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The Small Scale Effect on Nonlinear Vibration of Single Layer Graphene Sheets

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***Abstract*—**In the present article, nonlinear vibration analysis of single layer graphene sheets is presented and the effect of small length scale is investigated. Using the Hamilton's principle, the three coupled nonlinear equations of motion are obtained based on the von Karman geometrical model and Eringen theory of nonlocal continuum. The solutions of Free nonlinear vibration, based on a one term mode shape, are found for both simply supported and clamped graphene sheets. A complete analysis of graphene sheets with movable as well as immovable in-plane conditions is also carried out. The results obtained herein are compared with those available in the literature for classical isotropic rectangular plates and excellent agreement is seen. Also, the nonlinear effects are presented as functions of geometric properties and small scale parameter.

***Keywords*—**Small scale, Nonlinear vibration, Graphene sheet, Nonlocal continuum

I. Introduction

THE large amplitude vibration analysis has become increasingly important particularly in thin-walled structures such as plates and shells and several attempts have been made to obtain a solution for nonlinear vibration of such structures. The large amplitude vibration analysis has become increasingly important particularly in thin-walled structures such as plates and shells and several attempts have been made to obtain a solution for nonlinear vibration of such structures.

The large amplitude vibrations of plates of various geometries have been investigated by several authors. The general solutions of the large amplitude vibration of thin elastic plates were obtained by Chu and Herrman [1] and Yamaki [2]. The nonlinear free vibration behavior of rectangular cross ply laminates was investigated by Singh et al. [3] using direct numerical integration. Leung and Mao [4] studied simply supported rectangular plates with movable edges using the Galerkin method. Theoretical and experimental studies for geometrically nonlinear vibrations of rectangular plates were developed by Amabili [5]. Nonlinear free axisymmetric vibration of simply supported isotropic circular plates was investigated by Haterboucha and Benamar [6] using the energy method and a multimode approach. An analytical solution was provided by Woo et al. [7] for the nonlinear free vibration behavior of plates made of functionally graded materials. Amabili and Farhadi [8] used the classical von Karman and the first order shear deformation theories for studying the nonlinear forced vibrations of isotropic and laminated composite rectangular plates.Due to the vast computational expenses of nano-structures analyses when using atomic lattice dynamics and molecular dynamic simulations, there is a great interest in applying continuum mechanics for analysis of nano-structures. Many studies have been carried out for vibration analysis of nano-structures such as nanotubes and nano-plates. Lim and He [9] developed a von Karman type nonlinear model for ultra-thin, elastically isotropic films with surface effect. Kitipornchai et al. [10] used the continuum plate model for mechanical analysis of graphene sheets. Based on the continuum mechanics and a multiple-elastic beam model, Fu et al. [11] investigated the nonlinear free vibration analysis of embedded carbon nanotubes. Pradhan and Phadikar [12] presented classical and first-order shear deformation plate theories for vibration of nano-plates. Their approach was based on the Navier solution and for a nano-plate with all edges simply supported. Ke et al. [13] investigated the nonlinear free vibration of embedded double-walled carbon nanotubes based on the Eringen's nonlocal elasticity theory and von Karman geometric nonlinearity using differential quadrature method. Dong and Lim [14] studied the nonlinear free vibrations of a nano-beam with simply supports boundary conditions based on nonlocal elasticity theory. Murmu and Pradhan [15] developed a single elastic beam model for thermo-mechanical vibration of a single-walled carbon nanotube embedded in an elastic medium based on nonlocal elasticity theory. Nonlinear free vibration of single-walled carbon nanotubes based on the Timoshenko beam model was studied by Yang et al. [16]. An elastic continuum approach for modeling the nonlinear vibration of double-walled carbon nanotubes under harmonic excitation was investigated by Hawwa and Qahtani [17].Practically all of the problems in mechanics are nonlinear and linearization is commonly an approximation. Therefore, the infinitesimal deformation model is invalid, and a geometrically nonlinear model is evidently needed. In this paper, the large amplitude vibration of a single layer graphene sheet is studied based on the nano-plate model. Considering the small scale effect in constitutive relations and using the von Karman nonlinear model, the governing equations of motion are obtained in form of two coupled nonlinear partial differential equations. The free vibration analysis are presented for both simply supported and clamped nano-plates with movable and immovable in-plane conditions. The effects of nonlocal parameter, boundary condition and aspect ratio on the nonlinear vibration behavior of graphene sheets are discussed in details.

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II. CONSTITUTIVE RELATIONS OF NONLOCAL CONTINUUM

According to nonlocal elasticity theory, the stress at a reference point X is considered to be a function of the strain field at every point X' in the body. The nonlocal stress tensor σ at point X can be expressed as

$$\sigma = \int_V K(|X'-X|, \tau)\sigma'(X')\,dX' \tag{1}$$

where σ' is the classical stress tensor and $K(|X'-X|)$ is the Kernel function represents the nonlocal modulus. While the constitutive equations of classical elasticity is an algebraic relation between stress and strain tensors, that of nonlocal elasticity involves spatial integrals which represent weighted averages of contributions of the strain of all points in the body to the stress at the given point. Eringen [18] showed that it is possible to represent the integral constitutive relation in an equivalent differential form as

$$(1-\mu\nabla^2)\sigma = \sigma' \tag{2}$$

where $\mu = (e_0 a)^2$ is the nonlocal parameter, a an internal characteristic length and e_0 a constant. Also, ∇^2 is the Laplacian operator.

III. FORMULATION

Consider a thin nano-plate of total thickness h with dimension $a \times b$ for modeling the single layer graphene sheet. The origin of the Cartesian coordinate system is located in the middle of the plate. Since the graphene sheet is assumed to have large amplitude motion, the von Karman type strain-displacement relations are used as

$$\begin{aligned} \varepsilon_x &= \varepsilon_{x0} + z\kappa_x \\ \varepsilon_y &= \varepsilon_{y0} + z\kappa_y \\ \gamma_{xy} &= \gamma_{xy0} + z\kappa_{xy}, \end{aligned} \tag{3}$$

where

$$\begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}(\dfrac{\partial w}{\partial x})^2 \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}(\dfrac{\partial w}{\partial y})^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} \end{Bmatrix}, \qquad \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ -2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{4}$$

Here the time dependent variables u, v and w are the mid-plane displacement components in the x, y and z directions respectively. As it can be seen, the strain-displacement relations are nonlinear with respect to transverse displacement. According to Hamilton's principle, the equations of motion of the nano-plate can be given by

$$\delta\int_{t_1}^{t_2} L\,dt = 0 \tag{5}$$

where L is the Lagrangian and t is the time variable. Expressing the Lagrangian parameter based on the von Karman theory and considering the small scale effect from Eq. (2) in constitutive relations, the non-linear equations of motion for a nano-plate can be obtained as follows

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = (1-\mu\nabla^2)(I_0\ddot{u} - I_1\frac{\partial \ddot{w}}{\partial x}) \tag{6a}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = (1-\mu\nabla^2)(I_0\ddot{v} - I_1\frac{\partial \ddot{w}}{\partial y}) \tag{6b}$$

$$\begin{aligned} &\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial}{\partial x}(N_x\frac{\partial w}{\partial x} + N_{xy}\frac{\partial w}{\partial y}) + \\ &\frac{\partial}{\partial y}(N_{xy}\frac{\partial w}{\partial x} + N_y\frac{\partial w}{\partial y}) + P = \\ &(1-\mu\nabla^2)(I_0\ddot{w} + I_1(\frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y}) - I_2(\frac{\partial^2 \ddot{w}}{\partial x^2} + \frac{\partial^2 \ddot{w}}{\partial y^2})) \end{aligned} \tag{6c}$$

where a dot denotes differentiation with respect to time and P is the load acting on the plate in z direction. The resultant forces (N_x, N_y, N_{xy}) and resultant moments (M_x, M_y, M_{xy}) can be defined in terms of strains as

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \frac{Eh}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{h^2}{12} & \dfrac{h^2}{12}\nu & 0 \\ 0 & 0 & 0 & \dfrac{h^2}{12}\nu & \dfrac{h^2}{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{(1-\nu)h^2}{24} \end{bmatrix} \begin{bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \tag{7}$$

where E and ν are respectively the Young modulus and Poisson ratio of the nano-plate. Also, the inertia parameters (I_0, I_1, I_2) can be expressed in term of density of the plate, ρ, as follow

$$(I_0, I_1, I_2) = \int_{-h/2}^{h/2} \rho(1, z, z^2)\,dz \tag{8}$$

As it can be concluded, the parameter I_1 is identically zero for a symmetric nano-plate with respect to z axis.

Raju et al. [19] showed that the effect of longitudinal or in-plane inertia on large amplitude vibration of thin-walled structures is negligible. Vanishing the in-plane inertia terms, it can be seen that two first equations (6) will be exactly satisfied if a stress function φ is defined such as

$$N_x = \frac{\partial^2\varphi}{\partial y^2}, \quad N_y = \frac{\partial^2\varphi}{\partial x^2}, \quad N_{xy} = -\frac{\partial^2\varphi}{\partial x \partial y} \tag{9}$$

Substituting these relations into Eq. (6c) and expressing the resultant moments in terms of transverse displacement yield

$$\begin{aligned} &D\nabla^4 w + (I_0 - I_2)(1-\mu\nabla^2)(\ddot{w} + \nabla^2\ddot{w}) - P = \\ &\frac{\partial^2 w}{\partial x^2}\frac{\partial^2\varphi}{\partial y^2} - 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2\varphi}{\partial x \partial y} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2\varphi}{\partial x^2} \end{aligned} \tag{10}$$

where D denotes the flexural rigidity of the nano-plate $(D = \dfrac{Eh^3}{12(1-\nu^2)})$. Eq. (10) is a nonlinear fourth order partial differential equation in terms of transverse displacement and

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stress function, and it needs to be augmented with a compatibility equation. Eliminating u and v from Eq. (4) and expressing the in-plane strain components in terms of stress function, one can obtained the compatibility equation as

$$\nabla^4\varphi = Eh\left((\frac{\partial^2 w}{\partial x \partial y})^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \right) \tag{11}$$

Eqs. (10) and (11) are the starting equations for analyzing the nonlinear vibration analysis of graphene sheets. As it can be seen these equations are nonlinear partial differential equations with a total degree of eight. Thus, the nonlinear behavior can be considered as two kinds, one from the in-plane forces due to the nature of edge restrains and the other due to the interaction of displacement components affected in the in-plane differential equations as well as compatibility conditions.

Also, the in-plane displacement components can be expressed in terms of w and φ by help of Eqs. (4), (7) and (9) by the following relations

$$u = \int_0^x \left(\frac{1}{Eh}(\frac{\partial^2\varphi}{\partial y^2} - \nu\frac{\partial^2\varphi}{\partial x^2}) - \frac{1}{2}(\frac{\partial w}{\partial x})^2 \right) dx \tag{12a}$$

$$v = \int_0^y \left(\frac{1}{Eh}(\frac{\partial^2\varphi}{\partial x^2} - \nu\frac{\partial^2\varphi}{\partial y^2}) - \frac{1}{2}(\frac{\partial w}{\partial y})^2 \right) dy \tag{12b}$$

IV. Free vibration analysis

For large amplitude free vibration analysis of graphene sheets, the external load is assumed to be zero $(P = 0)$. Four boundary conditions are considered for the nano-plate as all edges simply supported boundary condition with either movable or immovable in-plane edges and all edges clamped boundary condition with either movable or immovable in-plane edges.

i. An all edges simply supported (**SSSS**) graphene sheet

Let us consider a simply supported nano-plate with the following boundary conditions

a) Simply supported with movable in-plane edge (**SSSS1**)

$$\begin{aligned} &w = M_x = \frac{\partial^2\varphi}{\partial x\partial y} = \int_{-b/2}^{b/2}\frac{\partial^2\varphi}{\partial y^2}dy = 0 \qquad \text{at } x = \pm\frac{a}{2} \\ &w = M_y = \frac{\partial^2\varphi}{\partial x\partial y} = \int_{-a/2}^{a/2}\frac{\partial^2\varphi}{\partial x^2}dx = 0 \qquad \text{at } y = \pm\frac{b}{2} \end{aligned} \tag{13a}$$

b) Simply supported with immovable in-plane edge (**SSSS2**)

$$\begin{aligned} &w = M_x = \frac{\partial^2\varphi}{\partial x\partial y} = u = 0 \qquad \text{at } x = \pm\frac{a}{2} \\ &w = M_y = \frac{\partial^2\varphi}{\partial x\partial y} = v = 0 \qquad \text{at } y = \pm\frac{b}{2} \end{aligned} \tag{13b}$$

The movable in-plane edge is the edge which is kept straight by a distribution of normal stresses and therefore the resultant stresses on the edge is zero.

The nonlinear free vibration response of the simply supported nano-plate can be obtained by introducing the following admissible function for transverse deflection

$$w = hW(t)\cos(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b}) \tag{14}$$

where n and m are the numbers of half cosine waves in the x and y directions respectively and $W(t)$ is a function of time only. It is obvious that Eq. (14) satisfies the first two boundary conditions in Eqs. (13a) and (13b) of each edge. Substituting Eq. (14) in the right side of Eq. (11), the general solution for stress function φ can be obtained as

$$\varphi = C_1 x^2 + C_2 y^2 + \frac{Eh^2}{32n^2m^2a^2b^2}\cdot\begin{pmatrix} m^4a^4\cos(2n\pi x/a) + \\ n^4b^4\cos(2m\pi y/b) \end{pmatrix} W(t)^2 \tag{15}$$

The integral constant C_1 and C_2 should be determined whereas the two last conditions (in-plane boundary conditions) of Eqs. (13a) or (13b) are satisfied. It is easy to show that for movable simply supported nano-plate (**SSSS1**), the constant coefficients C_1 and C_2 should be equal to zero and for immovable simply supported nano-plates (**SSSS2**) these coefficients are as follows

$$\begin{aligned} C_1 &= \frac{Eh^2\pi^2(a^2 + \nu b^2)}{16(1-\nu^2)a^2b^2}W(t)^2 \\ C_2 &= \frac{Eh^2\pi^2(b^2 + \nu a^2)}{16(1-\nu^2)a^2b^2}W(t)^2 \end{aligned} \tag{16}$$

The transverse displacement w and the stress function φ can be introduced from Eqs. (14) and (15) into the basic Eq. (10) and, in according to the Galerkin procedure, the integral

$$\int_A \Gamma(w,\varphi)\Psi\, dA \tag{17}$$

can be computed over the area of the nano-plate. $\Gamma(w,\varphi)$ is the nonlinear equation (10) and Ψ is the spatial part of admissible function (14). A simple but lengthy calculation of the above integral leads to a modal time differential equation of Duffing's type, for both cases, which can be written as

$$\frac{d^2W(t)}{dt^2} + \alpha W(t) + \beta W(t)^3 = 0 \tag{18}$$

where the parameters α and β are defined in terms of graphene sheet properties as

$$\alpha = \pi^4\frac{12D(a^2+b^2)^2}{(12a^2b^2 + a^2h^2\pi^2 + b^2h^2\pi^2)(a^2b^2 + a^2\mu\pi^2 + b^2\mu\pi^2)\rho h} \tag{19a}$$

$$\beta = \pi^4\frac{3Eh^2(a^4+b^4)}{4(12a^2b^2 + a^2h^2\pi^2 + b^2h^2\pi^2)(a^2b^2 + a^2\mu\pi^2 + b^2\mu\pi^2)\rho} \quad \text{for } \mathbf{SSSS1} \tag{19b}$$

$$\beta = \pi^4\frac{9D(a^4\nu^2 + b^4\nu^2 - 3a^4 - 3b^4 - 4a^2b^2\nu)}{(12a^2b^2 + a^2h^2\pi^2 + b^2h^2\pi^2)(a^2b^2 + a^2\mu\pi^2 + b^2\mu\pi^2)\rho h} \quad \text{for } \mathbf{SSSS2} \tag{19c}$$

For linear vibration of the graphene sheet in which term $\beta W(t)^3$ can be neglected, the corresponding linear natural frequency is given by $\omega_l = \sqrt{\alpha}$. For periodic motion with amplitude w_0, the nonlinear natural frequency of Duffing Eq. (18) can be expressed in term of Jacobi elliptic function as [20]

$$\omega_{nl}=\frac{\pi\sqrt{\alpha+\beta w_0^2}}{2K\left(\frac{\beta w_0^2}{2(\alpha+\beta w_0^2)}\right)} \tag{20}$$

where K is the complete elliptic integral of the first kind defined as

$$K(m)=\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-m\sin^2\theta}} \tag{21}$$

ii. An all edges clamped (**CCCC**) graphene sheet

The boundary conditions for a clamped edges nano-plate may be identified as

*a) Clamped movable edge (**CCCC1**)*

$$\begin{aligned} w=\frac{\partial w}{\partial x}=\frac{\partial^2\varphi}{\partial x\partial y}=\int_{-b/2}^{b/2}\frac{\partial^2\varphi}{\partial y^2}dy=0 \qquad \text{at } x=\pm\frac{a}{2}\\ w=\frac{\partial w}{\partial y}=\frac{\partial^2\varphi}{\partial x\partial y}=\int_{-a/2}^{a/2}\frac{\partial^2\varphi}{\partial x^2}dx=0 \qquad \text{at } y=\pm\frac{b}{2}\end{aligned} \tag{22a}$$

*b) Clamped immovable edge (**CCCC2**)*

$$\begin{aligned} w=\frac{\partial w}{\partial x}=\frac{\partial^2\varphi}{\partial x\partial y}=u=0 \qquad \text{at } x=\pm\frac{a}{2}\\ w=\frac{\partial w}{\partial y}=\frac{\partial^2\varphi}{\partial x\partial y}=v=0 \qquad \text{at } y=\pm\frac{b}{2}\end{aligned} \tag{22b}$$

Assume that a solution of transverse displacement can be expressed in the following form

$$w=hW(t)\cos^2(\frac{n\pi x}{a})\cos^2(\frac{m\pi y}{b}) \tag{23}$$

It is easy to show that the function (23) exactly satisfies the two first boundary conditions in clamped edges. Doing the same procedure as the previous case, the Duffing's equation can be obtained as Eq. (18) with different coefficients.

V. Numerical results

In order to verify the accuracy of the present formulations, the results are compared with the available results in literature for a special case of classical rectangular plates in which the nonlocal effect is neglected $(\mu=0)$. Fig. 1 presents a comparative study of the non-linear to linear time period ratio versus the non-dimensional amplitude for a square simply supported rectangular plate for both immovable and movable edges with $\nu=0.3$. It can be observed that the results of the present study are sufficiently accurate with the results in Ref. [2].

For numerical results, the following properties are assumed for the graphene sheet

$$E=5.864\,eV/A^{\circ 3},\qquad \nu=0.19,\qquad h=1.317\,A^{\circ},$$
$$\rho=0.13\,mg/A^{\circ} \tag{24}$$

In order to recognize the hardening or softening state of the graphene sheet, the parameter β is depicted in Fig. 2 versus the nonlocal parameter μ. It can be said that the coefficient of $W(t)^3$ is always positive and therefore the graphene sheet has a hard stiffness. Also, it can be concluded that as the small scale effect increases the hardening stiffness of the nano-plate rapidly decreases. i.e. the small scale effect makes the nano-plate more flexible as the nonlocal model may be viewed as atoms linked by elastic springs while the local continuum model assumes the spring constant to take on an infinite value. The nonlinear to linear frequencies ratio versus amplitude is plotted to study the effects of physical properties of graphene sheets. Backbone curves (nonlinear to linear frequency-amplitude curve) are plotted for all edge simply supported and all edges clamped graphene sheets in Fig. 3 and 4, respectively. It is found that the backbone curves do not depend on the nonlocal parameter. Also, it can be seen that the effect of aspect ratio on backbone curves of simply supported nano-plates is more significant than that of clamped nano-plates. Besides, it can be concluded that the in-plane boundary conditions are very important in nonlinear analysis, as these can change the backbone curves quite significantly.

VI. Conclusion

The free nonlinear vibration analysis of single layer graphene sheets has been studied and the small scale effect on the nonlinear behavior of the nano-plates has been investigated. Considering the Eringen nonlocal theory and von Karman hypothesis, the nonlinear equations of motion have been obtained using the Hamilton's principle. The solutions for the large amplitude vibrations of a nano-plate have been found. The effects of nonlocal parameter, boundary conditions, aspect ratio on the nonlinear vibration of graphene sheets have been discussed. It has been seen that the small length scale decreases the nonlinear behavior of the graphene sheets. Also, it has been shown that the backbone curves of the graphene sheets do not depend on nonlocal parameter.

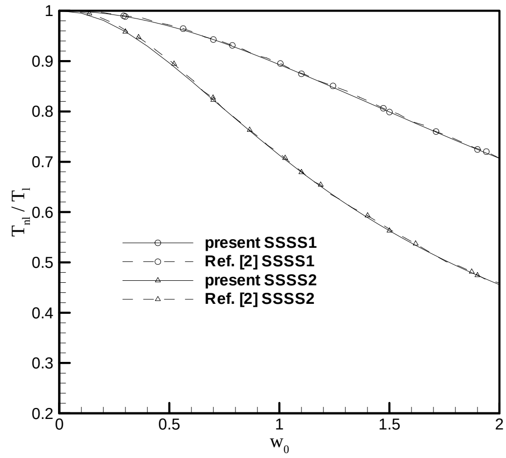

Fig. 1 The comparison of backbone curves of a rectangular plate

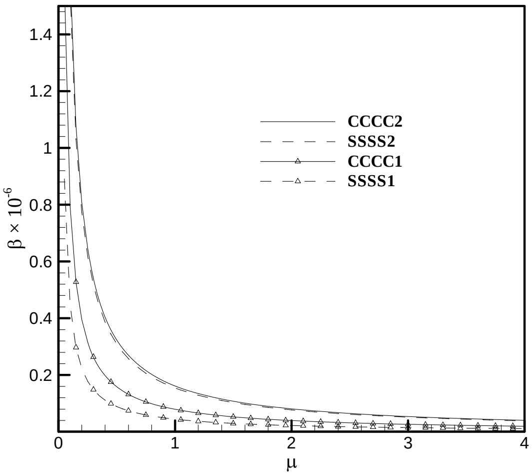

Fig. 2 Influence of nonlocal parameter on the nonlinear behavior of graphene sheets $a = b = 0.01h$

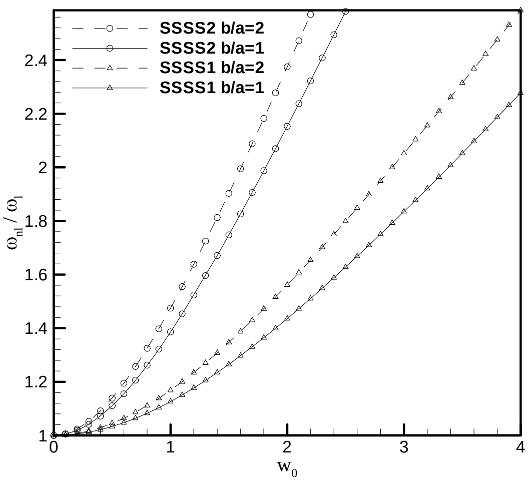

Fig. 3 Backbone curves for all edges simply supported graphene sheets

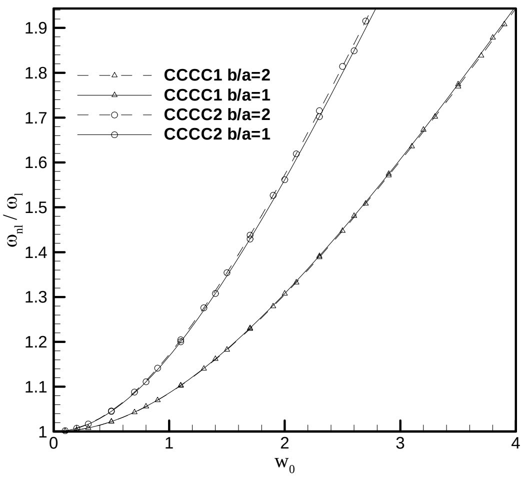

Fig. 4 Backbone curves for all edges clamped graphene sheets

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