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Differential Sandwich Theorems with Generalised Derivative Operator

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***Abstract*—In this paper, a generalized derivatives operator $^{n}_{\lambda,\beta}f$ introduced by the authors will be discussed. Some subordination and superordination results involving this operator for certain normalized analytic functions in the open unit disk will be investigated. Our results extend corresponding previously known results.

***Keywords*—Analytic functions, Univalent functions, Derivative operator, Differential subordination, Differential superordination.

I. Introduction

Denote by $\mathbb{U}$ the unit disk of the complex plane:

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathcal{H}(\mathbb{U})$ be the space of analytic function in $\mathbb{U}$. Let

$$\mathcal{A}_n = \{f \in \mathcal{H}(\mathbb{U}),\ f(z) = z + a_{n+1}z^{n+1} + \cdots,$$

for $(z \in \mathbb{U})$ with $\mathcal{A}_1 = \mathcal{A}$.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we let

$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(\mathbb{U}),\ f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots,\}$$

$(z \in \mathbb{U})$.

If functions f and F are analytic in $\mathbb{U}$, then we say that f is subordinate to F, and write $f \prec F$, if there exists a Schwarz function w analytic in $\mathbb{U}$ with $|w(z)| < 1$ and $w(0) = 0$ such that $f(z) = F(w(z))$ in $\mathbb{U}$. Furthermore, if the function $F(z)$ is univalent in $\mathbb{U}$, then $f(z) \prec F(z)\,(z \in \mathbb{U}) \Leftrightarrow f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.

A function f, analytic in $\mathbb{U}$, is said to be convex if it is univalent and $f(\mathbb{U})$ is convex.

Let $p, h \in \mathcal{H}(\mathbb{U})$ and let $\psi(r,s,t;z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. If p and $\psi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the (second-order) differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \quad (z \in \mathbb{U}) \tag{1}$$

then p is called a solution of the differential superordination (1). (If f subordinate to F, then F is superordinate to f).

An analytic function q is called a subordinant of the differential superodination, or more simply a subordinant if $q \prec p$ for all p satisfying (1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1) is said to be the best subordinant. (Note that the best subordinant is unique up to a rotation of $\mathbb{U}$). Recently Miller and Mocanu [5] obtained conditions on h, q and ψ for which the following implication holds:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \Rightarrow\ q(z) \prec p(z) \qquad (z \in \mathbb{U}).$$

We now state the following definition.

Definition 1.1: [3] Let function f in $\mathcal{A}$, then for $n, \lambda \in \mathbb{N}_0$ and $\beta > 0$, we define the following differential operator

$$\mathfrak{D}^n_{\lambda,\beta}f(z) = z + \sum_{k=2}^{\infty}[1+\beta(k-1)]^n C(k,\lambda)a_k z^k, \quad (z \in \mathbb{U}),$$

where $C(\lambda,k) = \binom{k+\lambda-1}{\lambda}$.
Special cases of this operator includes the Ruscheweyh derivative operator $\mathfrak{D}^0_{\lambda,1}f(z) \equiv D_\lambda$ [6], the Salagean derivative operator $\mathfrak{D}^n_{0,1}f(z) \equiv D^n$ [2], the generalized Salagean derivative operator $\mathfrak{D}^n_{0,\beta}f(z) \equiv D^n_\beta$ [1] and the generalized Ruscheweyh derivative operator $\mathfrak{D}^1_{\lambda,\beta}f(z) \equiv D_{\lambda,\beta}$ [4].

For $n, \lambda \in \mathbb{N}_0$ and $\beta > 0$, we obtain the following inclusion relations:

$$\mathfrak{D}^{n+1}_{\lambda,\beta}f(z) = (1-\beta)\mathfrak{D}^n_{\lambda,\beta}f(z) + \beta z(\mathfrak{D}^n_{\lambda,\beta}f(z))', \tag{2}$$

and

$$z(\mathfrak{D}^n_{\lambda,\beta}f(z))' = (1+\lambda)\mathfrak{D}^n_{\lambda+1,\beta}f(z) - \lambda\mathfrak{D}^n_{\lambda,\beta}f(z) \tag{3}$$

The main object of the present paper is to find sufficient condition for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{D^{n+1}_{\lambda,\beta}f(z)}{D^n_{\lambda,\beta}f(z)} \prec q_2(z),$$

where $n, \lambda \in \mathbb{N}_0, \beta > 0$ and q_1, q_2 are given univalent functions in $\mathbb{U}$. Also, we obtain the number of known results as their special cases.

In order to prove the original results we use shall need the following definition and lemmas. In this paper unless otherwise mentioned α, δ are complex numbers.

Definition 1.2: [5] Denote by $\underline{Q}$, the set of all functions f that are analytic and injective on $\mathbb{U} - E(f)$, where

$$E(f) = \left\{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} f(z) = \infty\right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} - E(f)$.

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Lemma 1.3: [7] Let q be univalent in the unit disk $\mathbb{U}$ and θ and ϕ be analytic in a domain $\mathcal{D}$ containing $q(\mathbb{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set

$$\psi(z) = zq'(z)\phi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + \psi(z).$$

Suppose that

1) $\psi(z)$ is starlike univalent in $\mathbb{U}$, and
2) $\mathrm{Re}\left\{\frac{zh'(z)}{\psi(z)}\right\} > 0$ for $z \in \mathbb{U}$.

If p is analytic with $p(0) = q(0)$, $p(\mathbb{U}) \subseteq \mathcal{D}$, and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \qquad (4)$$

then

$$p(z) \prec q(z)$$

and q is the best dominant.

Lemma 1.4: [8] Let q be convex univalent in the unit disk $\mathbb{U}$ and ϑ and φ be analytic in a domain $\mathcal{D}$ containing $q(\mathbb{U})$.

Suppose that

1) $\mathrm{Re}\left\{\frac{\vartheta'(q(z))}{\varphi(q(z))}\right\} > 0$ for $z \in \mathbb{U}$ and
2) $\psi(z) = zq'(z)\varphi(q(z))$ is starlike univalent in $\mathbb{U}$.

If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$ with $p(\mathbb{U}) \subseteq \mathcal{D}$, and

$\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in $\mathbb{U}$ and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \qquad (5)$$

then $q(z) \prec p(z)$ and q is best subordinant.

II. Subordination results

Using Lemma 1.3, we first prove the following theorem.

Theorem 2.1: Let $n, \lambda \in \mathbb{N}_0$, $\beta > 0$ and $q(z)$ be convex univalent in $\mathbb{U}$ with $q(0) = 1$. Further, assume that

$$\mathrm{Re}\left\{\frac{2\delta q(z)}{\alpha} + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0 \quad (z \in \mathbb{U}). \qquad (6)$$

Let

$$\begin{aligned}\Psi(n,\lambda,\beta,\delta,\alpha;z) \;=\; & \frac{\delta[2-\beta(2+\lambda)]}{\beta}\frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)} \\ & +\delta\beta(\lambda+2)(\lambda+1)\frac{\mathfrak{D}^{n}_{\lambda+2,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)} \\ & -\delta\beta(\lambda+1)^2\frac{\mathfrak{D}^{n}_{\lambda+1,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)} \\ & +[\alpha+\delta(1-\frac{1}{\beta})]\left(\frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)}\right)^2 \\ & -\frac{\delta(1-\beta)[1-\beta(\lambda+1)]}{\beta} \qquad (7)\end{aligned}$$

If $f \in \mathcal{A}$ satisfies

$$\Psi(\lambda,\delta,\alpha;z) \prec \delta zq'(z) + (\delta+\alpha)(q(z))^2 \qquad (8)$$

then

$$\frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)} \prec q(z)$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) = \frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)} \qquad (z \in \mathbb{U}). \qquad (9)$$

Then the function $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. Therefore, by making use of (2), (3) and (4), we obtain

$$\begin{aligned} & \frac{\delta[2-\beta(2+\lambda)]}{\beta}\frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)} + \delta\beta(\lambda+2)(\lambda+1)\frac{\mathfrak{D}^{n}_{\lambda+2,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)} \\ & -\delta\beta(\lambda+1)^2\frac{\mathfrak{D}^{n}_{\lambda+1,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)} + [\alpha+\delta(1-\frac{1}{\beta})]\left(\frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)}\right)^2 \\ & -\frac{\delta(1-\beta)[1-\beta(\lambda+1)]}{\beta} \\ & = \delta zp'(z) + (\delta+\alpha)(p(z))^2 \qquad (10)\end{aligned}$$

By using (10) in (8), we have

$$\delta zp'(z) + (\delta+\alpha)(p(z))^2 \prec \delta zq'(z) + (\delta+\alpha)(q(z))^2.$$

By setting $\theta(w) = \delta w^2$ and $\phi(w) = \alpha$, it can be easily observed that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C} - \{0\}$ and that $\phi(w) \neq 0$. Hence the result now follows by an application if Lemma 1.3.

Corollary 2.2: Let $q(z) = \frac{1+Az}{1+Bz}\,(-1 \leq B < A \leq 1)$ in Theorem 2.1, further assuming that (6) holds. If $f \in \mathcal{A}$ then,

$$\Psi(n,\lambda,\beta,\delta,\alpha;z) \prec \delta\frac{(A-B)z}{(1+Bz)^2} + (\delta+\alpha)\left(\frac{1+Az}{1+Bz}\right)^2,$$

$\Rightarrow \frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)} \prec \frac{1+Az}{1+Bz}$, and $\frac{1+Az}{1+Bz}$ is the best dominant.

Also, let $q(z) = \frac{1+z}{1-z}$, then for $f \in \mathcal{A}$ we have,

$$\Psi(n,\lambda,\beta,\delta,\alpha;z) \prec \frac{2\delta z}{(1-z)^2} + (\delta+\alpha)\left(\frac{1+z}{1-z}\right)^2,$$

$\Rightarrow \frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)} \prec \frac{1+z}{1-z}$, and $\frac{1+z}{1-z}$ is the best dominant.

By taking $q(z) = \left(\frac{1+z}{1-z}\right)^\mu$, $(0 < \mu \leq 1)$, for $f \in \mathcal{A}$, we have

$$\Psi(n,\lambda,\beta,\delta,\alpha;z) \prec \frac{2\delta\mu z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^{\mu-1} + (\alpha+\delta)\left(\frac{1+z}{1-z}\right)^{2\mu},$$

$\Rightarrow \frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)} \prec \left(\frac{1+z}{1-z}\right)^\mu$, and $\left(\frac{1+z}{1-z}\right)^\mu$ is the best dominant.

III. Superordination and sandwich results

Now, by applying Lemma 1.4, we prove the following theorem.

Theorem 3.1: Let q be convex univalent in $\mathbb{U}$ with $q(0) = 1$. Assume that

$$\mathrm{Re}\left\{\frac{2(\delta+\alpha)q(z)q'(z)}{\delta}\right\} > 0. \qquad (11)$$

Let $f \in \mathcal{A}$, $\frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)} \in \mathcal{H}[q(0),1] \cap Q$.

Further, let $\Psi(n,\lambda,\beta,\delta,\alpha;z)$ given by (7) be univalent in $\mathbb{U}$ and

$$(\delta+\alpha)(q(z))^2+\delta zq'(z)\prec\Psi(n,\lambda,\beta,\delta,\alpha;z)$$

then

$$q(z)\prec\frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)},$$

and q is the best subordinant.

Proof. Theorem 3.1 follows by using the same technique to prove Theorem 2.1 and by an application of Lemma 1.4.

By using Theorem 3.1, we have the following corollary.

Corollary 3.2: Let $q(z)=\frac{1+Az}{1+Bz}(-1\le B<A\le 1),\ f\in\mathcal{A}$ and $\frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)}\in\mathcal{H}[q(0),1]\cap Q$. Further assuming that (11) satisfies. If

$$(\delta+\alpha)\left(\frac{1+Az}{1+Bz}\right)^2+\frac{\delta(A-B)z}{(1+Bz)^2}\prec\Psi(n,\lambda,\beta,\delta,\alpha;z)$$

then

$$\frac{1+Az}{1+Bz}\prec\frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)}$$

and $\frac{1+Az}{1+Bz}$, is the best subordinant.

Also, by let $q(z)=\frac{1+z}{1-z},\ f\in\mathcal{A}$ and $\frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)}\in\mathcal{H}[q(0),1]\cap Q$. Further assuming that (11) satisfies. If

$$\frac{2\delta z}{(1-z)^2}+(\delta+\alpha)\Big(\frac{1+z}{1-z}\Big)^2\prec\Psi(n,\lambda,\beta,\delta,\alpha;z),$$

$\Rightarrow \frac{1+z}{1-z}$ and $\frac{1+z}{1-z}\prec\frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)}$, is the best subordinant.

Finally, by taking $q(z)=\left(\frac{1+z}{1-z}\right)^{\mu}$, $(0<\mu\le 1),\ f\in\mathcal{A}$ and $\frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)}\in\mathcal{H}[q(0),1]\cap Q$. Further assuming that (11) satisfies. If

$$\frac{2\delta\mu z}{(1-z)^2}\Big(\frac{1+z}{1-z}\Big)^{\mu-1}+(\alpha+\delta)\Big(\frac{1+z}{1-z}\Big)^{2\mu}\prec\Psi(n,\lambda,\beta,\delta,\alpha;z),$$

$\Rightarrow\left(\frac{1+z}{1-z}\right)^{\mu}$, and $\left(\frac{1+z}{1-z}\right)^{\mu}\prec\frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)}$ is the best subordinant.

Combining the results of differential subordination and superordination, we state the following Sandwich Theorems .

Theorem 3.3: Let q_1 and q_2 be convex univalent in $\mathbb{U}$ and satisfies (11) and (6), respectively. If $f\in\mathcal{A}$, $\frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)}\in\mathcal{H}[q(0),1]\cap Q$ and $\Psi(n,\lambda,\beta,\delta,\alpha;z)$ is univalent in $\mathbb{U}$, and

$$\delta zq_1'(z)+(\delta+\alpha)(q_1(z))^2\prec\Psi(n,\lambda,\beta,\delta,\alpha;z)\delta zq_2'(z)+(\delta+\alpha)(q_2(z))^2, \tag{12}$$

then

$$q_1(z)\prec\frac{D^{n+1}_{\lambda,\beta}f(z)}{D^{n}_{\lambda,\beta}f(z)}\prec q_2(z),$$

and q_1 and q_2 are respectively the best subordinant and best dominant.

For $q_1(z)=\frac{1+A_1z}{1+B_1z}$, $q_2(z)=\frac{1+A_2z}{1+B_2z}$,

where $(-1\le B_2\le B_1<A_1\le A_2\le 1)$,

we have the following corollary.

Corollary 3.4: If $f\in\mathcal{A}$, $\frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)}\in\mathcal{H}[q(0),1]\cap Q$ and

$$\begin{aligned}&\Psi_1(A_1,B_1,n,\lambda,\beta,\delta,\alpha;z)\\&\prec\Psi(n,\lambda,\beta,\delta,\alpha;z)\\&\prec\Psi_2(A_2,B_2,n,\lambda,\beta,\delta,\alpha;z)\end{aligned}$$

then

$$\frac{1+A_1z}{1+B_1z}\prec\frac{\mathfrak{D}^{n+1}_{\lambda,\beta}f(z)}{\mathfrak{D}^{n}_{\lambda,\beta}f(z)}\prec\frac{1+A_2z}{1+B_2z}$$

where

$$\Psi_1(A_1,B_1,n,\lambda,\beta,\delta,\alpha;z):=(\delta+\alpha)\left(\frac{1+A_1z}{1+B_1z}\right)^2+\frac{\delta(A_1-B_1)z}{(1+B_1z)^2},$$

$$\Psi_2(A_2,B_2,n,\lambda,\beta,\delta,\alpha;z):=(\delta+\alpha)\left(\frac{1+A_2z}{1+B_2z}\right)^2+\frac{\delta(A_2-B_2)z}{(1+B_2z)^2}.$$

Hence $\frac{1+A_1z}{1+B_1z}$ and $\frac{1+A_2z}{1+B_2z}$ are respectively the best subordinant and best dominant.

Remark 3.5: For special cases of the above results follows by choosing different values of n,λ and β.

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