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How are Equalities Defined, Strong or Weak on a Multiple Algebra ?

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***Abstract*—**For the purpose of finding the quotient structure of multiple algebras such as groups, Abelian groups and rings, we will state concepts of (strong or weak) equalities on multiple algebras, which will lead us to research on how (strong or weak) are equalities defined on a multiple algebra over the quotients obtained from it.

In order to find a quotient structure of multiple algebras such as groups, Abelian groups and loops, a part of this article has been allocated to the concepts of equalities (strong and weak) of the defined multiple functions on multiple algebras. This leads us to do research on how defined equalities (strong and weak) are made in the multiple algebra on its resulted quotient.

***Keywords*—**multiple algebra, mathematics , universal algebra

I. Introduction

OF other sections of mathematics that are related to super structures, one can mention ordered sets, binary relations, fuzzy sets, etc. Our view point on the relation between multiple algebras and other fields of study has been developed by an article of Corsini and Leoreanu [1].

Quasi- super groups, semi- super groups and super groups were the first multiple algebras to be studied. Gradually, other super structures including super rings, super modules and super lattices appeared. In this respect, the super lattices of the Romanian mathematician, Mihail Bonado, have played an important role. Gratzer and Pickett have provided important articles on the theory of multiple algebras.

In those articles, multiple algebras are seen as relational systems that are an extension of the theory of the study provided by Gratzer and Pickett[2]-[4].

II. Equalities on Multiple Algebras, Complete Multiple Algebras

in what follows $\mathcal{A}=\left(A,\left(f_r\right)_{r<o(\tau)}\right)$ is a multiple algebra, for every $a\in A$, we will denote the equivalence class $a^*<a>$ by the notation ā.

Let $\mathcal{A}$ be a multiple algebra of type τ and $\mathbf{r},\mathbf{q}\in\mathbf{P}^n(\tau)$ be two n- nomials of type τ . The strong equality $\mathbf{q}=\mathbf{r}$ is satisfied on the multiple algebra $\mathcal{A}$ if for every $a_\circ,\ldots,a_{n-1}\in A.$

$$q\left(a_\circ,\ldots,a_{n-1}\right)=r\left(a_\circ,\ldots,a_{n-1}\right)$$

Where $q=(\mathbf{q})_{\wp^*(\mathcal{A})}$ and $=(\mathbf{r})_{\wp^*(\mathcal{A})}$. the weak quality $\mathbf{q}\cap\mathbf{r}\neq\varnothing$ is satisfied on the multiple algebra $\mathcal{A}$ if for every $a_\circ,\ldots,a_{n_r-1}\in A.$

$$q\left(a_\circ,\ldots,a_{n-1}\right)\cap r\left(a_\circ,\ldots,a_{n-1}\right)\neq\varnothing.$$

Let K be a set of multiple algebras of type τ . the strong equality $\mathbf{q}=\mathbf{r}$ is satisfied on K , if for every $\mathcal{A}\in K$ and $a_\circ,\ldots,a_{n-1}\in A.$

$$q\left(a_\circ,\ldots,a_{n-1}\right)=r\left(a_\circ,\ldots,a_{n-1}\right)$$

Such that $q=(\mathbf{q})_{\wp^*(\mathcal{A})}$ and $r=(\mathbf{r})_{\wp^*(\mathcal{A})}$. also, the weak equality $\mathbf{q}\cap\mathbf{r}\neq\varnothing$ is satisfied on K, if for every $\mathcal{A}\in K$ and every $a_\circ,\ldots,a_{n-1}\in A.$

$$q\left(a_\circ,\ldots,a_{n-1}\right)\cap r\left(a_\circ,\ldots,a_{n-1}\right)\neq\varnothing.$$

Special multiple algebras are defined throuth the two concepts of strong and weak equalities of n- nomial functions on multiple algebras.

As said above, one of the issues suggested by Gratzer in [2] was studying the quotient structure of multiple algebras such as groups, Abelian groups, rings , as well as investigating about how the equalities satisfied on the algebra $\mathcal{B}$ are also satisfied on the multiple algebra $\mathcal{B}/\rho$. The following remark gives a general response to the issues raised above.

Remark 1

Let ß a universal algebra, $n\in\mathbb{N}$ and $\mathbf{q},\mathbf{r}\in\mathbf{P}^{(n)}(\tau)$. also, let ρ be an equivalence relation on B . if the strong equality $\mathbf{q}=\mathbf{r}$ is satisfied on $\mathcal{B}$, then for every $b_\circ,\ldots,b_{n-1}\in B$. the equivalence class $q\left(b_\circ,\ldots,b_{n-1}\right)\cap r\left(b_\circ,\ldots,b_{n-1}\right)$ module ρ lies in

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$q(\rho < b_\circ >,\ldots,\rho < b_{n-1} >) \cap r(\rho < b_\circ >,\ldots,\rho < b_{n-1} >)$

Therefore, the weak equality $\mathbf{q} \cap \mathbf{r} \neq \varnothing$ is satisfied on the multiple algebra $\mathcal{B} / \rho$.

Also, one may encounter strong equalities on the universal algebra $\mathcal{B}$ that are also strongly satisfied on the quotient multiple algebra $\mathcal{B} / \rho$ (ρ is an arbitrary equivalence relation). For instance, one can mention the equalities relating to the commutation property of operations defines on ß . for example. If $(G,.)$ is a commutative group , then for every $a,b \in G$.

$$\rho < a > .\rho < b > = \left\{ \rho < c > | c = a^{'}.b^{'}, a^{'} \rho a, b^{'} \rho b \right\}$$

$$= \left\{ \rho < c > | c = b^{'}.a^{'}, b^{'} \rho b, a^{'} \rho a \right\}$$

$$= \rho < a > .\rho < b >$$

III. Studying kinds of quotient

a quotient of semi-groups

let $(S,.)$ be a semi-group and ρ be an equivalence relation on S. by remark 1, the association property is weakly satisfied on the super groupoid (S/ ρ , .). Therefore (S/ ρ , .) is a H_v semi- group.

a quotient of groups

let $(G,.)$ be a group and ρ be an equivalence relation on G . existence and uniqueness of solutions to equations $ya = b$ and $ax = b$ permit us to define the operations / and \ on G as follows:

$$b / a = \{ y \in G | b = ya \}, a \backslash b = \{ x \in G | b = ax \}.$$

therefore, the group G can be considers as the universal algebra $(G,.,/,\backslash)$ on which the following equalities are satisfied.

$$(x_\circ.x_1).x_2 = x_\circ(x_1.x_2), x_1 = x_\circ.(x_\circ \backslash x_1), x_1 = (x_1 / x_\circ).x_\circ$$

$$x_1 = x_\circ \backslash (x_\circ.x_1), x_1 = (x_1.x_\circ) / x_\circ$$

G / ρ along with multiple operations, . ,\,/ constitutes a multiple algebra such that the above equalities are weakly satisfied on it. On the other hand , since $(G,.)$ is a group , therefore,

$$\rho < a > .G / \rho = \bigcup_{b \in G} \rho < a > .\rho < b$$

$$= \bigcup \left\{ \rho < c > | c = a^{'}.b, a^{'} \rho a, b \in G \right\}$$

$$= \bigcup \{ \rho < c > | c \in a.G \} = G / \rho$$

Similarly, one can show that $G / \rho.\rho < a > = G / \rho$. therefore, (G/ ρ , .) is a $-H_v$ group. Generally, $-H_v$ group have no identity element. In this case, the element $\rho < 1 >$, where 1 is the identity element of group G , satisfies the following condition.

$$\rho < a > \in \rho < a > .\rho < 1 > \cap \rho < 1 > .\rho < a > \quad \forall a \in A$$

Therefore, the following weak equalities are satisfied on the $-H_v$ group (G / ρ , .).

$$x_\circ.1 \cap x_\circ \neq \varnothing, 1.x_\circ \cap x_\circ \neq \varnothing$$

$\rho < 1 >$ is called an identity element of the H_v-group (G/ ρ ,.). moreover, for every $a \in G$,

$$\rho < 1 > \in \rho < a^{-1} > .\rho < a > \cap \rho < a > .\rho < a^{-1} > .$$

$\rho < a^{-1} >$ is called the inverse of the element $\rho < a >$ in the H_v-group (G/ ρ ,.) (a^{-1} is the inverse of the element a in the group). If (G ,.) is a commutative algebra, then the - H_v-group (G / ρ ,.) is also commutative.

a quotient of rings

the super structure $(R,+,.)$ is called a $-H_v$-ring, if $(R,+)$ is a H_v-group and $(R,.)$ is a H_v-semi group and for every $a,b,c \in R$.

$$a.(b+c) \cap (a.b + a.c) \neq \varnothing, (b+c).a \cap (b.a + c.a) \neq \varnothing$$

according to the two cases stated, it is seen easily that the algebra derived from a quotient of a ring is a H_v-ring. Considering the congruence and strongly regular relations , these relations are equivalent on universal algebras. Sometimes, the equalities present on the quotient algebra $\mathcal{B} / \rho$ that are induced from the algebra ß are dependent on the equivalence relation ρ , too.

Corollary 1

Let $\mathcal{A}$ be a multiple algebra and $\mathbf{q},\mathbf{r} \in \mathbf{P}^{(n)}(\tau)(n \in \mathbb{N})$ if the strong equality $\mathbf{q} = \mathbf{r}$ is satisfied on $\mathcal{A}$, then the strong equality $\mathbf{q} = \mathbf{r}$ is also satisfied on $\bar{\mathcal{A}}$.

Corollary 2

Let K be a class of multiple algebras. We consider $\bar{K}$ as the set of all basic algebras corresponding to the elements of K . if the weak equality $\mathbf{q} \cap \mathbf{r} \neq \varnothing$ is satisfied on K , then the strong equality $\mathbf{q} = \mathbf{r}$ is satisfied on $\bar{K}$.

One good example is when the basic algebra has a single element, therefore, every two n-ary polynomial function on the basic algebra are equal while they are not necessarily equal on the multiple algebra dependent on this basic algebra in the strong or weak case. Here, one can find multiple algebras with the property that the equalities existing on their basic algebra are satisfied weakly or strongly on these multiple algebras as well.

IV. EQUALITIES AND QUOTIENT MULTIPLE ALGEBRAS

Let $\mathcal{A}$ be a multiple algebras, and for $\rho \in E_{ua}(\mathcal{A})$ every weak or strong equality on $\mathcal{A}$, is satisfied as a strong equality on $\mathcal{A}/\rho$, now by definition of the relation $R_{qr}\left(\mathbf{q},\mathbf{r} \in \mathbf{P}^{(n)}(\tau), n \in \mathbb{N}\right)$ we examine the equalities and quotient multiple algebras that clame this issue.

For every $\mathbf{q},\mathbf{r} \in \mathbf{P}^{(n)}(\tau)(n \in \mathbb{N})$ We define the relation R_{qr} as follows

$$R_{qr} = \{(x,y) \in A \times A | x \in q(a_\circ,\ldots,a_{n-1}), y \in r(a_\circ,\ldots,a_{n-1}), a_\circ,\ldots,a_{n-1} \in A\}.$$

.

Suppose that there is a weak equality of $\left(\mathbf{q},\mathbf{r} \in \mathbf{P}^{(n)}(\tau)\right)\mathbf{q} \cap \mathbf{r} \neq \varnothing$ on the multiple algebra of A. Therefore for every member like A, there is $a \in A$, such that

$$a \in q(a_\circ,\ldots,a_{n-1}) \cap r(a_\circ,\ldots,a_{n-1}).$$

On the other hand there are the members of $a_\circ,\ldots,a_{n-1} \in A$ for each $(x,y) \in R_{qr}$ such that So there is a member like $a \in A$ for every $(x,y) \in R_{qr}$.

$$x\alpha^* a, y\alpha^* a$$

So that; $x\alpha^* y$ and therefore $R_{qr} \subseteq \alpha^*$. Therefore for each for each $\bar{y} \in r\left(\bar{a}_\circ,\ldots,\bar{a}_{n-1}\right)$ and $\bar{x} \in q\left(\bar{a}_\circ,\ldots,\bar{a}_{n-1}\right)$ $xa^* y$ means $\bar{x} = \bar{y}$. So, there is a strong equity $\mathbf{q} = \mathbf{r}$ on $\bar{\mathcal{A}}$.

We can make, without making weak equity of $\left(\mathbf{q},\mathbf{r} \in \mathbf{P}^{(n)}(\tau)\right)\mathbf{q} \cap \mathbf{r} \neq \varnothing$ on a desired multiple algebra such as $\mathcal{A}$, a quotient of it which is a global algebra and it has a strong equity of $\mathbf{q} = \mathbf{r}$ on it.

Lemma 1

Suppose that $\mathcal{A}$ is a multiple algebra. For each relation $\rho \in E_{ua}(\mathcal{A})$, there is a strong equity of $\mathbf{q} = \mathbf{r}$ on $\mathcal{A}/\rho$ if and only if $R_{qr} \subseteq \rho$.

Proof

Suppose that $\rho \in E_{ua}(\mathcal{A})$ and $R_{qr} \subseteq \rho$, for each $a_\circ,\ldots,a_{n-1} \in A$

$$q(\rho < a_\circ >,\ldots,\rho < a_{n-1} >) = \rho < a >, a \in q(a_\circ,\ldots,a_{n-1})$$

and

$$r(\rho < a_\circ >,\ldots,\rho < a_{n-1} >) = \rho < b >, b \in r(a_\circ,\ldots,a_{n-1})$$

Also for each $a \in q(a_\circ,\ldots,a_{n-1})$ and $(a,b) \in \rho, b \in r(a_\circ,\ldots,a_{n-1})$,therefore $\rho < a >= \rho < b >$. As the result, for each $a_\circ,\ldots,a_{n-1} \in A$

$$q(\rho < a_\circ >,\ldots,\rho < a_{n-1} >) = r(\rho < a_\circ >,\ldots,\rho < a_{n-1} >)$$

Conversely suppose that $\rho \in E_{ua}(\mathcal{A})$ and there is a equity of $\left(\mathbf{q},\mathbf{r} \in \mathbf{P}^{(n)}(\tau)\right)\mathbf{q} = \mathbf{r}$ on $\mathcal{A}/\rho$. If $(x,y) \in R_{qr}$, then there are members of $a_\circ,\ldots,a_{n-1} \in A$ that

$$x \in q(a_\circ,\ldots,a_{n-1}), y \in r(a_\circ,\ldots,a_{n-1}).$$

on the other hand for each $b_\circ,\ldots,b_{n-1} \in A$,

$$\begin{aligned} \rho < q(b_\circ,\ldots,b_{n-1}) >&= q(\rho < b_\circ > \rho < b_{n-1} >) \\ &= r(\rho < b_\circ > \rho < b_{n-1} >) \\ &= \rho < r(b_\circ,\ldots,b_{n-1}) >. \end{aligned}$$

therefore, $\rho < x >= \rho < y >$, which resulted in $x\rho y$.

Corollary 3

Suppose that $\mathcal{A}$ is a multiple algebra and $\left(n \in \mathbb{N}, \mathbf{q},\mathbf{r} \in \mathbf{P}^{(n)}(\tau)\right)$. If the weak equity of $\mathbf{q} \cap \mathbf{r} \neq \varnothing$ is being established on $\mathcal{A}$, then for each $\rho \in E_{ua}(\mathcal{A})$, there is a strong equity of $\mathbf{q} = \mathbf{r}$ on $\mathcal{A}/\rho$.

Proof

For each $\rho \in E_{ua}(\mathcal{A}), R_{qr} \subseteq \alpha^* \subseteq \rho$

Considering the definition of relation $\alpha(R)$ (R is a desired relation on A), the minimal equivalence relation of $E_{ua}(\mathcal{A})$ on the resulted quotient of which there is a strong equity of $\mathbf{q} = \mathbf{r}$, is $\alpha(R_{qr})$ relation and is shown by α^*_{qr}

Remark 2

By the definition of the relation $\alpha(R)$ (R is an arbitrary relation on A) and α^*_{qr}.

$$\alpha^*_{\mathcal{A}} = \alpha(\varnothing) = \alpha(\delta_A) = \alpha^*_{X_\circ} X_\circ.$$

Also, for every $\mathbf{q},\mathbf{r} \in \mathbf{P}^{(n)}(\tau)\alpha^*_{\mathcal{A}} \subseteq \alpha^*_{qr}$.

Proof:

On the other hand, $\alpha\left(\delta_A\right)=\left\{\bigcap \rho \mid \rho \in E_{ua}\left(\mathcal{A}\right), \delta_A \in \rho\right\}$ for every, $\left(x, y\right) \in \delta_A, x=y$. Thus, for every $\rho \in E_{ua}\left(\mathcal{A}\right), \delta_A \subseteq \rho$. As a result, $\alpha\left(\delta_A\right)=\alpha^*$
Also,

$$R_{X_\circ X_\circ}=\left\{\left(x, y\right) \mid x=y\right\}=\delta_A \subseteq \alpha_{\mathcal{A}}^*,$$

Therefore, $\alpha^*{}_{X_\circ X_\circ}=\alpha\left(R_{X_\circ X_\circ}\right)=\alpha^*$

Remark 3

By the structure of polynomial functions belonging to the set $P_A^{(1)}\left(\wp^*\left(\mathcal{A}\right)\right)$ the ordinary composition of any two elements of $P_A^{(1)}\left(\wp^*\left(\mathcal{A}\right)\right)$ Is an element of $P_A^{(1)}\left(\wp^*\left(\mathcal{A}\right)\right)$, for every $f, p \in P_A^{(1)}\left(\wp^*\left(\mathcal{A}\right)\right)$,

$fop: p^*\left(A\right) \rightarrow p^*\left(A\right)$. and

1- if $a \in A$, $f=c_a^1$ and $X \in P^*\left(A\right)$ (arbitrary), then

$$fop\left(X\right)=c_a^1\left(p\left(X\right)\right)=a=c_a^1\left(X\right).$$

Therefore, $fop=c_a^1 \in P_A^{(1)}\left(\wp^*\left(\mathcal{A}\right)\right)$.

2- if $f=e_\circ^1$ and $X \in P^*\left(A\right)$ (arbitrary), then

$$fop\left(X\right)=e_\circ^1\left(p\left(X\right)\right)=p\left(X\right).$$

Therefore $fop=p$.

3- if $r<o\left(\tau\right)$ and for every $f^\circ, \ldots, f^{n_r-1} \in P_A^{(1)}\left(\wp^*\left(\mathcal{A}\right)\right)$

Then for $f=f_r\left(f^\circ, \ldots, f^{n_r-1}\right)$

$$fop\left(X\right)=f\left(p\left(X\right)\right)=f_r\left(f^\circ, \ldots, f^{n_r-1}\right)\left(P\left(X\right)\right)$$

$$=f_r\left(f^\circ\left(P\left(X\right)\right), \ldots, f^{n_r-1}\left(P\left(X\right)\right)\right)$$

$$=f_r\left(P_\circ\left(X\right), \ldots, p_{n_r-1}\left(X\right)\right)$$

$$=f_r\left(P_\circ, \ldots, p_{n_r-1}\right)\left(X\right),$$

Therefore,

$$fop=f_r\left(P_\circ, \ldots, p_{n_r-1}\right) \in P_A^{(1)}\left(\wp^*\left(\mathcal{A}\right)\right)$$

Thus, one can give a definition of the relation α_{qr}^*, based on 1- tuple polynomial functions of $P_A^{(1)}\left(\wp^*\left(\mathcal{A}\right)\right)$.

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