

New Classes of Salagean type Meromorphic Harmonic Functions

Hakan Bostancı, and Metin Öztürk

Abstract—In this paper, a necessary and sufficient coefficient are given for functions in a class of complex valued meromorphic harmonic univalent functions of the form $f = h + \bar{g}$ using Salagean operator. Furthermore, distortion theorems, extreme points, convolution condition and convex combinations for this family of meromorphic harmonic functions are obtained.

Keywords—Harmonic mappings, Meromorphic functions, Salagean operator.

I. INTRODUCTION

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain D if both u and v are real harmonic in D . In any simply connected domain $D \subset \mathbf{C}$ we can write $f = h + \bar{g}$, where h and g are analytic in D . A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [2]). In [3], Hengartner and Schober investigated functions harmonic in the exterior of the unit disc $\tilde{U} = \{z : |z| > 1\}$. They showed that complex valued, harmonic, sense preserving, univalent mapping f must admits the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z|,$$

where

$$h(z) = \alpha z + \sum_{k=1}^{\infty} a_k z^{-k}$$

and

$$g(z) = \beta \bar{z} + \sum_{k=1}^{\infty} b_k \bar{z}^{-k}$$

for $0 \leq |\beta| < |\alpha|$, $A \in \mathbf{C}$.

Let MH denote the class of functions

$$f(z) = h(z) + \overline{g(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \quad (1)$$

which are harmonic in the punctured unit disk $U \setminus \{0\}$. $h(z)$ and $g(z)$ are analytic in $U \setminus \{0\}$ and U , respectively, and $h(z)$ has a simple pole at the origin with residue 1 here.

For $f = h + \bar{g}$ given by (1), Jahangiri [4] defined the modified Salagean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)}; \quad n = 0, 1, 2, \dots, \quad (2)$$

where

$$D^n h(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n a_k z^k$$

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and

$$D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k.$$

A function $f(z) \in MH$ is said to be in the subclass MHS^* of meromorphically harmonic starlike in $U \setminus \{0\}$ if it satisfies the condition

$$Re \left\{ -\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \right\} > 0, \quad z \in U \setminus \{0\}.$$

Now we define a new class $MHS_S^*(n, \alpha)$ (see [1]).

Definition 1.1: For $0 \leq \alpha < 1$, we let $MHS_S^*(n, \alpha)$ denote the class of meromorphic harmonic functions f of the form (1) such that

$$Re \left\{ -\frac{2D^{n+1}f(z)}{D^n f(z) - D^n f(-z)} \right\} > \alpha, \quad z \in U \setminus \{0\}. \quad (3)$$

We let the subclass $\overline{MHS}_S^*(n, \alpha)$ consist of meromorphic harmonic functions $f_n = h_n + \bar{g}_n$ in $MHS_S^*(n, \alpha)$ so that h_n and g_n are of the form

$$h_n(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} a_k z^k \quad (4)$$

and

$$g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \quad (5)$$

where $a_k \geq 0, b_k \geq 0$.

In this paper, we have obtained the coefficient conditions for the classes $MHS_S^*(n, \alpha)$ and $\overline{MHS}_S^*(n, \alpha)$. Further a representation theorem, inclusion properties and distortion bound for the class $\overline{MHS}_S^*(n, \alpha)$ are established.

II. MAIN RESULTS

Theorem 2.1: Let f be of the form (1). If

$$\sum_{k=1}^{\infty} [(|a_{2k}| + |b_{2k}|)(2k)^{n+1} + ((2k-1+\alpha)|a_{2k-1}| + (2k-1-\alpha)|b_{2k-1}|)(2k-1)^n] \leq 1 - \alpha, \quad (6)$$

then f is harmonic univalent, sense preserving in $U \setminus \{0\}$ and $f \in MHS_S^*(n, \alpha)$.

Proof: For $0 < |z_1| \leq |z_2| < 1$ we have

$$\begin{aligned} & |f(z_1) - f(z_2)| \\ & \geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\ & \geq \frac{|z_1 - z_2|}{|z_1||z_2|} \\ & - |z_1 - z_2| \sum_{k=1}^{\infty} (|a_k| + |b_k|) |z_1^{k-1} + \dots + z_2^{k-1}| \\ & > \frac{|z_1 - z_2|}{|z_1||z_2|} \left[1 - |z_2|^2 \sum_{k=1}^{\infty} k(|a_k| + |b_k|) \right] \\ & = \frac{|z_1 - z_2|}{|z_1||z_2|} \left[1 - |z_2|^2 \left(\sum_{k=1}^{\infty} 2k(|a_{2k}| + |b_{2k}|) \right. \right. \\ & \quad \left. \left. + \sum_{k=1}^{\infty} (2k-1)(|a_{2k-1}| + |b_{2k-1}|) \right) \right] \\ & > \frac{|z_1 - z_2|}{|z_1||z_2|} \left[1 - \sum_{k=1}^{\infty} (2k)^{n+1} (|a_{2k}| + |b_{2k}|) \right. \\ & \quad - \sum_{k=1}^{\infty} (2k-1)^n [(2k-1+\alpha)|a_{2k-1}|] \\ & \quad \left. - \sum_{k=1}^{\infty} (2k-1)^n [(2k-1-\alpha)|b_{2k-1}|] \right]. \end{aligned}$$

This last expression is non negative by (6) and so f is univalent in $U \setminus \{0\}$. To show that f is sense preserving in $U \setminus \{0\}$, we need to show that $|h'(z)| \geq |g'(z)|$ in $U \setminus \{0\}$. We have

$$\begin{aligned} |h'(z)| & \geq 1 - \sum_{k=1}^{\infty} k|a_k||z|^{k-1} \\ & = 1 - \sum_{k=1}^{\infty} k|a_k|r^{k-1} > 1 - \sum_{k=1}^{\infty} k|a_k| \\ & \geq 1 - \sum_{k=1}^{\infty} (2k)^{n+1}|a_{2k}| \\ & \quad - \sum_{k=1}^{\infty} (2k-1)^n(2k-1+\alpha)|a_{2k-1}| \\ & \geq \sum_{k=1}^{\infty} (2k)^{n+1}|b_{2k}| \\ & \quad + \sum_{k=1}^{\infty} (2k-1)^n(2k-1-\alpha)|b_{2k-1}| \\ & \geq \sum_{k=1}^{\infty} 2k|b_{2k}| + \sum_{k=1}^{\infty} (2k-1)|b_{2k-1}| \\ & > \sum_{k=1}^{\infty} k|b_k|r^{k-1} = \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \geq |g'(z)|. \end{aligned}$$

Now, we will show that $f \in MHS_S^*(n, \alpha)$. According to (2) and (3), for $0 \leq \alpha < 1$, we have

$$Re \left\{ -\frac{2D^{n+1}f(z)}{D^n f(z) - D^n f(-z)} \right\}$$

$$= Re \left\{ -\frac{2D^{n+1}h(z) - 2(-1)^n \overline{D^{n+1}g(z)}}{T^n(z)} \right\} \geq \alpha,$$

where

$$T^n(z) = \frac{D^n h(z) + (-1)^n \overline{D^n g(z)}}{-D^n h(-z) - (-1)^n \overline{D^n g(-z)}}.$$

Using the fact that $Re\{w\} \geq \alpha$ if and only if $|1 - \alpha + w| \geq |1 + \alpha - w|$, it suffices to show that

$$\begin{aligned} & \left| 1 - \alpha - \frac{2D^{n+1}f(z)}{D^n f(z) - D^n f(-z)} \right| \\ & \geq \left| 1 + \alpha + \frac{2D^{n+1}f(z)}{D^n f(z) - D^n f(-z)} \right| \end{aligned}$$

which is equivalent to

$$\begin{aligned} & |2D^{n+1}f(z) - (1-\alpha)(D^n f(z) - D^n f(-z))| \\ & - |2D^{n+1}f(z) + (1+\alpha)(D^n f(z) - D^n f(-z))| \geq 0. \quad (7) \end{aligned}$$

Substituting for $D^n f(z)$ and $D^{n+1}f(z)$ in (7) yields

$$\begin{aligned} & \left| \frac{2(-1)^n}{z} - 2 \sum_{k=1}^{\infty} k^{n+1} a_k z^k + 2(-1)^n \sum_{k=1}^{\infty} k^{n+1} \bar{b}_k \bar{z}^k \right. \\ & \left. + (1-\alpha) \left[\frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n a_k z^k \right. \right. \\ & \left. \left. + (-1)^n \sum_{k=1}^{\infty} k^n \bar{b}_k \bar{z}^k + \frac{(-1)^n}{z} - \sum_{k=1}^{\infty} (-1)^k k^n a_k z^k \right. \right. \\ & \left. \left. - (-1)^n \sum_{k=1}^{\infty} (-1)^k k^n \bar{b}_k \bar{z}^k \right] \right| \\ & - \left| \frac{2(-1)^n}{z} - 2 \sum_{k=1}^{\infty} k^{n+1} a_k z^k \right. \\ & \left. + 2(-1)^n \sum_{k=1}^{\infty} k^{n+1} \bar{b}_k \bar{z}^k - (1+\alpha) \left[\frac{(-1)^n}{z} \right. \right. \\ & \left. \left. + \sum_{k=1}^{\infty} k^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} k^n \bar{b}_k \bar{z}^k + \frac{(-1)^n}{z} \right. \right. \\ & \left. \left. - \sum_{k=1}^{\infty} (-1)^k k^n a_k z^k - (-1)^n \sum_{k=1}^{\infty} (-1)^k k^n \bar{b}_k \bar{z}^k \right] \right| \\ & = \left| \frac{2(2-\alpha)(-1)^n}{z} \right. \\ & \left. - \sum_{k=1}^{\infty} (2k - (1-\alpha) + (-1)^k(1-\alpha)) k^n a_k z^k \right. \\ & \left. + (-1)^n \sum_{k=1}^{\infty} (2k + (1-\alpha) - (-1)^k(1-\alpha)) k^n \bar{b}_k \bar{z}^k \right| \\ & - \left| \frac{2\alpha(-1)^n}{z} + \sum_{k=1}^{\infty} (2k + (1+\alpha) - (-1)^k(1+\alpha)) k^n a_k z^k \right. \\ & \left. - (-1)^n \sum_{k=1}^{\infty} (2k - (1-\alpha) + (-1)^k(1+\alpha)) k^n \bar{b}_k \bar{z}^k \right| \\ & = \left| \frac{2(2-\alpha)(-1)^n}{z} \right. \end{aligned}$$

$$\begin{aligned}
 & -2 \sum_{k=1}^{\infty} (2k-2+\alpha)(2k-1)^n a_{2k-1} z^{2k-1} \\
 & -2 \sum_{k=1}^{\infty} (2k)^{n+1} a_{2k} z^{2k} + 2(-1)^n \sum_{k=1}^{\infty} (2k)^{n+1} \bar{b}_{2k} \bar{z}^{2k} \\
 & + 2(-1)^n \sum_{k=1}^{\infty} (2k-\alpha)(2k-1)^n \bar{b}_{2k-1} \bar{z}^{2k-1} \Big| \\
 & - \left| \frac{2\alpha(-1)^n}{z} + 2 \sum_{k=1}^{\infty} (2k)^{n+1} a_{2k} z^{2k} \right. \\
 & + 2 \sum_{k=1}^{\infty} (2k+\alpha)(2k-1)^n a_{2k-1} z^{2k-1} \\
 & - 2(-1)^n \sum_{k=1}^{\infty} (2k)^{n+1} \bar{b}_{2k} \bar{z}^{2k} \\
 & \left. - 2(-1)^n \sum_{k=1}^{\infty} (2k-2+\alpha)(2k-1)^n \bar{b}_{2k-1} \bar{z}^{2k-1} \right| \\
 \geq & \frac{2(2-\alpha)(-1)^n}{z} \\
 & - 2 \sum_{k=1}^{\infty} (2k-2+\alpha)(2k-1)^n |a_{2k-1}| |z|^{2k-1} \\
 & - 2 \sum_{k=1}^{\infty} (2k)^{n+1} |a_{2k}| |z|^{2k} - 2 \sum_{k=1}^{\infty} (2k)^{n+1} |b_{2k}| |z|^{2k} \\
 & - 2 \sum_{k=1}^{\infty} (2k-\alpha)(2k-1)^n |b_{2k-1}| |z|^{2k-1} \\
 & - \frac{2\alpha(-1)^n}{z} - 2 \sum_{k=1}^{\infty} (2k+\alpha)(2k-1)^n |a_{2k-1}| |z|^{2k-1} \\
 & - 2 \sum_{k=1}^{\infty} (2k)^{n+1} |a_{2k}| |z|^{2k} - 2 \sum_{k=1}^{\infty} (2k)^{n+1} |b_{2k}| |z|^{2k} \\
 & - 2 \sum_{k=1}^{\infty} (2k-2+\alpha)(2k-1)^n |b_{2k-1}| |z|^{2k-1} \\
 = & \frac{4(1-\alpha)}{z} \left[1 - \sum_{k=1}^{\infty} \left\{ \frac{(2k)^{n+1}}{1-\alpha} (|a_{2k}| + |b_{2k}|) \right. \right. \\
 & \left. \left. + \frac{(2k-1)^n}{1-\alpha} [(2k-1+\alpha)|a_{2k-1}| \right. \right. \\
 & \left. \left. + (2k-1-\alpha)|b_{2k-1}|] \right\} \right].
 \end{aligned}$$

This last expression is non-negative by (7), and so the proof is complete. ■

Theorem 2.2: Let $f_n = h_n + \bar{g}_n$ where h_n and g_n are of the form (4) and (5). Then $f_n \in \overline{MHS}_S^*(n, \alpha)$, if and only if

$$\begin{aligned}
 & \sum_{k=1}^{\infty} [(a_{2k} + b_{2k})(2k)^{n+1} + ((2k-1+\alpha)a_{2k-1} \\
 & + (2k-1-\alpha)b_{2k-1})(2k-1)^n] \leq 1-\alpha. \tag{8}
 \end{aligned}$$

Proof: Since $\overline{MHS}_S^*(n, \alpha) \subset MHS_S^*(n, \alpha)$, we only need to prove the (only if) part of the theorem. To this end,

for functions $f_n = h_n + \bar{g}_n$, we notice that condition

$$Re \left\{ -\frac{2D^{n+1}f(z)}{D^n f(z) - D^n f(-z)} \right\} \geq \alpha, \quad z \in U \setminus \{0\},$$

is equivalent to

$$Re \left\{ \frac{\frac{2(1-\alpha)}{z} - \sum_{k=1}^{\infty} (2k+\alpha - (-1)^k \alpha) k^n a_k z^k}{\phi(z)} \right. \\
 \left. + \frac{(-1)^n \sum_{k=1}^{\infty} (2k-\alpha + (-1)^k \alpha) k^n b_k \bar{z}^k}{\phi(z)} \right\} \geq 0$$

which implies

$$Re \left\{ \frac{\frac{2(1-\alpha)}{z} - 2 \sum_{k=1}^{\infty} (2k)^{n+1} a_{2k} z^{2k}}{\phi(z)} \right. \\
 - \frac{2 \sum_{k=1}^{\infty} (2k-1+\alpha)(2k-1)^n a_{2k-1} z^{2k-1}}{\phi(z)} \\
 - \frac{2(-1)^n \sum_{k=1}^{\infty} (2k-1-\alpha)(2k-1)^n b_{2k-1} \bar{z}^{2k-1}}{\phi(z)} \\
 \left. - \frac{2(-1)^n \sum_{k=1}^{\infty} (2k)^{n+1} b_{2k} \bar{z}^{2k}}{\phi(z)} \right\} \geq 0, \tag{9}$$

where

$$\begin{aligned}
 \phi(z) = & \frac{2}{z} + 2 \sum_{k=1}^{\infty} (2k-1)^n a_{2k-1} z^{2k-1} \\
 & + 2 \sum_{k=1}^{\infty} (2k-1)^n b_{2k-1} \bar{z}^{2k-1}.
 \end{aligned}$$

The condition (9) must hold for all z in $U \setminus \{0\}$. By choosing $0 < z = r < 1$, from the left hand (9), we have

$$\begin{aligned}
 & \frac{1-\alpha - \sum_{k=1}^{\infty} (2k-1+\alpha)(2k-1)^n a_{2k-1} r^{2k}}{r\phi(r)/2} \\
 & - \frac{\sum_{k=1}^{\infty} (2k)^{n+1} a_{2k} r^{2k+1} + (-1)^n \sum_{k=1}^{\infty} (2k)^{n+1} b_{2k} r^{2k+1}}{r\phi(r)/2} \\
 & - \frac{(-1)^n \sum_{k=1}^{\infty} (2k-1-\alpha)(2k-1)^n b_{2k-1} r^{2k}}{r\phi(r)/2}. \tag{10}
 \end{aligned}$$

If the condition (8) does not hold, then the number in (10) is negative for r sufficiently close to 1. Hence there exist

$z_0 = r_0$ in $(0, 1)$, for which the eqnarray in (10) is negative. This contradicts the required condition for and so the proof is complete. It is easily seen that $f_n(z) \in \overline{MHS}_S^*(n, \alpha)$. Thus we complete the of the Theorem 2.2. ■

Theorem 2.3: If $f_n = h_n + \bar{g}_n \in \overline{MHS}_S^*(n, \alpha)$ for $0 < |z| = r < 1$, then

$$\frac{1}{r} - \frac{1-\alpha}{2^{n+1}}r \leq |f_n(z)| \leq \frac{1}{r} + \frac{1-\alpha}{2^{n+1}}r.$$

Proof: Let $f_n = h_n + \bar{g}_n \in \overline{MHS}_S^*(n, \alpha)$. Taking the absolute value of f we obtain

$$\begin{aligned} |f_n(z)| &= \left| \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} \overline{b_k z^k} \right| \\ &\leq \frac{1}{r} + \sum_{k=1}^{\infty} (a_k + b_k) r^k \\ &\leq \frac{1}{r} + \sum_{k=1}^{\infty} (a_k + b_k) r \\ &\leq \frac{1}{r} + \frac{1-\alpha}{2^{n+1}} \sum_{k=1}^{\infty} \frac{2^{n+1}}{1-\alpha} (|a_k| + |b_k|) r \\ &\leq \frac{1}{r} + \frac{1-\alpha}{2^{n+1}} r \end{aligned}$$

and

$$\begin{aligned} |f_n(z)| &= \left| \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} \overline{b_k z^k} \right| \\ &\geq \frac{1}{r} - \sum_{k=1}^{\infty} (a_k + b_k) r^k \\ &\geq \frac{1}{r} - \sum_{k=1}^{\infty} (a_k + b_k) r \\ &\geq \frac{1}{r} - \frac{1-\alpha}{2^{n+1}} \sum_{k=1}^{\infty} \frac{2^{n+1}}{1-\alpha} (|a_k| + |b_k|) r \\ &\geq \frac{1}{r} - \frac{1-\alpha}{2^{n+1}} r. \end{aligned}$$

■

Corollary 2.4: Let $A = \left\{ w : |w| < \frac{2^{n+1}-1+\alpha}{2^{n+1}} \right\}$.

If $f_n = h_n + \bar{g}_n \in \overline{MHS}_S^*(n, \alpha)$, then

$$f_n(U) \subset A^t.$$

Theorem 2.5: $f_n = h_n + \bar{g}_n \in \overline{MHS}_S^*(n, \alpha)$ if and only if f_n can be expressed as

$$f_n(z) = \sum_{k=0}^{\infty} (x_k h_{n_k} + y_k g_{n_k}), \tag{11}$$

where for $k = 1, 2, \dots$

$$\begin{aligned} h_{n_0}(z) &= g_{n_0}(z) = \frac{1}{z} \\ h_{n_{2k-1}}(z) &= \frac{1}{z} + \frac{1-\alpha}{(2k-1)^n(2k-1+\alpha)} z^{2k-1} \\ h_{n_{2k}}(z) &= \frac{1}{z} + \frac{1-\alpha}{(2k)^{n+1}} z^{2k} \\ g_{n_{2k-1}}(z) &= \frac{1}{z} + \frac{1-\alpha}{(2k-1)^n(2k-1-\alpha)} \bar{z}^{2k-1} \\ g_{n_{2k}}(z) &= \frac{1}{z} + \frac{1-\alpha}{(2k)^{n+1}} \bar{z}^{2k} \end{aligned}$$

and

$$\sum_{k=0}^{\infty} (x_k + y_k) = 1, \quad x_k \geq 0 \text{ and } y_k \geq 0.$$

In particular, the extreme point of $\overline{MHS}_S^*(n, \alpha)$ are $\{h_{n_k}\}$ and $\{g_{n_k}\}$.

Proof: For functions $f_n = h_n + \bar{g}_n$, where h_n and g_n of the form (4) and (5), we have

$$\begin{aligned} f_n(z) &= \sum_{k=0}^{\infty} (x_k h_{n_k} + y_k g_{n_k}) \\ &= x_0 h_{n_0} + y_0 g_{n_0} + \sum_{k=1}^{\infty} (x_k + y_k) \frac{1}{z} \\ &\quad + \sum_{k=1}^{\infty} x_{2k-1} \frac{1-\alpha}{(2k-1)^n(2k-1+\alpha)} z^{2k-1} \\ &\quad + \sum_{k=1}^{\infty} x_{2k} \frac{1-\alpha}{(2k)^{n+1}} z^{2k} \\ &\quad + \sum_{k=1}^{\infty} y_{2k-1} \frac{1-\alpha}{(2k-1)^n(2k-1-\alpha)} \bar{z}^{2k-1} \\ &\quad + \sum_{k=1}^{\infty} y_{2k} \frac{1-\alpha}{(2k)^{n+1}} \bar{z}^{2k} \\ &= \sum_{k=0}^{\infty} (x_k + y_k) \frac{1}{z} \\ &\quad + \sum_{k=1}^{\infty} \frac{1-\alpha}{(2k-1)^n(2k-1+\alpha)} x_{2k-1} z^{2k-1} \\ &\quad + \sum_{k=1}^{\infty} \frac{1-\alpha}{(2k)^{n+1}} x_{2k} z^{2k} \\ &\quad + \sum_{k=1}^{\infty} \frac{1-\alpha}{(2k-1)^n(2k-1-\alpha)} y_{2k-1} \bar{z}^{2k-1} \\ &\quad + \sum_{k=1}^{\infty} \frac{1-\alpha}{(2k)^{n+1}} y_{2k} \bar{z}^{2k}. \end{aligned}$$

Then,

$$\begin{aligned} &\sum_{k=1}^{\infty} (2k-1+\alpha)(2k-1)^n \\ &\times \left\{ \frac{1-\alpha}{(2k-1)^n(2k-1+\alpha)} x_{2k-1} \right\} \end{aligned}$$

$$\begin{aligned}
 & + \sum_{k=1}^{\infty} (2k-1-\alpha)(2k-1)^n \\
 & \times \left\{ \frac{1-\alpha}{(2k-1)^n(2k-1-\alpha)} y_{2k-1} \right\} \\
 & + \sum_{k=1}^{\infty} (2k)^{n+1} \left\{ \frac{1-\alpha}{(2k)^{n+1}} (x_{2k} + y_{2k}) \right\} \\
 & = (1-\alpha) \sum_{k=1}^{\infty} (x_k + y_k) \\
 & = (1-\alpha)[1 - (x_0 + y_0)] \leq 1 - \alpha.
 \end{aligned}$$

So $f_n \in \overline{MHS}_S^*(n, \alpha)$.

Conversely, suppose that $f_n \in \overline{MHS}_S^*(n, \alpha)$.

Let, for $k = 1, 2, \dots$

$$\begin{aligned}
 x_{2k} &= \frac{1-\alpha}{(2k)^{n+1}} a_{2k}, y_{2k} = \frac{1-\alpha}{(2k)^{n+1}} b_{2k} \\
 x_{2k-1} &= \frac{1-\alpha}{(2k-1)^n(2k-1+\alpha)} a_{2k-1} \\
 y_{2k-1} &= \frac{1-\alpha}{(2k-1)^n(2k-1-\alpha)} b_{2k-1}.
 \end{aligned}$$

Then note that by the $\overline{MHS}_S^*(n, \alpha)$, $0 \leq x_k \leq 1$ and $0 \leq y_k \leq 1$ ($k = 1, 2, \dots$). We define $0 \leq x_0 \leq 1$ and

$$y_0 = 1 - x_0 - \sum_{k=1}^{\infty} (x_k + y_k).$$

Consequently, we obtain

$$f(z) = \sum_{k=0}^{\infty} (x_k h_{n_k} + y_k g_{n_k})$$

as required. ■

Theorem 2.6: If $f \in MHS_S^*(n, \alpha)$, then the diameter D_f of $\mathbb{C} \setminus f(U)$ satisfies

$$D_f \geq 2|1 + b_1|.$$

Proof: Let $D_f(R)$ be diameter of $f(|z| = R)$, $0 < R < 1$, and let $D_f^*(R) = \max_{|z|=R} |f(z) - f(-z)|$. Then $D_f(R) \rightarrow D_f$ as $R \rightarrow 1$ and $D_f(R) \geq D_f^*(R)$. Since

$$\begin{aligned}
 [D_f^*(R)]^2 &\geq \frac{1}{2\pi} \int_0^{2\pi} |f(Re^{i\theta}) - f(-Re^{i\theta})|^2 d\theta \\
 &= 4 \left[\frac{1}{R^2} + b_1 + \bar{b}_1 \right. \\
 &\quad \left. + \sum_{k=1}^{\infty} (|a_{2k-1}|^2 + |b_{2k-1}|^2) R^{2(2n-1)} \right] \\
 &\geq 4 \left[1 + 2Reb_1 \right. \\
 &\quad \left. + \sum_{k=1}^{\infty} (|a_{2k-1}|^2 + |b_{2k-1}|^2) \right]
 \end{aligned}$$

we conclude that $D_f \geq 2\sqrt{|1 + b_1|^2}$. ■

Note that if f and F are

$$f(z) = h(z) + \overline{g(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}$$

and

$$F(z) = H(z) + \overline{G(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} A_k z^k + \overline{\sum_{k=1}^{\infty} B_k z^k},$$

then the convolution (or Hadamard product) of f and F is defined to be the function

$$(f * F)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k A_k z^k + \overline{\sum_{k=1}^{\infty} b_k B_k z^k}. \quad (12)$$

Theorem 2.7: For $0 \leq \beta \leq \alpha < 1$, let $f_n \in \overline{MHS}_S^*(n, \alpha)$ and $F_n \in \overline{MHS}_S^*(n, \beta)$. Then the convolution function $f_n * F_n \in \overline{MHS}_S^*(n, \alpha) \subset \overline{MHS}_S^*(n, \beta)$.

Proof: For f_n and F_n as Theorem 2.7. Then the convolution $f_n * F_n$ is given by (12). We wish to show that the coefficients of $f_n * F_n$ satisfy the required condition given in Theorem 2.2. For $F_n \in \overline{MHS}_S^*(n, \beta)$, we note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Since $0 \leq \beta \leq \alpha < 1$ and $f_n \in \overline{MHS}_S^*(n, \alpha)$ for $f_n * F_n$, we obtain

$$\begin{aligned}
 & \sum_{k=1}^{\infty} \left[(a_{2k} A_{2k} + b_{2k} B_{2k}) \frac{(2k)^{n+1}}{1-\beta} \right. \\
 & \quad \left. + ((2k-1+\alpha)a_{2k-1} A_{2k-1} \right. \\
 & \quad \left. + (2k-1-\alpha)b_{2k-1} B_{2k-1}) \frac{(2k-1)^n}{1-\beta} \right] \\
 & \leq \sum_{k=1}^{\infty} \left[(a_{2k} + b_{2k}) \frac{(2k)^{n+1}}{1-\beta} \right. \\
 & \quad \left. + ((2k-1+\alpha)a_{2k-1} \right. \\
 & \quad \left. + (2k-1-\alpha)b_{2k-1}) \frac{(2k-1)^n}{1-\beta} \right] \\
 & \leq \sum_{k=1}^{\infty} \left[(a_{2k} + b_{2k}) \frac{(2k)^{n+1}}{1-\alpha} \right. \\
 & \quad \left. + ((2k-1+\alpha)a_{2k-1} \right. \\
 & \quad \left. + (2k-1-\alpha)b_{2k-1}) \frac{(2k-1)^n}{1-\alpha} \right] \\
 & \leq 1.
 \end{aligned}$$

Therefore $f_n * F_n \in \overline{MHS}_S^*(n, \alpha) \subset \overline{MHS}_S^*(n, \beta)$. ■

Theorem 2.8: The class $\overline{MHS}_S^*(n, \alpha)$ is closed under convex combination.

Proof: Suppose that $f_{n_i}(z) \in \overline{MHS}_S^*(n, \alpha)$ for $i = 1, 2, 3, \dots$, where f_{n_i} is given by

$$f_{n_i}(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} a_{i_k} z^k + (-1)^n \overline{\sum_{k=1}^{\infty} b_{i_k} z^k}.$$

Then by Theorem 2.2.

$$\sum_{k=1}^{\infty} [(a_{i_{2k}} + b_{i_{2k}})(2k)^{n+1} + ((2k-1+\alpha)a_{i_{2k-1}} \quad (13)$$

$$+(2k - 1 - \alpha)b_{i_{2k-1}}(2k - 1)^n] \leq 1 - \alpha.$$

For

$$\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1,$$

the convex combinations of f_{n_i} may be written as

$$\begin{aligned} \sum_{i=1}^{\infty} t_i f_{n_i}(z) &= \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{i_k} \right) z^{-n} \\ &\quad + (-1)^n \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{i_k} \right) z^{-n}. \end{aligned}$$

Then by (13),

$$\begin{aligned} &\sum_{k=1}^{\infty} \left[\frac{(2k)^{n+1}}{1 - \alpha} \sum_{i=1}^{\infty} t_i (a_{i_{2k}} + b_{i_{2k}}) \right. \\ &\quad + \frac{(2k - 1)^n}{1 - \alpha} \left(\frac{(2k - 1 + \alpha)}{1 - \alpha} \sum_{i=1}^{\infty} t_i a_{i_{2k-1}} \right. \\ &\quad \left. \left. + \frac{(2k - 1 - \alpha)}{1 - \alpha} \sum_{i=1}^{\infty} t_i b_{i_{2k-1}} \right) \right] \\ &\leq \sum_{i=1}^{\infty} t_i = 1. \end{aligned}$$

Thus

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) \in \overline{MHS}_S^*(n, \alpha).$$

■

Theorem 2.9: If $f_n \in \overline{MHS}_S^*(n, \alpha)$, then

$$\sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2) \leq 1 + 2Re\{b_1\}.$$

Equality occurs if and only if $C \setminus f(U)$ has area zero.

Proof: The area of the omitted set is

$$\begin{aligned} &\lim_{R \rightarrow 1} \lim_{2i} \frac{1}{2i} \int_{0 < |z|=R < 1} \bar{f} df \\ &= \lim_{R \rightarrow 1} \left[\frac{1}{2i} \int_{0 < |z|=R < 1} \bar{h} h' dz + \frac{1}{2i} \int_{0 < |z|=R < 1} g \bar{g}' d\bar{z} \right. \\ &\quad \left. + \frac{1}{2i} \int_{0 < |z|=R < 1} g h' dz + \frac{1}{2i} \int_{0 < |z|=R < 1} \bar{h} \bar{g}' d\bar{z} \right] \\ &= \pi \left[\sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2) R^{2k} - \frac{1}{R^2} - 2Re\{b_1\} \right]. \end{aligned}$$

For $0 < r < 1$ the curve $\Gamma_r = f(C_r)$ is a simple closed curve oriented clockwise. Hence, for $R \rightarrow 1$ we obtain

$$\sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2) - 1 - 2Re\{b_1\} \leq 0$$

and the result follows. ■

REFERENCES

[1] R.A.Al-Khal and H.A.Al-Kharsani. *Salagean-Type Harmonic Univalent Functions With Respect to Symmetric Points*. Vol 4, pp. 1-13, 2007.
 [2] J.Clumine and T.Sheil-Small. *Harmonic Functions*. Ann. Acad. Sci. Fenn. Ser. A1 Math. **9**(1984), 3-25.
 [3] W.Hengartner and G.Schober. *Univalent Harmonic Functions*. Trans. Amer. Math. Soc. **299**(1987), 1-31.
 [4] J.M.Jahangiri, G.Murugusundaramoorthy, and K.Vijaya. *Salagean-Type Harmonic Univalent Functions*. Southwest J. Pure Appl. Math. **2**(2002), 77-82.4.