Riemann-Stieltjes integral, which can be easily and cost effectively implemented using the sum expression of (2) (for example, with summing operational amplifiers).

Following is a detailed derivation of the PSD of PCM signals. A PCM signal is represented by the complex envelope

$$p(t) = \sum_{m=-\infty}^{\infty} a_k \phi(t - mT_b)\ , \tag{4}$$

where $T_b = R_b^{-1}$ is the bit period (reciprocal of the incoming bit rate R_b), a_k is the random voltage value of the *k*-th random bit, assumed to be wide-sense stationary, and $\phi(t)$ is the pulse shape modulated by a bit.

The autocorrelation function (ACF) is a very power tool that describes the correlation between random variables generated by a random signal at specific time instants separated by a deterministic interval of time. Knowledge of the ACF results in a thorough understanding of the PSD of the random signal. The ACF of the PCM signal is given by

$$\begin{aligned} E\left[p(t+\tau)p(t)\right] &= \\ & E\left[\sum_{m=-\infty}^{\infty} a_m \phi(t+\tau-mT_b)\sum_{n=-\infty}^{\infty} a_n\phi(t-nT_b)\right] \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} E\left(a_m a_n\right)\phi(t+\tau-mT_b)\phi(t-nT_b) \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} R_A(m-n)\phi(t+\tau-mT_b)\phi(t-nT_b). \end{aligned} \tag{5}$$

Making a change of variable from (m,n) to (k,l) results in

$$\left.\begin{array}{l} k=m-n \\ l=n \end{array}\right\} \Rightarrow \left\{\begin{array}{l} m=k+l \\ n=l \end{array}\right. . \tag{6}$$

The ACF thus becomes

$$\begin{aligned} & E\left[p(t+\tau)p(t)\right] = \\ & \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} R_A(k)\phi(t+\tau-(k+l)T_b)\phi(t-lT_b) \\ & = \frac{1}{T_b}\sum_{k=-\infty}^{\infty} R_A(k)\sum_{l=-\infty}^{\infty}\phi(t-lT_b+\tau-kT_b)\phi(t-lT_b)T_b \\ & = \frac{1}{T_b}\sum_{k=-\infty}^{\infty} R_A(k)\sum_{l=-\infty}^{\infty}\phi(t+lT_b+\tau-kT_b)\phi(t+lT_b)T_b \\ & (\text{ set } t_l = t+lT_b,\ \Delta t_l = T_b = t_l - t_{l-1}) \\ & = \frac{1}{T_b}\sum_{k=-\infty}^{\infty} R_A(k)\sum_{l=-\infty}^{\infty}\phi(t_l+\tau-kT_b)\phi(t_l)\Delta t_l,\ \left\|\Delta t_l\right\| = \left\|T_b\right\| \\ & = \frac{1}{T_b}\sum_{k=-\infty}^{\infty} R_A(k)\underbrace{\lim_{\|\Delta\|\to 0}\sum_{l=-\infty}^{\infty}\phi(t_l+\tau-kT_b)\phi(t_l)\Delta t_l}_{\substack{\text{Riemann-Stieltjes integral of } \phi(t+\tau-kT_b)\phi(t) \\ \text{with respect to } t}} \\ & = \frac{1}{T_b}\sum_{k=-\infty}^{\infty} R_A(k)\int_{-\infty}^{\infty}\phi(t+\tau-kT_b)\phi(t)dt \\ & = \frac{1}{T_b}\sum_{k=-\infty}^{\infty} R_A(k)\phi(\tau-kT_b)\otimes\phi(\tau), \end{aligned} \tag{7}$$

where $\otimes$ is the time correlation operator.

Since the PSD is the Fourier transform of the ACF, we have

$$S_P(f) = \frac{\left|\phi(f)\right|^2}{T_b}\sum_{k=-\infty}^{\infty} R_A(k)e^{-j2\pi kfT_b}\ , \tag{8}$$

where the autocorrelation of the data

$$R_A(k) = \sum_i \left(a_n a_{n+k}\right)_i\ p_i\ , \tag{9}$$

with $a_n a_{n+k}$ being the levels of the n-th and the $(n + k)$-th symbol position and p_i being the probability of having the i-th $\left(a_n a_{n+k}\right)$ product. Fig. 3 illustrates the power spectral density of some PCM line codes.

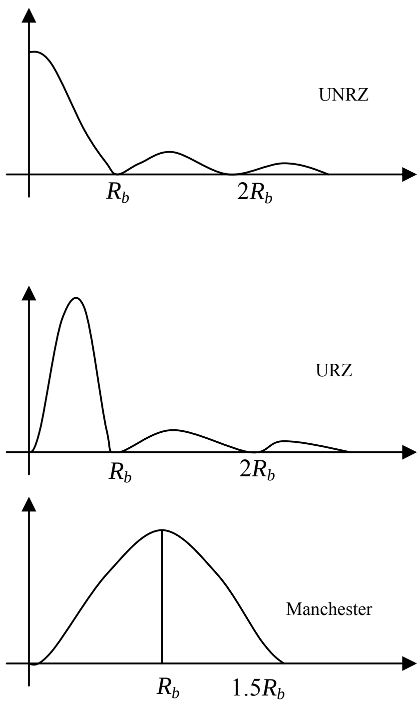

Fig. 3 Power spectral density of different PCM line codes

References

[1] K. C. Pohlmann, *Principles of Digital Audio*, 2nd edition, Sams/Prentice Hall Computer Publishing, Carmel, Indiana. 1985.

[2] L. W. Couch II, *Digital and Analog Communication Systems*, Fifth Edition, Prentice Hall, Upper Saddle River, NJ, 1997.

[3] S. Haykin, *Communication Systems*, John Wiley & Sons, 2000.

[4] B. Sklar, *Digital Communications Fundamentals and Applications*, Prentice Hall Editions, 2002.

[5] B. P. Lathi, *Modern Digital and Analog Communication Systems*, Oxford University Press, 2003.

[6] J. G. Proakis and M. Salehi, *Communication Systems Engineering*, Prentice Hall, 1st edition, 1994.