International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:5, No:10, 2011

Extremal Properties of Generalized Class of Close-to-convex Functions

Norlyda Mohamed, Daud Mohamad, and Shaharuddin Cik Soh

***Abstract*—**Let $G_{\alpha,\beta}(\gamma,\delta)$ denote the class of function $f(z)$, $f(0)=f'(0)-1=0$ which satisfied $\operatorname{Re} e^{i\delta}\{\alpha f'(z)+\beta z f''(z)\}>\gamma$ in the open unit disk $D=\{z\in\mathbb{C}:|z|<1\}$ for some $\alpha\in\mathbb{C}$ $(\alpha\neq 0)$, $\beta\in\mathbb{C}$ and $\gamma\in\mathbb{R}$ $(0\le\gamma<\alpha)$ where $|\delta|\le\pi$ and $\alpha\cos\delta-\gamma>0$. In this paper, we determine some extremal properties including distortion theorem and argument of $f'(z)$.

***Keywords*—**Argument of $f'(z)$, Carathéodory Function, Close-to-convex Function, Distortion Theorem, Extremal Properties

I. Introduction

We denote $G_{\alpha,\beta}(\gamma,\delta)$ the class of normalized analytic function f in the open unit disk, $D=\{z\in\mathbb{C}:|z|<1\}$ where

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n$$

satisfying $\operatorname{Re} e^{i\delta}\{\alpha f'(z)+\beta z f''(z)\}>\gamma,\ z\in D$ for some $\alpha\in\mathbb{C}$ $(\alpha\neq 0)$, $\beta\in\mathbb{C}$ and $\gamma\in\mathbb{R}$ $(0\le\gamma<\alpha)$.

Many of the subclasses of $G_{\alpha,\beta}(\gamma,\delta)$ have been studied by some other researchers as [1] for $G_{\alpha,\beta}(\gamma,0)$ of some $\alpha\in\mathbb{C}$, $\beta\in\mathbb{C}$ $(\beta\neq 0)$ and $\gamma\in\mathbb{R}$ $(0\le\gamma<\alpha)$, [2] for $G_{1,\beta}(\gamma,0)$ where $\alpha>0,\beta<1$, [3] for $G_{1,1}(\gamma,0)$, [4] for $G_{1,1}(0,0)$, [5] for $G_{1,0}(\gamma,\delta)$ where $|\delta|\le\pi$ and $\cos\delta-\gamma>0$, [6] for $G_{1,0}(0,\delta)$ where $|\delta|<\frac{\pi}{2}$ and [7] for $G_{1,0}(0,0)$.

There is a relationship of the class P in the form of $p(z)=1+\sum_{n=1}^{\infty}c_n z^n$ with the extremal information of each selected classes. Writing

$$\frac{e^{i\delta}(\alpha f'(z)+\beta z f''(z))-\gamma-i\alpha\sin\delta}{(a\cos\delta-\gamma)}=p(z)\quad(z\in D)$$

clearly $f\in G_{\alpha,\beta}(\gamma,\delta)$ if $p\in P$, the class of functions with positive real parts.

Norlyda Mohamed, Daud Mohamad, and Shaharuddin Cik Soh are with Department of Mathematics, Faculty of Computers and Mathematical Sciences, Universiti Teknologi MARA 40450 Shah Alam Selangor, Malaysia. (e-mail: norlyda_mohamed@yahoo.com, daud@tmsk.uitm.edu.my and shahar@tmsk.uitm.edu.my).

We make use the result of representation theorem

$$f(z)=\int_{|x|=1}\left[-e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)z-\frac{2e^{-i\delta}A}{(n\beta+\alpha)}\bar{x}\log(1-xz)\right]d\mu(x)$$

where $A=(\alpha\cos\delta-\gamma)$ given by [8] in order to determine the distortion theorem and argument of $f'(z)$ for this class of function.

II. Extremal Properties

We begin by finding the radius and centre of $G_{\alpha,\beta}(\gamma,\delta)$ that will be used for later results.

Theorem 3.1 *Let $f(z)\in G_{\alpha,\beta}(\gamma,\delta)$. Then $f'(z)$ maps $|z|\le r$ into disc D_r with centre and radius*

$$-e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)+\frac{2e^{-i\delta}AM}{1-r^2}\ \ and\ \ \frac{2AMr}{1-r^2}$$

where $A=\alpha\cos\delta-\gamma$, $M=\dfrac{1}{n\beta+\alpha}$ respectively.

Proof. If a and b are complex numbers with $|b|<1$ and if $0<r<1$, the range of the function $(1+arw)/(1+brw)$ where $|w|\le 1$ is a disc with center and radius respectively.

$$\frac{1-a\bar{b}r^2}{1-|b|^2r^2}\ ,\quad\frac{|a-b|r}{1-|b|^2r^2}$$

By taking $a=\bar{B}e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)xr$ and $b=xr$ where $|x|=1$, we see that maps $|z|\le r$ onto D_r. By convexity, any linear combination of functions of this form also maps D onto D_r. Since for some probability measure μ, we have

$$B\left\{\frac{1+\bar{B}e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)xz}{1-xz}\right\}$$

$$f'(z)=\int_{|x|=1} B\left\{\frac{1+\overline{B}e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)xz}{(1-xz)}\right\}d\mu(x)$$

where $B=\dfrac{\frac{2\gamma}{\alpha}n\beta e^{-i\delta}-n\beta e^{-2i\delta}+\alpha}{n\beta+\alpha}$, so, the result follows.

Corollary 3.1 *If* $f(z)\in G_{\alpha,\beta}(\gamma,\delta)$ *then*

$$f'(z)\prec B\left\{\frac{1+\overline{B}e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)z}{1-z}\right\},\ z\in D$$

The simple geometry of a circle enables us to deduce from Theorem 3.2, upper and lower bounds for $\operatorname{Re} f'(z)$, $\operatorname{Im} f'(z)$, $|f'(z)|$ and $\arg f'(z)$ when $f(z)=G_{\alpha,\beta}(\gamma,\delta)$.

Theorem 3.2 *If* $f(z)\in G_{\alpha,\beta}(\gamma,\delta)$, *then*

$$\frac{1+B+r^2(2AR-1)-2AMr}{1-r^2}\le \operatorname{Re} f'(z)\le \frac{1+B+r^2(2AR-1)+2AMr}{1-r^2}$$

where $B=\dfrac{2A(A+\gamma)(M\alpha-1)}{\alpha^2}$ *and* $R=\dfrac{(A+\gamma)}{\alpha^2}$, *and*

$$\frac{-2A\left(T\left(M-\frac{1}{\alpha}\right)+r\left(M+\frac{rT}{\alpha}\right)\right)}{1-r^2}\le \operatorname{Im} f'(z)\le \frac{2A\left(T\left(M-\frac{1}{\alpha}\right)+r\left(M+\frac{rT}{\alpha}\right)\right)r}{1-r^2}$$

where $T=\sqrt{1-\left(\frac{A+\gamma}{\alpha}\right)^2}$ *and all bounds are sharp for any extreme point* $f(z)$ *of* $G_{\alpha,\beta}(\gamma,\delta)$.

Proof. By Theorem 3.1, we can write

$$\left|f'(z)-\left\{-e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)+\frac{2e^{-i\delta}AM}{1-r^2}\right\}\right|\le \frac{2AMr}{1-r^2} \qquad (1)$$

So that

$$\frac{2AMr}{1-r^2}\le \operatorname{Re}\left\{f'(z)+e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)-\frac{2e^{-i\delta}AM}{1-r^2}\right\}\le \frac{2AMr}{1-r^2}$$

that gives ;

$$-\frac{2AMr}{1-r^2}\le \operatorname{Re} f'(z)-\left\{\frac{2AM\cos\delta-(1-r^2)\left(2\cos^2\delta-1-\frac{2\gamma}{\alpha}\cos\delta\right)}{1-r^2}\right\}$$

$$\le \frac{2AMr}{1-r^2}$$

and

$$-\frac{2AMr}{1-r^2}\le \operatorname{Im}\left\{f'(z)+e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)-\frac{2e^{-i\delta}AM}{1-r^2}\right\}\le \frac{2AMr}{1-r^2}$$

that gives

$$\frac{-2A\left(\sin\delta\left(M-\frac{1}{\alpha}\right)+r\left(M+\frac{r\sin\delta}{\alpha}\right)\right)}{1-r^2}\le \operatorname{Im} f'(z)\le$$

$$\frac{2A\left(\sin\delta\left(\frac{1}{\alpha}-M\right)+r\left(M-\frac{r\sin\delta}{\alpha}\right)\right)}{1-r^2}$$

Since $\cos\delta=\dfrac{A+\gamma}{\alpha}$ and $\sin\delta=\sqrt{1-\left(\frac{A+\gamma}{\alpha}\right)^2}$, we can write the inequalities in this form

$$-\frac{2AMr}{1-r^2}\le \operatorname{Re} f'(z)-\left\{\frac{1+2A(A+\gamma)\left(\frac{M\alpha-1}{\alpha^2}\right)+r^2\left(\frac{2A(A+\gamma)}{\alpha^2}-1\right)}{1-r^2}\right\}\le \frac{2AMr}{1-r^2}$$

and

$$\frac{-2A\left(\sqrt{1-\left(\frac{A+\gamma}{\alpha}\right)^2}\left(M-\frac{1}{\alpha}\right)+r\left(M+\frac{r\sqrt{1-\left(\frac{A+\gamma}{\alpha}\right)^2}}{\alpha}\right)\right)}{1-r^2}\le \operatorname{Im} f'(z)\le$$

$$\frac{2A\left(\sqrt{1-\left(\frac{A+\gamma}{\alpha}\right)^2}\left(\frac{1}{\alpha}-M\right)+r\left(M-\frac{r\sqrt{1-\left(\frac{A+\gamma}{\alpha}\right)^2}}{\alpha}\right)\right)}{1-r^2}$$

Letting $B=\dfrac{2A(A+\gamma)(M\alpha-1)}{\alpha^2}$, $R=\dfrac{(A+\gamma)}{\alpha^2}$ and $T=\sqrt{1-\left(\frac{A+\gamma}{\alpha}\right)^2}$, we obtain the above inequalities as required. It is clear that each inequality is sharp for some z on

$|z| = r$ when f is an extreme point.

Our next result is to obtain a distortion theorem for $f(z) \in G_{\alpha,\beta}(\gamma,\delta)$.

Theorem 3.3 *If $f(z) \in G_{\alpha,\beta}(\gamma,\delta)$, then $|f'(z)| \le |\Gamma(r)| + \frac{2AMr}{1-r^2}$ where*

$$C(r) = \left(1 - \frac{4\gamma A}{\alpha^2} + \frac{4AM}{\alpha(1-r^2)}\left\{\frac{AM\alpha}{(1-r^2)} + \gamma - A\right\}\right)^{\frac{1}{2}} \quad (2)$$

Proof. Let

$$\Gamma(r) = -e^{-i\delta}\left(e^{-i\delta} - \frac{2\gamma}{\alpha}\right) + \frac{2e^{-i\delta}AM}{1-r^2} \quad (3)$$

and from (1), we have

$$|f'(z) - \Gamma(r)| \le \frac{2AMr}{1-r^2}$$

so that

$$|f'(z)| \le |\Gamma(r)| + \frac{2AMr}{1-r^2} = C(r) + \frac{2AMr}{1-r^2}$$

as required.

If $\gamma \ge 0$, then f' is non-zero throughout D and has continuous argument whereas if $\gamma < 0$ and f_0 is any extreme function of $G_{\alpha,\beta}(\gamma,\delta)$, then at some points in D, f_0' has a zero, thus, there is no argument. We next obtain bounds for $\arg f'(z)$ when $f(z) \in G_{\alpha,\beta}(\gamma,\delta)$ with restricted value of $|z|$ for the case of $\gamma < 0$. Furthermore, we will use the following property for argument: for given δ in $[-\pi,\pi]$ and as x varies in some interval $[0,c]$, so that $e^{i\delta} + x \ne 0$, $\phi_\delta(x)$ is continuous argument of $e^{i\delta} + x \ne 0$ for which $\phi_\delta(0) = \delta$. We have

$$\phi_\delta(x) = \begin{cases} \tan^{-1}\left(\frac{\sin\delta}{\cos\delta + x}\right) & x + \cos\delta > 0 \\ \pi + \tan^{-1}\left(\frac{\sin\delta}{\cos\delta + x}\right) & x + \cos\delta < 0 \\ \frac{\pi}{2} & x + \cos\delta = 0 \end{cases}$$

when $0 < \delta < \pi$ and $-\pi < \delta < 0$ for $\delta = 0, \pm\pi$.

Theorem 3.4 *Let $f(z) \in G_{\alpha,\beta}(\gamma,\delta)$ and put $x(r) = 2\left(\frac{AM}{1-r^2} - \frac{A}{\alpha}\right)$ $(0 \le r \le 1)$. Let*

$$r_0 = \begin{cases} 1 & \gamma \ge 0 \\ \sqrt{1 - \frac{4\alpha AM(\alpha AM - A + \gamma)}{(4A\gamma - \alpha^2)}} & \gamma < 0 \end{cases}$$

Then, for $0 < |z| = r < r_0$, and for a suitable determination of argument

$$|\arg f'(z) + \delta - \phi_\delta(x(r))| \le \sin^{-1}\frac{2AMr}{(1-r^2)C(r)}$$

where $\phi_\delta(x)$ is defined on $[0, x(r_0))$ as above and $C(r)$ is given by (2). The result is sharp.

Proof. To make sure that $f'(z) \ne 0$, we restrict the values of $|z| = r$ by the condition

$$\left|\frac{2AM}{1-r^2} - \left(e^{-i\delta} - \frac{2\gamma}{\alpha}\right)\right| > \frac{2AMr}{1-r^2}$$

Squaring both sides, we have

$$\frac{4A^2M^2}{(1-r^2)} + 1 + \frac{4AM}{(1-r^2)}\left(\frac{2\gamma}{\alpha} - \cos\delta\right) - \frac{4\gamma\left(\cos\delta - \frac{\gamma}{\alpha}\right)}{\alpha} > 0$$

and since $A = \alpha\cos\delta - \gamma$, hence

$$1 + \frac{4AM}{(1-r^2)}\left(AM - \frac{(A-\gamma)}{\alpha}\right) - \frac{4A\gamma}{\alpha^2} > 0$$

The inequality holds for all r in $[0,1)$ if $\gamma \ge 0$ and for $0 \le r > \sqrt{1 - \frac{4\alpha AM(\alpha AM - A + \gamma)}{(4A\gamma - \alpha^2)}}$ if $\gamma < 0$. This establishes the restricted on $|z|$ in the statement of the theorem.

From (1) then with $\Gamma(r)$ given by (3) and $C(r) = |\Gamma(r)|$, we have

$$C(r) = \left(1 + \frac{4A^2M}{1-r^2}\left(\frac{M}{1-r^2} - \frac{1}{\alpha}\right) + \frac{4A(\alpha\cos\delta - A)}{\alpha^2}\left(\frac{M\alpha}{1-r^2} - 1\right)\right)^{\frac{1}{2}}$$

and deduced to $|\arg f'(z) - \arg\Gamma(r)| \le \sin^{-1}\frac{2AMr}{(1-r^2)C(r)}$

also

$$\arg\Gamma(r) = \arg\left(e^{-i\delta}\left(\frac{2AM}{1-r^2} - e^{-i\delta} + \frac{2\gamma}{\alpha}\right)\right)$$
$$= -\delta + \arg\left(e^{-i\delta} + 2\left(\frac{AM}{1-r^2} - \frac{A}{\alpha}\right)\right).$$

Put $x(r)=2\left(\frac{AM}{1-r^2}-\frac{A}{\alpha}\right)$, then $\arg\Gamma(r)=-\delta+\phi_\delta(x(r))$.

We obtain another theorem that replaced $\arg f'(z)$ with restricted range of $|z|$ as $\arg(f'(z)+k)$ for some real number k that satisfied $f'(z)+k\neq 0$ for $z\in D$ and $f\in G_{\alpha,\beta}(\gamma,\delta)$. By taking $|\delta|\neq\pi/2$ as any choice of k with $k\cos\delta+\gamma>0$ will ensure that above conditions are fulfilled and this is important for the following result to be valid. In the following theorem, for a given $\delta\in[-\pi,\pi]$ and as x varies in same interval $[0,c)$, so that $(k+1)e^{i\delta}+x\neq 0$, $\psi_\delta(\delta)$ is the continuous argument of $(k+1)e^{i\delta}+x$ for which $\psi_\delta(0)$ is principal.

Theorem 3.5 *For* $|\delta|\neq\pi/2$, $f(z)\in G_{\alpha,\beta}(\gamma,\delta)$ *and put* $x(r)=2\left(\frac{AM}{1-r^2}-\frac{A}{\alpha}\right)$ $(0\le r\le 1)$. *Let* $k\alpha\cos\delta+\gamma>0$ *where* k *is a real number. Then, for* $\psi_\delta(x)$ *defined on* $[0,\infty)$, $\left|\arg(f'(z)+k)+\delta-\psi_\delta(x(r))\right|\le\sin^{-1}\frac{2AMr}{(1-r^2)C_1(r)}$ *where*
$C_1(r)=\left[4AT\left(k\cos\delta+\frac{A}{\alpha}(T+1)+\frac{\gamma}{\alpha}\right)+(k+1)^2\right]$, $T=\frac{M}{1-r^2}-\frac{1}{\alpha}$

Proof. Let $|\delta|\neq\pi/2$ and k satisfied $k\alpha\cos\delta+\gamma>0$. Using (1), we have

$$\left|(f'(z)+k)-(\Gamma(r)+k)\right|\le\frac{2AMr^2}{1-r^2}$$

where

$$\Gamma(r)=-e^{-i\delta}\left(e^{-i\delta}-\frac{2\gamma}{\alpha}\right)+\frac{2e^{-i\delta}AM}{1-r^2}=1+\frac{2Ae^{-i\delta}\left(M-\frac{1}{\alpha}+\frac{r^2}{\alpha}\right)}{1-r^2}$$

Hence

$$\left|\arg(f'(z)+k)-\arg(\Gamma(r)+k)\right|\le\sin^{-1}\frac{2AMr}{(1-r^2)C_1(r)}\qquad(4)$$

where

$$C_1(r)=|\Gamma(r)+k|=\begin{bmatrix}(k+1)^2+4A^2\left(\frac{M}{1-r^2}-\frac{1}{\alpha}\right)^2+k\cos\delta+\cos\delta\\ +\frac{4A}{1-r^2}\left(\frac{M}{1-r^2}-\frac{1}{\alpha}\right)\left(A\left(\frac{M}{1-r^2}-\frac{1}{\alpha}\right)\right)\end{bmatrix}^{\frac{1}{2}}$$

Let $T=\frac{M}{1-r^2}-\frac{1}{\alpha}$, we have

$$C_1(r)=\left[4AT\left(k\cos\delta+\frac{A}{\alpha}(T+1)+\frac{\gamma}{\alpha}\right)+(k+1)^2\right]^{\frac{1}{2}}$$

Now

$$\begin{aligned}\arg(\Gamma(r)+k)&=\arg\left(e^{-i\delta}\left(\frac{2AM}{1-r^2}-e^{-i\delta}+\frac{2\gamma}{\alpha}+ke^{i\delta}\right)\right)\\&=-\delta+\arg\left((k+1)e^{i\delta}+2A\left(\frac{M}{1-r^2}-\frac{1}{\alpha}\right)\right)\\&=-\delta+\psi_\delta(x(r))\end{aligned}$$

and with (4) this completes the proof.

REFERENCES

[1] S. Owa, T. Hayami, and K. Kuroki, "A note on certain analytic functions," *J. Math. Soc. Japan*, vol. 1538, pp. 74–81, 2007.
[2] C. Y. Gao, and S. Q. Zhou, "Certain subclass of starlike functions," *Applied Math. and Computation*, vol. 187, pp. 176–182, 2007.
[3] H. Silverman, "A class of bounded starlike functions," *Internat. J. Math. & Math. Sci. (2nd Series)*, vol. 17, pp. 249–252, 1994.
[4] P. N. Chichra, "New subclasses of the class of close-to-convex functions," *Proceedings of the American Mathematical Society (First Series)*, vol. 62, pp. 37–43.
[5] D. Mohamad, "On a class of functions whose derivatives map the unit disc into a half plane," *Bull. Malaysian Math. Sc. Soc. (2nd Series)*, vol. 23, pp. 163–171, 2001.
[6] H. Silverman, and E. M. Silvia, "On α-close-to-convex function," *Publ. Math. Debrecen*, vol. 49, pp. 532–537, 1996.
[7] T. H. MacGregor, "Function whose derivative has a positive real part," *Trans. Amer. Math. Soc.*, vol. 104, pp. 532–537, 1962.
[8] N. Mohamed, D. Mohamad, and S. S. Cik, "Some results on generalized class of close-to-convex functions," unpublished.