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A Study of the Change of Damping Coefficient Regarding Minimum Displacement

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***Abstract*—**This research proposes the change of damping coefficient regarding minimum displacement. From the mass with external forced and damper problem, when is the constant external forced transmitted to the understructure in the difference angle between 30 and 60 degrees. This force generates the vibration as general known; however, the objective of this problem is to have minimum displacement. As the angle is changed and the goal is the same; therefore, the damper of the system must be varied while keeping constant spring stiffness. The problem is solved by using nonlinear programming and the suitable changing of the damping coefficient is provided.

***Keywords*—** Damping coefficient, Optimal control, Minimum Displacement and Vibration

I. Introduction

Most of the dynamic systems and advanced mobile machines nowadays are designed so that they are either optimized on their energy consumption or on their greatest smoothness of motion, [3]. Consequently, the trajectory planning and designs of these systems are done exclusively through many approaches such as the minimum energy and minimum jerk, [4]. Nevertheless, in some applications, the system is needed to work very smoothly in order to avoid damaging the specimen that the mechanical system is handling while consuming least amount of energy at the same time. In other words, we may want to minimize the jerk of the movement of such a system as to give it the smoothest motion as well as optimize that system in the energy consumption issue.

The general format of the dynamic problems is consisting of the equation of motion, the initial conditions, and the boundary conditions. The area of interest in this paper will involve the problems with two-point-boundary-value conditions. Each of the problems may contain many possible solutions depending on the objective of application.

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Obviously, the mechanical system that aims to run at lowest cost of energy will be designed to have the lowest actuator inputs during the motion. This is basically the optimization problem of the dynamic systems. Research shows that many of the researchers pay a lot of their attention on the minimization of energy while many tend to seek for the smoothness of the system. Therefore, this research paper aims to consider a spring-mass-damper system with defined external force and to minimize the displacement in the time interval between two given points.

II. Problem Statement

Dynamic systems can be described as the first order derivative function of state as

$$\dot{x}_i = f_i(x_1,...,x_n,u_1,...u_m,t); \quad i = 1,...,n \tag{1}$$

where $x \in R^n$, $u \in R^m$ and t are state, control input, and time respectively, [5]. The problem of interest is to find the states $x(t)$ and control inputs $u(t)$ that make our system operates according to the desired objective of minimum energy or minimum jerk. Note that this paper is focusing on the system with fixed end time and fixed end points. Therefore, states and control inputs that serve the necessary condition must also be able to bring the system from initial conditions $x(t_0)$ at initial time t_0 to the end point $x(t_f)$ at time t_f.

The optimization problem of minimum energy will take the form of

$$J = \int_{t_0}^{t_f} \sum_{i=1}^{m} u_i^2 dt \,, \tag{2}$$

where u_i is the control input, which can be force or torque applied to the system, and $i = 1,...,m$. J is the cost function of the energy consumed by the system from initial time t_0 to end time t_f.

III. Necessary Condition

In this paper, we use the calculus of variations in solving for the extremal solutions of the dynamic system, [1]. Representing the control input with u, the principle of calculus of variations helps us solve the optimization problem by finding the time history of the control input that would minimize the cost function of the form

$$J = \phi(t, x_1, ..., x_n)_{t_f} + \int_{t_0}^{t_f} L(t, x_1, ..., x_n, u_1, ..., u_m)\, dt \quad (3)$$

where

$$\phi(t, x_1, ..., x_n)_{t_f}\,, \quad (4)$$

is the cost based on the final time and the final states of the system, and

$$\int_{t_i}^{t_f} L(t, x_1, ..., x_n, u_1, ..., u_m)\, dt\,, \quad (5)$$

is an integral cost dependent on the time history of the state and control variables. Since the cost of the final states would be equal in all feasible time histories of the control input; therefore, the first term of (3) is omitted.

To find the extremum of the function, the dynamic equations are augmented via Lagrange Multipliers to the cost functional as follow:

$$J'(x_1, ..., x_n, u_1, ..., u_m) = \int_{t_i}^{t_f} L'(t, x_1, ..., x_n, u_1, ..., u_m) dt. \quad (6)$$

Where

$$L'(t, x_1, ..., x_n, u_1, ..., u_m) = L + \sum_{i=1}^{n} \lambda_i (f_i)\,, \quad (7)$$

and $\lambda_i(t)$ are Lagrange multipliers. Consequently, (6) becomes:

$$J'(x_1, ..., x_n, u_1, ..., u_m) = \int_{t_i}^{t_f} [L(t, x_1, ..., x_n, u_1, ..., u_m) + \sum_{i=1}^{n} \lambda_i(t)[\dot{x}_i - f_i(t, x_1, ..., x_n, u_1, ..., u_m)]]\, dt \quad (8)$$

Since the problem with fixed end time and end points are considered, initial time t_0, end time t_f, initial state $x(t_0)$, and final state $x(t_f)$ must be set prior to solving the problem. The differentiable functions are dependent on the boundary condition of $x(t_0) = x_0$, $x(t_f) = x_f$, $u(t_0) = u_0$ and $u(t_f) = u_f$ where time used falls in the interval $t_i \le t \le t_f$.

Let function $L(t, x_1, ..., x_n, u_1, ..., u_m, \dot{x}_1, ..., \dot{x}_n)$ be represented as a functional

$$J[x_1, ..., x_n, u_1, ..., u_m] = \int_{t_0}^{t_f} L(t, x_1, ..., x_n, u_1, ..., u_m, \dot{x}_1, ..., \dot{x}_n) dt \quad (9)$$

Let $x(t_0)$ be incremented by $h_{xj}(t_0)$, $u(t_0)$ be incremented by $h_{uk}(t_0)$, and still satisfy the boundary conditions, then $h_{xj}(t_0) = h_{xj}(t_f) = h_{uk}(t_0) = h_{uk}(t_f) = 0$. So, the change in functional ΔJ will be

$$\Delta J = J[x_1 + h_{x1}, ..., x_n + h_{xj}, u_1 + h_{u1}, ..., u_m + h_{uk}] - J[x_1, ..., x_n, u_1, ..., u_m]$$

$$= \int_{t_i}^{t_f} \left[L\begin{pmatrix} t, x_1 + h_{x1}, ..., x_n + h_{xj}, \dot{x}_1 + \dot{h}_{x1}, ..., \dot{x}_n + \dot{h}_{xj}, \\ u_1 + h_{u1}, ..., u_m + h_{uk}, \dot{u}_1 + \dot{h}_{u1}, ..., \dot{u}_m + \dot{h}_{uk} \end{pmatrix} - L(t, x_1, ..., x_n, \dot{x}_1, ..., \dot{x}_n, u_1, ..., u_m, \dot{u}_1, ..., \dot{u}_m) \right] dt \quad (10)$$

Applying Taylor's Series to (10), disregard the higher order terms, and apply it to the problem results in

$$\begin{aligned} \delta J' = & \int_{t_i}^{t_f} \sum_{j=1}^{n} \left(\frac{\partial L'}{\partial x_j} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{x}_j}\right) h_{x_j}\, dt \\ & + \int_{t_i}^{t_f} \sum_{k=1}^{m} \left(\frac{\partial L'}{\partial u_k} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{u}_k}\right) h_{u_k}\, dt \\ & + \sum_{k=1}^{m} \left(\frac{\partial L'}{\partial \dot{u}_k} h_{u_k}|_{t_f} - \frac{\partial L'}{\partial \dot{u}_k} h_{u_k}|_{t_i}\right) \\ & + \sum_{j=1}^{m} \left(\frac{\partial L'}{\partial \dot{x}_j} h_{x_j}|_{t_f} - \frac{\partial L'}{\partial \dot{x}_j} h_{x_j}|_{t_i}\right) \end{aligned} \quad (11)$$

Since $h_{x_j}|_{t_f} = h_{x_j}|_{t_i} = 0$ and $\frac{\partial L'}{\partial \dot{u}_k} = 0$, the last two terms of (11) become zero. In order that the cost functional of jerk in (8) can be solved for minimal solution, the condition that make $\delta J' = 0$ at arbitrary variation of h_{x_j} and h_{u_k} are needed. From (11), obviously the mentioned conditions are as follow:

$$\frac{\partial L'}{\partial x_j} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{x}_j} = 0, \quad (12)$$

and

$$\frac{\partial L'}{\partial u_k} - \frac{d}{dt}\frac{\partial L'}{\partial \dot{u}_k} = 0, \quad (13)$$

for $j = 1, \ldots, n$ and $k = 1, \ldots, m$.

Equations (12) and (13) are the necessary conditions that will lead to solve for Lagrange multipliers $\lambda_j(t)$, and control inputs $u_k(t)$. Alternatively, we can use the derived relationship below to solve for the unknowns necessary conditions:
For

$$\dot{x}_i = f_i(x_1, ..., x_n, u_1, ..u_m, t), \quad i = 1, ..., n \quad (14)$$

Necessary conditions are (14) and

$$\dot{\lambda}_j = -\frac{\partial L}{\partial x_j} - \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial x_j}, \quad j = 1, ..., n\,, \quad (15)$$

$$\frac{\partial L}{\partial u_k} + \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial u_k} = 0, \quad k = 1, ..., m\,. \quad (16)$$

As of above the necessary conditions are in the form of differential and algebraic equations which are known as two-point boundary valued problem, [2].

IV. Example

An analysis of vibration system in general must include effects of friction or damping to account for its true motion, otherwise, the solutions obtained with negligence to these effects would only describe the system's motion.

Fig. 1 shows a sample spring-mass-damper system with external forced and the system is assumed to move only along the vertical direction and supported by one spring and one damper.

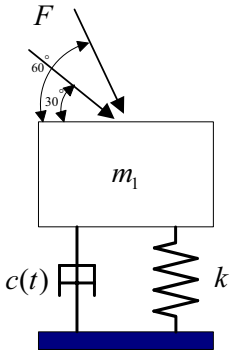

Fig. 1 Damped System with external forced

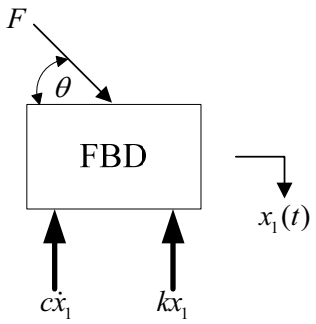

Fig. 2 Free Body Diagram

Assume system weights 20,000 kg is supported by one spring and one damper. The constant forced is 500 kg. The spring stiffness (k) is 2,000 $\mathrm{kg/m}$.

From the second law of Newton, it can be rewritten as

$$\sum F = m_1\ddot{x}_1 \tag{17}$$

Let

$$F\sin\theta = U_1 \tag{18}$$

therefore,

$$U_1 - c\dot{x}_1 - kx_1 = m\ddot{x}_1 \tag{19}$$

and an equation of motion becomes:

$$m\ddot{x}_1 + c\dot{x}_1 + kx_1 = U_1 \tag{20}$$

To have minimum displacement, the displacement square is selected as an integrand in order to assure about global minimum. The problem statement is to find damping coefficient as a function of time during $t_0 = 0$ to $t_f = 0.1$ such that

$$\min J = \int_{t=0}^{t=0.1} x_1^2 dt \tag{21}$$

subject to

$$20,000\ddot{x}_1 + c(t)\dot{x}_1 + 2,000x_1 = U_1 \tag{22}$$

with given two – point boundary values as

$$\begin{aligned} x_{1(t=0)} &= 0 \quad x_{1(t=0.1)} = 0 \\ \dot{x}_{1(t=0)} &= 0 \quad \dot{x}_{1(t=0.1)} = 0 \end{aligned} \tag{23}$$

The initial and terminal condition on function x(t) are necessarily to pin down the solution and we are looking for the general class of functions which will fulfill the transition equation (or Euler equation).

Minimum principle is used in optimal control theory in order to find the best possible control for taking a dynamic system from one state to another. By applying minimum principle, then:

$$L = x_1^2 + \lambda_1(20,000\ddot{x}_1 + c(t)\dot{x}_1 + 2,000x_1 - U_1) \tag{24}$$

Consider an infinitely-lived agent choosing a control variable from Eq. (24). The Euler equation from conventional from

$$\frac{\partial L}{\partial x_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{x}_1} = 0 \tag{25}$$

becomes

$$2x_1 + 2,000\lambda_1 - \dot{c}\lambda_1 - c\dot{\lambda}_1 + 20,000\ddot{\lambda}_1 = 0 \tag{26}$$

Therefore, the necessary conditions can be derived as

$$\frac{\partial L}{\partial c} = 0\,;\, \lambda_{1(t)}\dot{x}_1 = 0\,. \tag{27}$$

Eqs. (22), (23), (26) and (27) are solved together as Two-Point-Boundary-Value problem by transforming them to parameter optimization problem (nonlinear programming problem) [6]. The solutions from varying the angle of force that applied to the system as 30, 35, 40, 45, 50, 55 and 60 degrees are shown in Fig. 3-9, respectively.

From the solution, it can be seen quite obvious that displacements of each problem are quite small in magnitude and the damping coefficient increases versus time due to that minimum displacement since the system has only damper to reduce the vibration. This solution can be applied to the system with mechanical design on how to control the damping coefficient versus time.

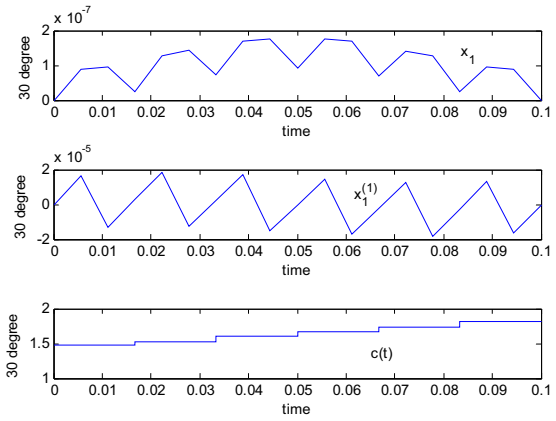

Fig. 3 30 degrees solution for displacement, rate of displacement and damping coefficient with respect to time interval from 0 to 0.1 second

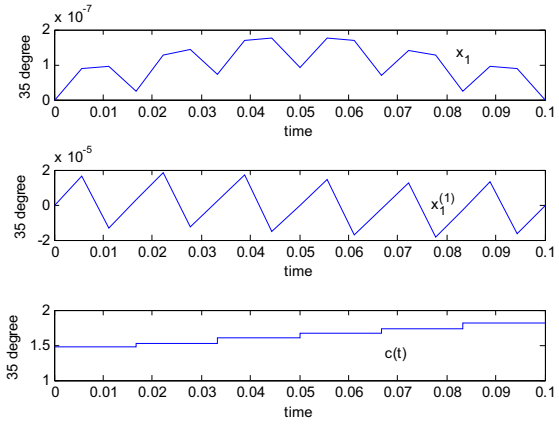

Fig. 4 35 degrees solution for displacement, rate of displacement and damping coefficient with respect to time interval from 0 to 0.1 second

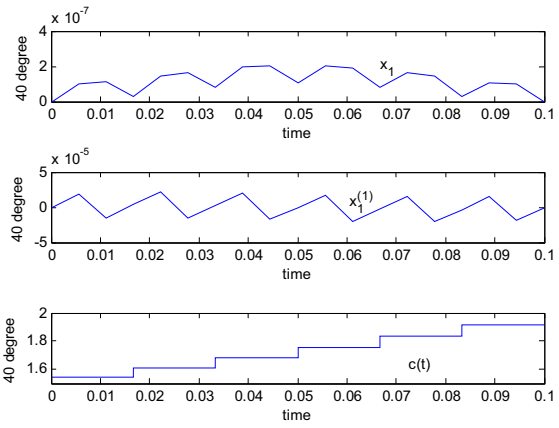

Fig. 5 40 degrees solution for displacement, rate of displacement and damping coefficient with respect to time interval from 0 to 0.1 second

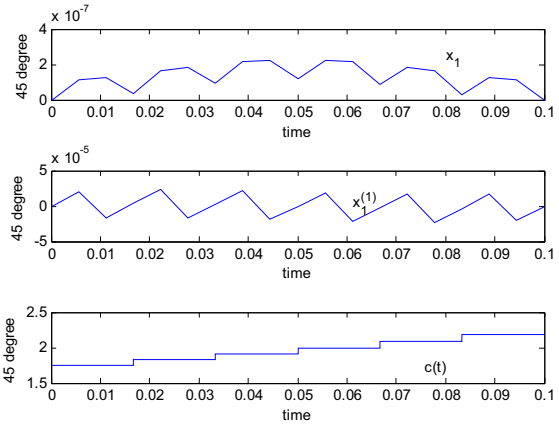

Fig. 6 45 degrees solution for displacement, rate of displacement and damping coefficient with respect to time interval from 0 to 0.1 second

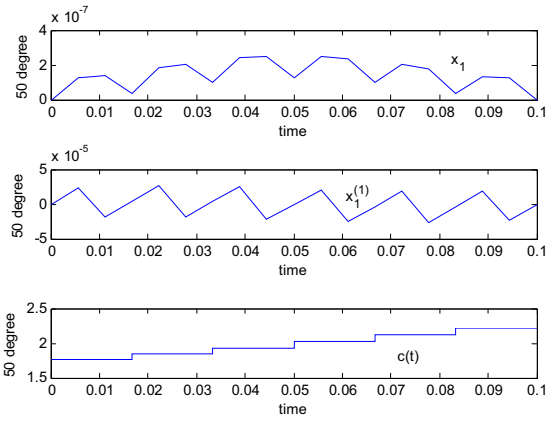

Fig. 7 50 degrees solution for displacement, rate of displacement and damping coefficient with respect to time interval from 0 to 0.1 second

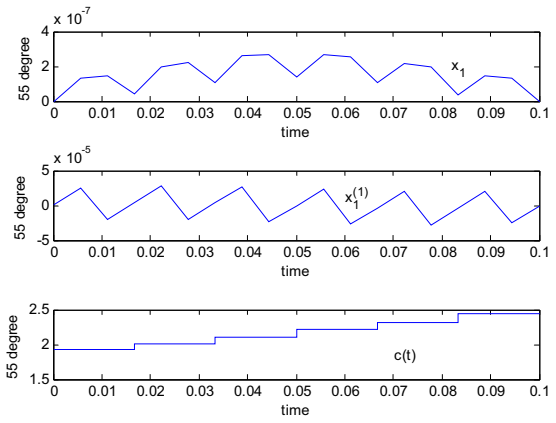

Fig. 8 55 degrees solution for displacement, rate of displacement and damping coefficient with respect to time interval from 0 to 0.1 second

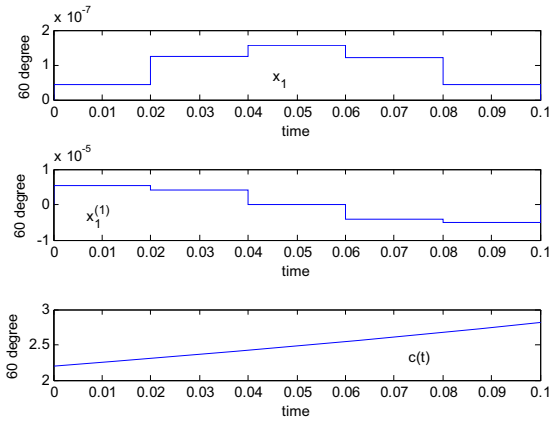

Fig. 9 60 degrees solution for displacement, rate of displacement and damping coefficient with respect to time interval from 0 to 0.1 second

V. Conclusion

A conclusion can be made here that the idea of finding damping coefficient as a function of time in order to minimize the displacement of a dynamic system can be applied to such a system such as automobile, artillery cannon, etc. A good example is artillery cannon that might have to fire shells as many in a period of time. During that short time interval between the launched and launching shells, the displacement or system vibration must be vanished in time to make sure that the next shell will stay on the target. However, as state above that controlling this damping coefficient versus time is open problem to the future work.

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