

Laplace Transformation on Ordered Linear Space of Generalized Functions

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Abstract—Aim. We have introduced the notion of order to multi-normed spaces and countable union spaces and their duals. The topology of bounded convergence is assigned to the dual spaces. The aim of this paper is to develop the theory of ordered topological linear spaces $\mathcal{L}'_{a,b}$, $\mathcal{L}'(w, z)$, the dual spaces of ordered multi-normed spaces $\mathcal{L}_{a,b}$, ordered countable union spaces $\mathcal{L}(w, z)$, with the topology of bounded convergence assigned to the dual spaces. We apply Laplace transformation to the ordered linear space of Laplace transformable generalized functions. We ultimately aim at finding solutions to non-homogeneous n th order linear differential equations with constant coefficients in terms of generalized functions and comparing different solutions evolved out of different initial conditions.

Method. The above aim is achieved by

- Defining the spaces $\mathcal{L}_{a,b}$, $\mathcal{L}(w, z)$.
- Assigning an order relation on these spaces by identifying a positive cone on them and studying the properties of the cone.
- Defining an order relation on the dual spaces $\mathcal{L}'_{a,b}$, $\mathcal{L}'(w, z)$ of $\mathcal{L}_{a,b}$, $\mathcal{L}(w, z)$ and assigning a topology to these dual spaces which makes the order dual and the topological dual the same.
- Defining the adjoint of a continuous map on these spaces and studying its behaviour when the topology of bounded convergence is assigned to the dual spaces.
- Applying the two-sided Laplace Transformation on the ordered linear space of generalized functions W and studying some properties of the transformation which are used in solving differential equations.

Result. The above techniques are applied to solve non-homogeneous n -th order linear differential equations with constant coefficients in terms of generalized functions and to compare different solutions of the differential equation.

Keywords—Laplace transformable generalized function, positive cone, topology of bounded convergence.

I. THE SPACES $\mathcal{L}_{a,b}$, $\mathcal{L}(w, z)$ AND THEIR DUALS

We have associated the notion of 'order' to multi-normed spaces, countable union spaces (see [3], for details of multi-normed spaces, countable union spaces) and their duals, by identifying a positive cone on these spaces. Also, the topology of bounded convergence is assigned to the dual spaces so that the order dual and the topological dual become identical [1].

In this section we define the spaces $\mathcal{L}_{a,b}$, $\mathcal{L}(w, z)$ and apply the above ideas to these spaces and to their duals.

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Let $\mathcal{L}_{a,b}$ denote the linear space of all complex valued smooth functions defined on \mathbb{R} . Let

$$\begin{aligned} L_{a,b}(t) &= e^{at}, & 0 \leq t < \infty \\ &= e^{bt}, & -\infty < t < 0. \end{aligned}$$

(K_m) be a sequence of compact subsets of \mathbb{R} such that $K_1 \subseteq K_2 \subseteq \dots$ and such that each compact subset of \mathbb{R} is contained in one K_m . Define

$$\gamma_{K_m, k}(\phi) = \sup_{t \in K_m} |L_{a,b}(t) D^k \phi(t)|, \quad k = 0, 1, 2, \dots$$

$\{\gamma_{K_m, k}\}_{k=0}^{\infty}$ is a multinorm on \mathcal{L}_{a,b, K_m} where \mathcal{L}_{a,b, K_m} is the subspace of $\mathcal{L}_{a,b}$ whose elements have their support in K_m . The above multinorm generates the topology τ_{a,b, K_m} on \mathcal{L}_{a,b, K_m} . $\mathcal{L}_{a,b}$ is equipped with the inductive limit topology $\tau_{a,b}$ as K_m varies over all compact sets K_1, K_2, \dots . $\mathcal{L}_{a,b}$ is complete for $\tau_{a,b}$. For each fixed s , $e^{-st} \in \mathcal{L}_{a,b}$ if and only if $a \leq \operatorname{Re} s \leq b$. For each positive integer k , $t^k e^{-st} \in \mathcal{L}_{a,b}$ if and only if $a < \operatorname{Re} s < b$.

We recall the notions of a positive cone, normal cone and strict b -cone that have been defined in [1].

Definition 1: Let V be a multi-normed space whose field of scalars is \mathbb{R} . A subset C or $C(V)$ is a positive cone in V if

- $C + C \subseteq C$
- $\alpha C \subseteq C$ for all scalars $\alpha > 0$
- $C \cap (-C) = \{0\}$

When the field of scalars is \mathbb{C} , $C + iC$ is the positive cone in V which is also denoted as C . C defines an order relation on V , $\phi \leq \psi$ if $\psi - \phi \in C$.

Definition 2: Let $V(\tau)$ be an ordered multi-normed space with positive cone C . C is normal for the topology τ generated by the multinorm S if there is a neighbourhood basis of 0 for τ consisting of full sets.

Definition 3: Let \mathcal{G} be a saturated class of τ -bounded subsets of an ordered multi-normed space $V(\tau)$ such that $V = \cup\{s : s \in \mathcal{G}\}$. The positive cone C in $V(\tau)$ is a strict \mathcal{G} -cone if the class $\mathcal{G}_\epsilon = \{(S \cap C) - (S \cap C) : S \in \mathcal{G}\}$ is a fundamental system for \mathcal{G} . A strict \mathcal{G} -cone for the class \mathcal{G} of all τ -bounded sets in $V(\tau)$ is called a strict b -cone.

Definition 4: The positive cone C of $\mathcal{L}_{a,b}$ when $\mathcal{L}_{a,b}$ is restricted to real-valued functions is the set of all non-negative functions in $\mathcal{L}_{a,b}$. Then $C + iC$ is the positive cone in $\mathcal{L}_{a,b}$ which is also denoted as C .

Now we prove that the cone of $\mathcal{L}_{a,b}$ is not normal but is a strict b -cone.

Theorem 1: The cone C of $\mathcal{L}_{a,b}$ is not normal.

Proof: Let $\mathcal{L}_{a,b}$ be restricted to real-valued functions. Let m be a fixed positive integer and (ϕ_i) be a sequence of

functions in $\mathcal{L}_{a,b,K_m} \cap C$ such that $\lambda_i = \sup\{\phi_i(t) : t \in K_m\}$ converges to 0 but (ϕ_i) does not converge to 0 for \mathcal{L}_{a,b,K_m} .

Define

$$\begin{aligned} \psi_i &= \lambda_i, \quad t \in K_{m+1} \\ &= 0, \quad t \notin K_{m+1}. \end{aligned}$$

Let ξ_i be the regularization of ψ_i defined by

$$\xi_i(t) = \int_{\mathbb{R}} \theta_\alpha(t-t')\psi_i(t')dt'.$$

Then $\xi_i \in \mathcal{L}_{a,b,K_{m+2}}$, $\forall i$.

Also, $0 \leq \phi_i \leq \xi_i$, $\forall i$ and (ξ_i) converges to 0 for $\tau_{a,b}$. In Proposition 1.3, Chapter 1, [2], it has been proved that the positive cone in an ordered linear space $E(\tau)$ is normal if and only if for any two nets $\{x_\beta : \beta \in I\}$ and $\{y_\beta : \beta \in I\}$ in $E(\tau)$ if $0 \leq x_\beta \leq y_\beta$ for all $\beta \in I$ and if $\{y_\beta : \beta \in I\}$ converges to 0 for τ then $\{x_\beta : \beta \in I\}$ converges to 0 for τ . So we conclude that C is not normal for the Schwartz topology. It follows that $C + iC$ is not normal for $\mathcal{L}_{a,b}$, we conclude that the cone of $\mathcal{L}_{a,b}$ is not normal. ■

Theorem 2: The cone C is a strict b -cone in $\mathcal{L}_{a,b}$.

Proof: Assume that $\mathcal{L}_{a,b}$ is restricted to real-valued functions. Let \mathbb{B} be the saturated class of all bounded circled subsets of $\mathcal{L}_{a,b}$ for $\tau_{a,b}$. Then $\mathcal{L}_{a,b} = \cup_{B \in \mathbb{B}} B$ where $\mathbb{B}_c = \{(B \cap C) - (B \cap C) : B \in \mathbb{B}\}$ is a fundamental system for \mathbb{B} and C is a strict b -cone since \mathbb{B} is the class of all τ -bounded sets in $\mathcal{L}_{a,b}$.

Let B be a bounded circled subset of $\mathcal{L}_{a,b}$ for $\tau_{a,b}$. Then all functions in B have their support in some compact set K_{m_0} and there exists a constant $M > 0$ such that $|\phi(t)| \leq M$, $\forall \phi \in B$, $t \in K_{m_0}$. Let

$$\begin{aligned} \psi(t) &= M, \quad t \in K_{m_0+1} \\ &= 0, \quad t \notin K_{m_0+1}. \end{aligned}$$

Then the regularization ξ of ψ is defined by

$$\xi(t) = \int_{\mathbb{R}} \theta_\alpha(t-t')\psi(t')dt'$$

and ξ has its support in K_{m_0+2} .

Also, $B \subseteq (B + \xi) - \{\xi\} \subseteq (B + \xi) \cap C - (B + \xi) \cap C$.

It follows that C is a strict b -cone. We conclude that $C + iC$ is a strict b -cone in $\mathcal{L}_{a,b}$. ■

Order and topology on the dual $\mathcal{L}'_{a,b}$ of $\mathcal{L}_{a,b}$.

Let $\mathcal{L}'_{a,b}$ denote the linear space of all linear functionals defined on $\mathcal{L}_{a,b}$. $\mathcal{L}'_{a,b}$ is ordered by the dual cone (see [2]) of C in $\mathcal{L}_{a,b}$. We assign the topology of bounded convergence (see [1] for details) to $\mathcal{L}'_{a,b}$. The class of all B^0 , the polars of B as B varies over all $\sigma(\mathcal{L}_{a,b}, \mathcal{L}'_{a,b})$ -bounded subsets of $\mathcal{L}_{a,b}$ is a neighbourhood basis of 0 in $\mathcal{L}'_{a,b}$ for the locally convex topology $\beta(\mathcal{L}'_{a,b}, \mathcal{L}_{a,b})$. When $\mathcal{L}'_{a,b}$ is ordered by the dual cone C' and is equipped with the topology $\beta(\mathcal{L}'_{a,b}, \mathcal{L}_{a,b})$ it follows that C' is a normal cone since C is a strict b -cone in $\mathcal{L}_{a,b}$ for $\tau_{a,b}$ by Corollary 1.26, Chapter 2, [2].

Theorem 3: The order dual $\mathcal{L}'_{a,b}$ is the same as the topological dual $\mathcal{L}'_{a,b}$ when $\mathcal{L}'_{a,b}$ is equipped with the topology of bounded convergence $\beta(\mathcal{L}'_{a,b}, \mathcal{L}_{a,b})$.

Proof: Every positive linear functional is continuous for the Schwartz topology $\tau_{a,b}$ on $\mathcal{L}_{a,b}$.

$$\mathcal{L}'_{a,b} = C(\mathcal{L}_{a,b}, \mathbb{R}) - C(\mathcal{L}_{a,b}, \mathbb{R})$$

where $C(\mathcal{L}_{a,b}, \mathbb{R})$ is the linear subspace consisting of all non-negative order bounded linear functionals of $L(\mathcal{L}_{a,b}, \mathbb{R})$, the linear space of all order bounded, linear functionals on $\mathcal{L}_{a,b}$. It follows that $\mathcal{L}'_{a,b} \subseteq \mathcal{L}'_{a,b}$.

On the other hand, the space $\mathcal{L}'_{a,b}$ equipped with the topology of bounded convergence $\beta(\mathcal{L}'_{a,b}, \mathcal{L}_{a,b})$ and ordered by the dual cone C' of the cone C in $\mathcal{L}_{a,b}$ is a reflexive space ordered by a closed normal cone. Hence if D is a directed set (\leq) of generalized functions that is either majorized in $\mathcal{L}'_{a,b}$ or contains a $\beta(\mathcal{L}'_{a,b}, \mathcal{L}_{a,b})$ -bounded section, then $\sup D$ exists in $\mathcal{L}'_{a,b}$ and the filter $\mathcal{F}(D)$ of sections of D converges to $\sup D$ for $\beta(\mathcal{L}'_{a,b}, \mathcal{L}_{a,b})$. Hence $\mathcal{L}'_{a,b} = \mathcal{L}'_{a,b}$ and we conclude that $\mathcal{L}'_{a,b}$ with respect to the topology $\beta(\mathcal{L}'_{a,b}, \mathcal{L}_{a,b})$ is both order complete and topologically complete. (See Proposition 1.8, Chapter 4, [2]). ■

Theorem 4: Let $a \leq c$, $d \leq b$. The restriction of any $f \in \mathcal{L}'_{a,b}$ to $\mathcal{L}_{c,d}$ is $\mathcal{L}'_{c,d}$ when $\mathcal{L}'_{a,b}$, $\mathcal{L}'_{c,d}$ are assigned the topology of bounded convergence.

Proof: When $a \leq c$, $d \leq b$, $\mathcal{L}_{c,d} \subseteq \mathcal{L}_{a,b}$. The topology $\tau_{c,d}$ of $\mathcal{L}_{c,d}$ is stronger than the topology induced on $\mathcal{L}_{c,d}$ by $\mathcal{L}_{a,b}$. By Zemanian [3], the restriction of any $f \in \mathcal{L}'_{a,b}$ to $\mathcal{L}_{c,d}$ is in $\mathcal{L}'_{c,d}$ when $\mathcal{L}'_{c,d}$ and $\mathcal{L}'_{a,b}$ are assigned the topology of pointwise convergence. It follows that the above result is true when $\mathcal{L}'_{c,d}$ and $\mathcal{L}'_{a,b}$ are assigned the topology of bounded convergence by Theorem 2.15, [1]. ■

Theorem 5: If $a < c$ or $d < b$. $\mathcal{L}'_{a,b}$ cannot be identified in one-to-one correspondence with a subspace of $\mathcal{L}'_{c,d}$.

Proof: Zemanian [3] has illustrated that two different members of $\mathcal{L}'_{a,b}$ have the same restriction to $\mathcal{L}_{c,d}$ if $a < c$ or $d < b$ and when $\mathcal{L}'_{a,b}$ is assigned the topology of pointwise convergence. By Theorem 2.15, [1], it follows that the same result is true when $\mathcal{L}'_{a,b}$ is assigned the topology of bounded convergence. ■

The Ordered Countable Union Space $\mathcal{L}(w, z)$ and its dual $\mathcal{L}'(w, z)$.

Definition 5: Let w = a real number or $-\infty$ and z = a real number or $+\infty$. Let (a_i) , (b_i) be sequences of real numbers such that $a_i \rightarrow w^+$, $b_i \rightarrow z^-$. Let $\mathcal{L}(w, z) = \cup_{i=1}^{\infty} \mathcal{L}_{a_i, b_i}$. $\mathcal{L}(w, z)$ is a countable union spaces.

We have defined the spaces \mathcal{D} , \mathcal{E} in [1] as follows.

Definition 6: Let \mathcal{D} denote the linear space of all complex-valued smooth functions with compact support in \mathbb{R}^n . Let K be an arbitrary compact set in \mathbb{R}^n . Let \mathcal{D}_K denote the subspace of \mathcal{D} consisting of functions with support in K . The topology on \mathcal{D}_K is induced by the multinorm

$$\gamma_{K,k}(\phi) = \sup\{|D^p \phi(t)| : t \in K, |p| \leq k\}.$$

The topology induced by $\{\gamma_{K,k}\}_{k=0}^{\infty}$ is a complete metrizable locally convex topology on \mathcal{D}_K . \mathcal{D} is assigned the inductive limit topology with respect to the family $\{\mathcal{D}_{K_m}\}_{m=1}^{\infty}$ of linear subspaces of \mathcal{D} where

$$K_m = \{t = (t_1, \dots, t_n) \in \mathbb{R}^n, |t_i| \leq m, i = 1, \dots, n\}.$$

Restricting \mathcal{D} to the real-valued functions an order relation is defined on \mathcal{D} by identifying the positive cone to be the set of all non-negative functions on \mathcal{D} . The cone $C + iC$ is the positive cone of \mathcal{D} .

Definition 7: Let \mathcal{E} be the linear space of all complex-valued smooth functions defined on \mathbb{R}^n and (K_m) a sequence of compact subsets of \mathbb{R}^n such that $K_1 \subseteq K_2 \subseteq \dots$ and such that each compact subset of \mathbb{R}^n is contained in some K_m . Let $\gamma_{K_m,k}(\phi) = \sup_{t \in K_m} |D^k \phi(t)|$, $\phi \in \mathcal{E}$, $k = 0, 1, 2, \dots$. $\{\gamma_{K_m,k}\}_{k=0}^\infty$ is a multinorm on \mathcal{E} and \mathcal{E} is equipped with the topology generated by it.

Note For details on \mathcal{D} , \mathcal{E} , refer [1], [3].

Result 1: $\mathcal{D} \subseteq \mathcal{L}_{a,b} \subseteq \mathcal{E}$ and \mathcal{D} is not dense in $\mathcal{L}_{a,b}$ but \mathcal{D} is dense in $\mathcal{L}(w, z)$ for every w, z . \mathcal{D} is dense in \mathcal{E} also. It follows that $\mathcal{L}_{a,b}$ is dense in \mathcal{E} (refer [3]). Equipped with the Schwartz topology $\mathcal{L}(w, z)$ is complete since it is a countable union space. For $k = 0, 1, 2, \dots$ $t^k e^{-st} \in \mathcal{L}(w, z)$ if and only if $w < \text{Res} < z$. $\mathcal{L}(w, z)$ is ordered by the cone

$$C(\mathcal{L}(w, z)) = \cup_{i=1}^\infty C_i(\mathcal{L}_{a_i, b_i})$$

The dual $\mathcal{L}'(w, z)$ of $\mathcal{L}(w, z)$ is ordered by the dual cone $C'(\mathcal{L}'(w, z))$ of $C(\mathcal{L}(w, z))$ and is assigned the topology of bounded convergence $\beta(\mathcal{L}'(w, z), \mathcal{L}(w, z))$. From the definitions of $\mathcal{L}(w, z)$, $\mathcal{L}'(w, z)$ and Theorems 1, 2, 3 we conclude the following.

Theorem 6: The cone $C(\mathcal{L}(w, z))$ is a strict b -cone.

Theorem 7: The dual cone $C'(\mathcal{L}'(w, z))$ of $\mathcal{L}'(w, z)$ is a normal cone.

Theorem 8: $\mathcal{L}'(w, z)$ is topologically complete and order complete.

II. LINEAR MAPS ON ORDERED MULTINORMED SPACES, ORDERED COUNTABLE UNION SPACES AND THEIR ADJOINTS

In this section we study the properties of linear maps defined on ordered multinormed spaces, ordered countable union spaces and the adjoints of these maps defined on their duals when the dual spaces are assigned the topology of bounded convergence. Also, we apply these results to some linear maps on $\mathcal{L}_{a,b}$, $\mathcal{L}(w, z)$ and their adjoints.

Definition 8: Let U, V be ordered multinormed spaces or ordered countable union spaces with positive cones $C(U)$, $C(V)$ respectively. A linear map

$$T : U \rightarrow V$$

is

- (i) positive if $T(C(U)) \subseteq C(V)$, i.e., $T(\phi) \geq 0$ whenever $\phi > 0$, $\phi \in U$.
- (ii) strictly positive if $T(\phi) > 0$ whenever $\phi > 0$.
- (iii) order bounded if T maps each order bounded set in U into an order bounded set in V .

Note. Every strictly positive linear map is positive and every positive linear map is order bounded.

Definition 9: Let U, V be ordered multinormed spaces or ordered countable union spaces and let $T : U \rightarrow V$ be continuous and linear. $T' : V' \rightarrow U'$ defined by $(T'f)(\phi) = f(T(\phi))$, $f \in V'$, $\phi \in U$ is the adjoint of T .

Theorem 9: If $T : U \rightarrow V$ is linear and continuous its adjoint $T' : V' \rightarrow U'$ is also linear and continuous where U, V are ordered multinormed spaces or ordered countable union spaces.

Proof: For $\phi, \psi \in C(U)$ and α, β any two complex numbers, we have

$$(T'f)(\alpha\phi + \beta\psi) = \alpha(T'f)(\phi) + \beta(T'f)(\psi)$$

so that $T'f$ is a linear functional on U .

Let $(\phi_\alpha)_{\alpha \in J}$ be a net in $C(U)$ which converges to 0. Since $T : U \rightarrow V$ is continuous, $T(\phi_\alpha) \rightarrow 0$ as $\alpha \rightarrow \infty$. $(T'f)(\phi_\alpha) = f(T(\phi_\alpha)) \rightarrow 0$ as $\alpha \rightarrow \infty$ since $f \in U'$.

Thus $T'f$ is a continuous linear functional on U . i.e. T' is a mapping on V' to U' .

Also, $T'(\alpha f + \beta g)(\phi) = (\alpha T'(f) + \beta T'(g))(\phi)$, $f, g \in C'(V')$, $\phi \in C(U)$, α, β any two complex numbers. So, $T' : V' \rightarrow U'$ is linear.

Let $(f_\alpha)_{\alpha \in J}$ be a net converging to 0 and let B_V^0 be a neighbourhood basis of 0 for the topology of bounded convergence in V' . B_V^0 is the polar of B_V , a $\sigma(V', V)$ -bounded subset of V . Since $T : U \rightarrow V$ is linear and continuous, $B_U = T^{-1}(B_V)$ is a $\sigma(U', U)$ -bounded subset of U and B_U^0 , the polar of B_U is such that $T(f_\alpha) \in B_U^0$, the polar of B_U , $\forall \alpha \in J$. It follows that $(T'f_\alpha)_{\alpha \in J}$ converges to 0 and that $T' : V' \rightarrow U'$ is continuous with respect to the topology of bounded convergence in U', V' . ■

Note. The above results are true if U, V are countable union spaces also.

Theorem 10: Let U, V be ordered multinormed spaces. If $T : U \rightarrow V$ is strictly positive $T' : V' \rightarrow U'$ is also strictly positive.

Proof: Let $f > 0$. For $\phi > 0$, $T(\phi) > 0$ since T is strictly positive. $f(T(\phi)) > 0 \Rightarrow (T'f)(\phi) > 0$ for $\phi > 0$. Thus $f > 0 \Rightarrow T'f > 0$, T' is strictly positive. It follows that T' is positive and order bounded. ■

A linear partial differential operator and its adjoint on generalized functions.

Let I be an open set in \mathbb{R}^n or \mathbb{C}^n and let $t = (t_1, \dots, t_n) \in I$. Let $\theta_i(t)$, $i = 0, 1, \dots, m$ be complex valued smooth functions on I . Consider the linear partial differential operator

$$T = (-1)^{|k|} \theta_0 D^{k_1} \theta_1 D^{k_2} \dots \theta_{m-1} D^{k_m} \theta_m$$

where k_i denote non-negative integers and $|k| = k_1 + \dots + k_m$. T denotes a sequence of operators: multiply by θ_m , differentiate according to D^{k_m} , multiply by θ_{m-1} etc. Let $U(I)$, $V(I)$ be testing function spaces on I . Let T' denote the adjoint of T

$$(T'f)(\phi) = f(T(\phi)) = \int_I f(t) T(\phi(t)) dt.$$

By successive integration by parts this becomes

$$\int_I \phi(\theta_m D^{k_m} \dots D^{k_1} \theta_0 f) dt.$$

Thus T' may be identified as the operator

$$\theta_m D^{k_m} \dots D^{k_1} \theta_0$$

where D denotes the conventional differentiation. When T' acts upon $f \in V'(I)$, D denotes the generalized differentiation.

Theorem 11: The generalized differentiation is a continuous linear mapping of $C'(\mathcal{L}'_{a,b})$ into itself and of $C'(\mathcal{L}'(w, z))$ into itself and hence of $\mathcal{L}'_{a,b}$ and $\mathcal{L}'(w, z)$ when the topology of bounded convergence is assigned to these dual spaces.

Proof: By the definition of seminorms $\gamma_k, \gamma_k(-D\phi) = \gamma_{k+1}(\phi)$. By Lemma 1.10.1, [3], $(-D)$ is a continuous linear mapping of $\mathcal{L}_{a,b}$ into itself. By Theorem 9 its adjoint operator D which is a generalized differentiation is a continuous linear mapping of $\mathcal{L}'_{a,b}$ into itself. It follows that $(-D)$ is a continuous linear mapping of $\mathcal{L}(w, z)$ into itself and that its adjoint operator D , the generalized differentiation is a continuous linear mapping of $\mathcal{L}'(w, z)$ into itself. ■

Definition 10: Let S be a linear space of smooth functions defined as follows: $\theta(t) \in S$ if and only if it is smooth on $(-\infty, \infty)$ and for each non-negative integer k there exists an integer N_k such that,

$$(1 + t^2)^{-N_k} D^k(\theta(t))$$

is bounded on $(-\infty, \infty)$.

Theorem 12: For $\theta \in S$ the mapping $f \rightarrow \theta f$ is a continuous linear mapping of $C'(\mathcal{L}'(w, z))$ into itself and hence of $\mathcal{L}'(w, z)$ into itself.

Proof: Let $\phi \in C(\mathcal{L}_{c,d})$ for some $c, d \in \mathbb{R}$ and let $a, b \in \mathbb{R}$ such that $a < c, d < b$. Then for any $\theta \in S$, $\phi \rightarrow \theta\phi$ is a continuous linear mapping of $C(\mathcal{L}_{c,d})$ into $\mathcal{L}_{a,b}$. Thus $\theta\phi \in \mathcal{L}_{a,b}$ whenever $\phi \in C(\mathcal{L}_{c,d})$. Let (ϕ_i) be a sequence in $\mathcal{L}(w, z)$ that converges in $\mathcal{L}(w, z)$. Then there exists real numbers a, b, c, d such that $w < a < c, d < b < z$ such that (ϕ_i) converges in $\mathcal{L}_{c,d}$. By what proved above $(\theta\phi_i)$ converges in $\mathcal{L}_{a,b}$ and hence in $\mathcal{L}(w, z)$. Thus $\theta \rightarrow \theta\phi$ is a continuous linear mapping of $C(\mathcal{L}(w, z))$ to $\mathcal{L}(w, z)$ and hence of $\mathcal{L}(w, z)$ into itself. The adjoint of this map is $f \rightarrow \theta f$, $f \in C'(\mathcal{L}'(w, z))$ and we conclude from Theorem 9 that $f \rightarrow \theta f$ is a continuous linear map of $\mathcal{L}'(w, z)$ into itself. ■

Theorem 13: Let α be a fixed complex number and $r = \text{Re } \alpha$. The mapping $f \rightarrow e^{-\alpha t} f$ from $C'(\mathcal{L}'_{a,b})$ onto $C'(\mathcal{L}'_{a-r, b-r})$ is linear, continuous, strictly positive and hence order bounded. Its inverse is also continuous, strictly positive and order bounded. Hence the map is a strictly positive, order bounded isomorphism from $\mathcal{L}'_{a,b}$ onto $\mathcal{L}'_{a-r, b-r}$. The map is a strictly positive, order bounded isomorphism from $C'(\mathcal{L}'(w, z))$ onto $C'(\mathcal{L}'(w-r, z-r))$ and hence from $\mathcal{L}'(w, z)$ onto $\mathcal{L}'(w-r, z-r)$.

Proof: The map $\phi(t) \rightarrow e^{-\alpha t} \phi(t)$ and its inverse are linear and strictly positive. Zemanian [3] has proved that the map from $\mathcal{L}_{a-r, b-r}$ onto $\mathcal{L}_{a,b}$ as well as its inverse are continuous. The adjoint map $f \rightarrow e^{-\alpha t} f$, $f \in C'(\mathcal{L}'_{a-r, b-r})$ is linear, continuous and its inverse is also continuous by Theorem 9. It follows that the adjoint map and its inverse are strictly positive by Theorem 10 and hence are order bounded. We conclude that the map is a strictly positive, order bounded isomorphism from $\mathcal{L}'_{a,b}$ onto $\mathcal{L}'_{a-r, b-r}$.

From the definition of $\mathcal{L}(w, z)$, corresponding results follow in the case of the map from $\mathcal{L}'(w, z)$ onto $\mathcal{L}'(w-r, z-r)$.

Theorem 14: Let λ be a fixed real number. For every $a, b, w, z, f(t) \rightarrow f(t-\lambda)$ from $C'(\mathcal{L}'_{a,b})$ to itself is linear, continuous, strictly positive and order bounded. Its inverse is continuous and order bounded. Hence the map is an order bounded automorphism on $\mathcal{L}'_{a,b}$. The map is an order bounded automorphism on $\mathcal{L}'(w, z)$.

Proof: $\phi(t) \rightarrow \phi(t+\lambda)$ is linear, continuous and strictly positive on $\mathcal{L}_{a,b}$ and hence on $\mathcal{L}(w, z)$. The adjoint of this map $f(t) \rightarrow f(t+\lambda)$ on $C'(\mathcal{L}'_{a,b})$ is continuous by Theorem 9 and the map is strictly positive and hence order bounded by Theorem 2.4. It follows that the map is an order bounded automorphism on $\mathcal{L}'_{a,b}$. From the definition of $\mathcal{L}(w, z)$ it follows that the map is an order bounded automorphism on $\mathcal{L}'(w, z)$. ■

III. THE TWO-SIDED LAPLACE TRANSFORMATION: DEFINITION AND SOME BASIC PROPERTIES.

In this section we introduce the two-sided Laplace Transformation and discuss some properties of the transformation and derive results which are used in solving differential equations.

Definition 11: Let $d(f)$ be a linear space of conventional functions and f be a linear functional defined on $d(f)$ which satisfies the following properties

- (i) f is additive i.e. $f(\phi + \psi) = f(\phi) + f(\psi)$, $\phi, \psi \in d(f)$
- (ii) $\mathcal{L}_{a,b} \subseteq d(f)$ for at least one pair of real numbers a, b with $a < b$
- (iii) For every $\mathcal{L}_{a,b} \subseteq d(f)$ the restriction of f to $\mathcal{L}_{c,d}$ is in $\mathcal{L}'_{c,d}$.

f is called a Laplace transformable generalized function.

Note. The set W of all such functionals form a linear space under the usual operations of addition of functions and multiplication by complex numbers and is ordered by the positive cone of non-negative functionals.

Definition 12: Let f be a Laplace transformable generalized function. The set of all real numbers r such that there exists two real numbers a_r, b_r depending on r such that $a_r < r < b_r$ and $\mathcal{L}_{a_r, b_r} \subseteq d(f)$ is denoted as \wedge_f .

Note. \wedge_f is an open set.

Theorem 15: Let $\sigma_1 = \inf \wedge_f, \sigma_2 = \sup \wedge_f$. Given a functional f defined on a linear space $d(f)$ of conventional functions f can be extended to a functional f_1 on $d(f) \cup \mathcal{L}(\sigma_1, \sigma_2)$ such that

- (i) The restriction of f_1 to $\mathcal{L}(\sigma_1, \sigma_2)$ is a member of $\mathcal{L}'(\sigma_1, \sigma_2)$.
- (ii) The restriction of f_1 to $d(f)$ coincides with f .

Proof: Since $\sigma_1 = \inf \wedge_f, \sigma_2 = \sup \wedge_f$ there exists two sequences $(c_i), (d_i)$ such that

$$c_i \rightarrow \sigma_1+, d_i \rightarrow \sigma_2-, c_i, d_i \in \wedge_f, c_i < d_i, \forall i.$$

Then $f \in \mathcal{L}'(c_i, c_i), f \in \mathcal{L}'(d_i, d_i), \forall i$.

Let $\lambda(t)$ be a fixed smooth function on \mathbb{R} such that

$$\begin{aligned} \lambda(t) &= 0, & \text{for } t < -1 \\ &= 1, & \text{for } t > 1. \end{aligned}$$

f can be extended to $C(\mathcal{L}_{c_i, d_i})$ and to \mathcal{L}_{c_i, d_i} as follows: Let $\psi \in C(\mathcal{L}_{c_i, d_i})$.

Define $f(\psi) = f(\lambda\psi) + f((1-\lambda)\psi)$.
 f is continuous and linear on $C(\mathcal{L}_{c_i, d_i})$ and hence on \mathcal{L}_{c_i, d_i} .
 Using the same methods f may be extended to $\mathcal{L}(\sigma_1, \sigma_2)$. This extension of f is unique. ■

Definition 13: Let f be a Laplace transformable generalized function, $\sigma_1 = \inf \Lambda_f$, $\sigma_2 = \sup \Lambda_f$,

$$\Omega_f = \{s : \sigma_1 < \text{Re } s < \sigma_2\}.$$

The Laplace transformation F of f is defined as a conventional function on Ω_f by $F(s) = (\mathcal{L}f)(s) = \langle f(t), e^{-st} \rangle$, $s \in \Omega_f$.
 For any fixed $s \in \Omega_f$ the right hand side has a meaning as the application of $f \in \mathcal{L}'(\sigma_1, \sigma_2)$ to $e^{-st} \in \mathcal{L}(\sigma_1, \sigma_2)$.

Theorem 16: If $f(t)$ is a positive locally integrable function such that $\frac{f(t)}{L_{a,b}(t)}$ is absolutely integrable on $-\infty < t < \infty$, then $f(\phi) = \int_{-\infty}^{\infty} f(t)\phi(t)dt$ is a member of $C'(\mathcal{L}_{a,b})$.

Proof: Follows from Zemanian [3], Chapter 2, Section 3.2(v) and Theorem 2.15 of [1]. ■

Theorem 17: If $\mathcal{L}f = F(s)$ for $s \in \Omega_f$ then $F(s)$ is analytic on Ω_f and $DF(s) = \langle f(t), -te^{-st} \rangle$, $s \in \Omega_f$.

Theorem 18: The Laplace transform is strictly positive and is an order bounded linear map.

Proof: The linear space W of all Laplace transformable generalized functions is an ordered linear space. We define an order relation on the field of complex numbers by identifying the positive cone to be the set of complex numbers $\alpha + i\beta$, $\alpha > 0$, $\beta > 0$. Since $\mathcal{L}f > 0$ for $f > 0$, $f \in W$ it follows that \mathcal{L} is strictly positive and hence is order bounded. ■

Notation.

$$D \equiv \frac{d}{dt}, D_t \equiv \frac{\partial}{\partial t}, D_s \equiv \frac{\partial}{\partial s}.$$

Result 2: $\mathcal{L}(D^k f(t)) = s^k F(s)$, $s \in \Omega_f$, $k = 1, 2, \dots$, $\mathcal{L}f = F(s)$.

By Theorem 11 the generalized differentiation is a continuous linear mapping of $\mathcal{L}'(\sigma_1, \sigma_2)$ into itself. For $f \in \mathcal{L}'(\sigma_1, \sigma_2)$

$$\begin{aligned} \langle D^k f(t), e^{-st} \rangle &= \langle f(t), (-D_t)^k e^{-st} \rangle \\ &= \langle f(t), s^k e^{-st} \rangle \\ &= s^k F(s), \quad s \in \Omega_f, k = 1, 2, \dots \end{aligned}$$

Result 3: $\mathcal{L}(t^k f(t)) = (-D_s)^k F(s)$, $s \in \Omega_f$, $k = 1, 2, 3, \dots$, $t^k \in S$ for $k = 1, 2, 3, \dots$

Let S be a fixed complex number such that for $\sigma_1 < \text{Re } s < \sigma_2$,

$$\begin{aligned} \langle t f(t), e^{-st} \rangle &= \langle f(t), t e^{-st} \rangle \\ \text{i.e., } \mathcal{L}f(t f(t)) &= -D_s F(s), \quad s \in \Omega_f. \end{aligned}$$

For $k = 1, 2, 3, \dots$

$$\begin{aligned} \langle t^k f(t), e^{-st} \rangle &= \langle t^{k-1} f(t), t e^{-st} \rangle \\ &= -D_s \langle t^{k-1} f(t), t e^{-st} \rangle \\ &= -D_s \langle t^{k-2} f(t), t e^{-st} \rangle \\ &= (-D_s)^2 \langle t^{k-2} f(t), e^{-st} \rangle \\ &= \dots \\ &= (-D_s)^k \langle f(t), e^{-st} \rangle \end{aligned}$$

$$\text{i.e., } \mathcal{L}(t^k f(t)) = (-D_s)^k F(s), \quad s \in \Omega_f, k = 1, 2, 3, \dots$$

Result 4: For a fixed complex number α if $\mathcal{L}f = F(s)$,

$$\mathcal{L}(e^{-\alpha t} f(t)) = F(s + \alpha), \quad s + \alpha \in \Omega_f.$$

Result 5: If $\mathcal{L}f = F(s)$, for a fixed real number β

$$\mathcal{L}(f(t - \beta)) = e^{-s\beta} F(s), \quad s \in \Omega_f.$$

IV. INVERSION AND UNIQUENESS

The results on inversion of the Laplace transform and Uniqueness Theorems proved by Zemanian [3] hold good in the ordered dual spaces when they are assigned the topology of bounded convergence. In some situations the domain is restricted to the positive cone of the respective spaces. We state the theorems without proof.

Lemma 1: Let $\mathcal{L}f = F(s)$ for $\sigma_1 < \text{Re } s < \sigma_2$, let $\phi \in C(\mathcal{D})$ and set $\psi(s) = \int_{-\infty}^{\infty} \phi(t)e^{st} dt$. Then for any fixed real number r , with $0 < r < \infty$

$$\int_{-r}^r \langle f(t), e^{-s\tau} \rangle \psi(s) dw = \langle f(\tau), \int_{-r}^r e^{-S\tau} \psi(S) dw \rangle$$

where $s = \sigma + iw$ and σ is fixed with $\sigma_1 < \sigma < \sigma_2$.

Lemma 2: Let a, b, σ and r be real numbers with $a < \sigma < b$, $\phi \in C(\mathcal{D})$. Then

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \phi(t + \tau) e^{\sigma t} \frac{\sin rt}{t} dt$$

converges in $C(\mathcal{L}_{a,b})$ to $\phi(\tau)$ and hence in $\mathcal{L}_{a,b}$ as $r \rightarrow \infty$.

Theorem 19: Let $\mathcal{L}f = F(s)$ for $\sigma_1 < \sigma < \sigma_2$, r be a real variable. Then for a fixed real number σ , $\sigma_1 < \sigma < \sigma_2$,

$$f(t) = \lim_{r \rightarrow \infty} \frac{1}{2\pi i} \int_{\sigma - ir}^{\sigma + ir} F(s) e^{st} ds$$

where the convergence is with respect to the topology of bounded convergence in \mathcal{D}' , the dual of \mathcal{D} . (For details see [1]).

Theorem 20: If $\mathcal{L}f = F(s)$ for $s \in \Omega_f$ and $\mathcal{L}h = H(s)$ for $s \in \Omega_h$ and if $\Omega_f \cap \Omega_h \neq \emptyset$ and if $F(s) = H(s)$ for $s \in \Omega_f \cap \Omega_h$ then $f = h$ in the sense of equality in $\mathcal{L}'(w, z)$ where the interval $w < \sigma < z$ is the intersection of $\Omega_f \cap \Omega_h$ with the real axis.

V. OPERATIONAL CALCULUS AND SOLUTIONS OF DIFFERENTIAL EQUATIONS

The following characterization for the Laplace transform has been modified to suit the present situation of ordered Laplace transformable generalized functions where we consider only the elements of the positive cone we have defined.

Theorem 21: Let $F(s)$ be a strictly positive function. The necessary and sufficient condition for $F(s)$ to be the Laplace transform of a positive generalized function f and for the corresponding strip of definition to be

$$\Omega_f = \{s : \sigma_1 < \text{Re } s < \sigma_2\}$$

is that $F(S)$ be analytic on Ω_f and for each closed substrip $\{s : a < \text{Re } s \leq b\}$ of Ω_f where $\sigma_1 < a < b < \sigma_2$ is that there be a polynomial P such that

$$F(s) \leq P(|s|) \text{ for } a < \text{Re } s < b$$

The polynomial will in general depend on the choices of a and b .

Corollary 1: Let $F(s)$ be a strictly positive function and let $\mathcal{L}f = F(s)$ for $s \in \Omega_f$. Choose three fixed real numbers a , σ and b such that $a < \sigma < b$ and choose a polynomial $Q(s)$ that has no zeros for $a \leq \operatorname{Re} s \leq b$ and such that

$$\frac{F(s)}{|Q(s)|} \leq \frac{k}{|s|^2}, \quad a < \operatorname{Re} s < b, \quad k \text{ a constant.}$$

Then in the sense of equality in $\mathcal{L}'(a, b)$

$$F(t) = Q(D_t) \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{F(s)}{Q(s)} e^{st} ds, \quad a < \sigma < b.$$

where D_t represents generalized differentiation in $\mathcal{L}'(a, b)$ and the integral converges in the conventional sense to a continuous function that generates a regular member of $C'(\mathcal{L}(a, b))$.

Operational Calculus

Consider the linear differential equation

$$\mathcal{L}u(t) = (a_n D^n + a_{n-1} D^{n-1} + \dots + a_0)u(t) = g(t)$$

where the a_i 's are constants, $a_n \neq 0$ and $g(t)$ is a given Laplace transformable generalized function. Applying Laplace transform to both sides,

$$B(s) \cup(s) = G(s)$$

where $B(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_0$

$$\cup(s) = \mathcal{L}u$$

$$G(s) = \mathcal{L}g, \quad s \in \Omega_g = \{s : \sigma_{g_1} < \operatorname{Re} s < \sigma_{g_2}\}$$

If $B(s)$ has no zeros in Ω_g then by Theorem 21 there exists a generalized function $u(t)$ whose Laplace transform is $\frac{G(s)}{B(s)}$ on Ω_g . $u(t)$ is a unique member of $\mathcal{L}(\sigma_{g_1}, \sigma_{g_2})$ and satisfies the given equation. If $B(s)$ has a finite number of zeros in Ω_g there exists a set of m adjoint substrips

$\sigma_{g_1} = \sigma_0 < \operatorname{Re} s < \sigma_1, \sigma_1 < \operatorname{Re} s < \sigma_2, \dots, \sigma_{m-1} < \operatorname{Re} s < \sigma_m = \sigma_{g_2}$ on which $\frac{G(s)}{B(s)}$ is analytic and satisfies the growth condition of Corollary 1. For any given substrip say $\sigma_i < \operatorname{Re} s < \sigma_{i+1}$ there exists a unique member $u(t)$ of $\mathcal{L}(\sigma_i, \sigma_{i+1})$ and whose Laplace transform is $\frac{G(s)}{B(s)}$ on $\sigma_i < \operatorname{Re} s < \sigma_{i+1}$. For any other choice of the substrip there will be a different solution.

If $u(t), v(t)$ are two solutions of the above differential equation such that $u(t)$ for all t lies to the left of the line say $\operatorname{Re} s = \alpha$ and $v(t)$ lies to the right for all t then $u \leq v$ by the order relation we have defined on the Laplace transformable generalized functions.

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