Switching rule for the exponential stability and stabilization of switched linear systems with interval time-varying delays

Kreangkri Ratchagit

Abstract—This paper is concerned with exponential stability and stabilization of switched linear systems with interval time-varying delays. The time delay is any continuous function belonging to a given interval, in which the lower bound of delay is not restricted to zero. By constructing a suitable augmented Lyapunov-Krasovskii functional combined with Leibniz-Newton's formula, a switching rule for the exponential stability and stabilization of switched linear systems with interval time-varying delays and new delay-dependent sufficient conditions for the exponential stability and stabilization of the systems are first established in terms of LMIs. Numerical examples are included to illustrate the effectiveness of the results.

Keywords—Switching design, exponential stability and stabilization, switched linear systems, interval delay, Lyapunov function, linear matrix inequalities.

I. INTRODUCTION

Stability analysis of linear systems with time-varying delays $\dot{x}(t) = Ax(t) + Dx(t - h(t))$ is fundamental to many practical problems and has received considarable attention [1-5]. Most of the known results on this problem are derived assuming only that the time-varing delay $h(t)$ is a continuously differentialbe function, satisfying some boundedness condition on its derivative: $\dot{h}(t) \leq \delta < 1$. In delay–dependent stability criteria, the main concerns is to enlarge the feasible region of stability criteria in given time-delay interval. Interval time-varying delay means that a time delay varies in an interval in which the lower bound is not restricted to be zero. By constracting a suitable argumented Lyapunov functionalsand utilizing free weigpht matrices, some less conservative conditions for asymptotic stability are derived in [6-16] for systems with time delay varying in an interval. However, the shortcoming of the method used in these works is that the delay function is assumed to be differential and its derivative is still bounded: $\dot{h}(t) \leq \delta$. This paper gives the improved results for the exponential stability and stabilization of switched linear systems with interval time-varying delay. The time delay is assumed to be a time-varying continuous function belonging to a given interval, but not necessary to be differentiable. Specifically, our goal is to develop a constructive way to design switching rule to the exponential stability and stabilization of switched linear systems with interval time-varying delay. By constructing argumented Lyapunov functionals combined with LMI technique, we propose new criteria for the exponential stability and stabilization of the switched linear system. The

delay-dependent stability conditions are formulated in terms of LMIs, being thus solvable by utilizing Matlab's LMI Control Toolbox available in the literature to date.

The paper is organized as follows: Section 2 presents definitions and some well-known technical propositions needed for the proof of the main results. Delay-dependent exponential stability and stabilization conditions of the switched linear system with numerical examples showing the effectiveness of proposed method are presented in Section 3.

II. PRELIMINARIES

The following notations will be used in this paper. R^+ denotes the set of all real non-negative numbers; R^n denotes the $n-$dimensional space with the scalar product $\langle .,. \rangle$ and the vector norm $\| \, . \, \|$; $M^{n \times r}$ denotes the space of all matrices of $(n \times r)-$dimensions; A^T denotes the transpose of matrix A; A is symmetric if $A = A^T$; I denotes the identity matrix; $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\min/\max}(A) = \min/\max\{\text{Re}\lambda; \lambda \in \lambda(A)\}$; $x_t := \{x(t + s) : s \in [-h, 0]\}$, $\|x_t\| = \sup_{s \in [-h,0]} \| x(t + s) \|$; $C([0, t], R^n)$ denotes the set of all R^n-valued continuous functions on $[0, t]$; Matrix A is called semi-positive definite $(A \geq 0)$ if $\langle Ax, x \rangle \geq 0$, for all $x \in R^n$; A is positive definite $(A > 0)$ if $\langle Ax, x \rangle > 0$ for all $x \neq 0$; $A > B$ means $A - B > 0$. $*$ denotes the symmetric term in a matrix.

Consider a linear system with interval time-varying delay of the form

$$\dot{x}(t) = A_\gamma x(t) + D_\gamma x(t - h(t)), \quad t \in R^+,$$
$$x(t) = \phi(t), t \in [-h_2, 0], \tag{1}$$

where $x(t) \in R^n$ is the state; $\gamma(.) : R^n \to \mathcal{N} := \{1, 2, \ldots, N\}$ is the switching rule, which is a function depending on the state at each time and will be designed. A switching function is a rule which determines a switching sequence for a given switching system. Moreover, $\gamma(x(t)) = i$ implies that the system realization is chosen as the ith system, $i = 1, 2, ..., N$. It is seen that the system (1) can be viewed as an autonomous switched system in which the effective system changes when the state $x(t)$ hits predefined boundaries. $A_i, D_i \in M^{n \times n}, i = 1, 2, ..., N$ are given constant matrices, and $\phi(t) \in C([-h_2, 0], R^n)$ is the initial function with the norm $\| \phi \| = \sup_{s \in [-h_2, 0]} \| \phi(s) \|$; The time-varying delay function $h(t)$ satisfies

$$0 \leq h_1 \leq h(t) \leq h_2, \quad t \in R^+.$$

Kreangkri Ratchagit is with the Department of Mathematics, Maejo University, Chiangmai 50290, Thailand e-mail: kreangkri@mju.ac.th.

The stability problem for switched system (1) is to construct a switching rule that makes the system exponentially stable.

Remark 2.1. It is worth noting that the time delay is a time-varying function belonging to a given interval, in which the lower bound of delay is not restricted to zero.

Definition 2.1. Given $\alpha > 0$. The switched linear system (1) is $\alpha-$exponentially stable if there exists a switching rule $\gamma(.)$ such that every solution $x(t, \phi)$ of the system satisfies the following condition

$$\exists N > 0: \quad \| x(t, \phi) \| \leq N e^{-\alpha t} \| \phi \|, \quad \forall t \in R^+.$$

We end this section with the following technical well-known propositions, which will be used in the proof of the main results.

Definition 2.2. The system of matrices $\{J_i\}, i = 1, 2, \ldots, N$, is said to be strictly complete if for every $x \in R^n \backslash \{0\}$ there is $i \in \{1, 2, \ldots, N\}$ such that $x^T J_i x < 0$.

It is easy to see that the system $\{J_i\}$ is strictly complete if and only if

$$\bigcup_{i=1}^{N} \alpha_i = R^n \backslash \{0\},$$

where

$$\alpha_i = \{x \in R^n : \quad x^T J_i x < 0\}, i = 1, 2, ..., N.$$

We end this section with the following technical well-known propositions, which will be used in the proof of the main results.

Proposition 2.1. [17] *The system* $\{J_i\}, i = 1, 2, \ldots, N$, *is strictly complete if there exist* $\delta_i \geq 0, i = 1, 2, \ldots, N, \sum_{i=1}^{N} \delta_i > 0$ *such that*

$$\sum_{i=1}^{N} \delta_i J_i < 0.$$

If $N = 2$ then the above condition is also necessary for the strict completeness.

Proposition 2.2. (Cauchy inequality) *For any symmetric positive definite marix $N \in M^{n \times n}$ and $a, b \in R^n$ we have*

$$\pm a^T b \leq a^T N a + b^T N^{-1} b.$$

Proposition 2.3. [18] *For any symmetric positive definite matrix $M \in M^{n \times n}$, scalar $\gamma > 0$ and vector function $\omega : [0, \gamma] \to R^n$ such that the integrations concerned are well defined, the following inequality holds*

$$\left(\int_0^\gamma \omega(s)\, ds \right)^T M \left(\int_0^\gamma \omega(s)\, ds \right) \leq \gamma \left(\int_0^\gamma \omega^T(s) M \omega(s)\, ds \right).$$

Proposition 2.4. [19] *Let E, H and F be any constant matrices of appropriate dimensions and $F^T F \leq I$. For any $\epsilon > 0$, we have*

$$EFH + H^T F^T E^T \leq \epsilon E E^T + \epsilon^{-1} H^T H.$$

Proposition 2.5. (Schur complement lemma [20]). *Given constant matrices X, Y, Z with appropriate dimensions satisfying $X = X^T, Y = Y^T > 0$. Then $X + Z^T Y^{-1} Z < 0$ if and only if*

$$\begin{pmatrix} X & Z^T \\ Z & -Y \end{pmatrix} < 0 \quad \text{or} \quad \begin{pmatrix} -Y & Z \\ Z^T & X \end{pmatrix} < 0.$$

III. MAIN RESULTS

Let us set

$$\mathcal{M}_i = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ * & M_{22} & 0 & M_{24} & S_2 \\ * & * & M_{33} & M_{34} & S_3 \\ * & * & * & M_{44} & M_{45} \\ * & * & * & * & M_{55} \end{bmatrix},$$

$$J_i = -S_1 A_i - A_i^T S_1^T, \tag{2}$$

$$\alpha_i = \{x \in R^n : \quad x^T J_i x < 0\}, \quad i = 1, 2, ..., N,$$

$$\bar{\alpha}_1 = \alpha_1, \quad \bar{\alpha}_i = \alpha_i \setminus \bigcup_{j=1}^{i-1} \bar{\alpha}_j, \quad i = 2, 3, \ldots, N,$$

$$\lambda_1 = \lambda_{\min}(P),$$

$$\lambda_2 = \lambda_{\max}(P) + (h_2 - h_1)^2 \lambda_{\max}(U),$$

$$M_{11} = A_i^T P + P A_i + 2\alpha P,$$

$$M_{12} = -S_2 A_i, \quad M_{13} = -S_3 A_i,$$

$$M_{14} = P D_i - S_1 D_i - S_4 A_i, \quad M_{15} = S_1 - S_5 A_i,$$

$$M_{22} = -e^{-2\alpha h_2} U, \quad M_{24} = e^{-2\alpha h_2} U - S_2 D_i,$$

$$M_{33} = -e^{-2\alpha h_2} U, \quad M_{34} = e^{-2\alpha h_2} U - S_3 D_i,$$

$$M_{44} = -S_4 D_i - 2e^{-2\alpha h_2} U, \quad M_{45} = S_4 - S_5 D_i,$$

$$M_{55} = S_5 + S_5^T + (h_2 - h_1)^2 U.$$

The main result of this paper is summarized in the following theorem.

Theorem 3.1. *Given $\alpha > 0$. The zero solution of the switched linear system (1) is $\alpha-$exponentially stable if there exist symmetric positive definite matrices P, U, and matrices $S_i, i = 1, 2, ..., 5$ such that satisfying the following conditions*

(i) $\exists \delta_i \geq 0, i = 1, 2, \ldots, N, \quad \sum_{i=1}^{N} \delta_i > 0 : \sum_{i=1}^{N} \delta_i J_i < 0.$
(ii) $\mathcal{M}_i < 0, \quad i = 1, 2, ..., N.$

Moreover, the solution $x(t, \phi)$ of the system satisfies

$$\| x(t, \phi) \| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} e^{-\alpha t} \| \phi \|, \quad \forall t \in R^+.$$

Proof: We consider the following Lyapunov-Krasovskii functional for the system (1)

$$V(t, x_t) = V_1 + V_2,$$

where

$$V_1 = x^T(t) P x(t),$$

$$V_2 = (h_2 - h_1) \int_{t-h_2}^{t-h_1} \int_{t+s}^{t} e^{2\alpha(\tau - t)} \dot{x}^T(\tau) U \dot{x}(\tau)\, d\tau\, ds.$$

It easy to check that

$$\lambda_1 \parallel x(t) \parallel^2 \le V(t, x_t) \le \lambda_2 \parallel x_t \parallel^2, \quad \forall t \ge 0, \qquad (3)$$

Taking the derivative of V_1 along the solution of system (1) we have

$$\dot{V}_1 = 2x^T(t)P\dot{x}(t)$$
$$= x^T(t)[A_i^T P + A_i P]x(t) + 2x^T(t)PD_i x(t - h(t));$$
$$\dot{V}_2 = (h_2 - h_1)^2 \dot{x}^T(t)U\dot{x}(t)$$
$$- (h_2 - h_1)e^{-2\alpha h_2} \int_{t-h_2}^{t-h_1} \dot{x}^T(s)U\dot{x}(s)\,ds - 2\alpha V_2.$$

Using Proposition 2.2 gives

$$[h_2 - h(t)] \int_{t-h_2}^{t-h(t)} \dot{x}^T(s)U\dot{x}(s)ds \ge$$

$$\Big[\int_{t-h_2}^{t-h(t)} \dot{x}(s)ds \Big]^T U \Big[\int_{t-h_2}^{t-h(t)} \dot{x}(s)ds \Big] \ge$$

$$[x(t - h(t)) - x(t - h_2)]^T U[x(t - h(t)) - x(t - h_2)].$$

Since $h_2 - h(t) \le h_2 - h_1$, we have

$$[h_2 - h_1] \int_{t-h_2}^{t-h(t)} \dot{x}^T(s)U\dot{x}(s)ds \ge$$

$$[x(t - h(t)) - x(t - h_2)]^T U[x(t - h(t)) - x(t - h_2)],$$

then

$$-(h_2 - h_1) \int_{t-h_2}^{t-h(t)} \dot{x}^T(s)U\dot{x}(s)ds \le$$

$$-[x(t - h(t)) - x(t - h_2)]^T U[x(t - h(t)) - x(t - h_2)].$$

Similarly, we have

$$-(h_2 - h_1) \int_{t-h(t)}^{t-h_1} \dot{x}^T(s)U\dot{x}(s)\,ds \le$$

$$-[x(t - h_1) - x(t - h(t))]^T U[x(t - h_1) - x(t - h(t))].$$

Therefore, we have

$$\dot{V}(.) + 2\alpha V(.) \le x^T(t)[A_i^T P + A_i P + 2\alpha P]x(t)$$
$$+ 2x^T(t)PD_i x(t - h(t))$$
$$+ \dot{x}^T(t)[(h_2 - h_1)^2 U]\dot{x}(t)$$
$$- e^{-2\alpha h_2}[x(t - h(t)) - x(t - h_2)]^T U$$
$$[x(t - h(t)) - x(t - h_2)]$$
$$- e^{-2\alpha h_2}[x(t - h_1) - x(t - h(t))]^T U$$
$$[x(t - h_1) - x(t - h(t))].$$

(4)

By using the following identity relation

$$\dot{x}(t) - A_i x(t) - D_i x(t - h(t)) = 0,$$

we have

$$2x^T(t)S_1\dot{x}(t) - 2x^T(t)S_1 A_i x(t)$$
$$-2x^T(t)S_1 D_i x(t - h(t)) = 0$$
$$2x^T(t - h_1)S_2\dot{x}(t) - 2x^T(t - h_1)S_2 A_i x(t)$$
$$-2x^T(t - h_1)S_2 D_i x(t - h(t)) = 0$$
$$2x^T(t - h_2)S_3\dot{x}(t) - 2x^T(t - h_2)S_3 A_i x(t)$$
$$-2x^T(t - h_2)S_3 D_i x(t - h(t)) = 0$$
$$2x^T(t - h(t))S_4\dot{x}(t) - 2x^T(t - h(t))S_4 A_i x(t)$$
$$-2x^T(t - h(t))S_4 D_i x(t - h(t)) = 0$$
$$2\dot{x}^T(t)S_5\dot{x}(t) - 2\dot{x}^T(t)S_5 A_i x(t)$$
$$-2\dot{x}^T(t)S_5 D_i x(t - h(t)) = 0$$

(5)

Adding all the zero items of (5) into (4), we obtain

$$\dot{V}(.) + 2\alpha V(.) \le x^T(t)[A_i^T P + PA_i + 2\alpha P - S_1 A_i$$
$$- A_i^T S_1^T]x(t)$$
$$+ 2x^T(t)[-S_2 A_i]x(t - h_1)$$
$$+ 2x^T(t)[-S_3 A_i]x(t - h_2)$$
$$+ 2x^T(t)[PD_i - S_1 D_i - S_4 A_i]x(t - h(t))$$
$$+ 2x^T(t)[S_1 - S_5 A_i]\dot{x}(t)$$
$$+ x^T(t - h_1)[-e^{-2\alpha h_2}U]x(t - h_1)$$
$$+ 2x^T(t - h_1)[e^{-2\alpha h_2}U - S_2 D_i]x(t - h(t))$$
$$+ 2x^T(t - h_1)S_2\dot{x}(t)$$
$$+ x^T(t - h_2)[-e^{-2\alpha h_2}U]x(t - h_2)$$
$$+ x^T(t - h_2)[e^{-2\alpha h_2}U - S_3 D_i]x(t - h(t))$$
$$+ 2x^T(t - h_2)S_3\dot{x}(t)$$
$$+ x^T(t - h(t))[-S_4 D_i - 2e^{-2\alpha h_2}U]x(t - h(t))$$
$$+ 2x^T(t - h(t))[S_4 - S_5 D_i]\dot{x}(t)$$
$$+ \dot{x}^T(t)[S_5 + S_5^T + (h_2 - h_1)^2 U]\dot{x}(t)$$
$$= x^T(t)J_i x(t) + \zeta^T(t)\mathcal{M}_i\zeta(t),$$

(6)

where

$$\zeta(t) = [x(t), x(t - h_1), x(t - h_2), x(t - h(t)), \dot{x}(t)],$$
$$J_i = -S_1 A_i - A_i^T S_1^T,$$

$$\mathcal{M}_i = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ * & M_{22} & 0 & M_{24} & S_2 \\ * & * & M_{33} & M_{34} & S_3 \\ * & * & * & M_{44} & M_{45} \\ * & * & * & * & M_{55} \end{bmatrix},$$

$$M_{11} = A_i^T P + PA_i + 2\alpha P,$$
$$M_{12} = -S_2 A_i, \quad M_{13} = -S_3 A_i,$$
$$M_{14} = PD_i - S_1 D_i - S_4 A_i, \quad M_{15} = S_1 - S_5 A_i,$$
$$M_{22} = -e^{-2\alpha h_2}U, \quad M_{24} = e^{-2\alpha h_2}U - S_2 D_i,$$
$$M_{33} = -e^{-2\alpha h_2}U, \quad M_{34} = e^{-2\alpha h_2}U - S_3 D_i,$$
$$M_{44} = -S_4 D_i - 2e^{-2\alpha h_2}U, \quad M_{45} = S_4 - S_5 D_i,$$
$$M_{55} = S_5 + S_5^T + (h_2 - h_1)^2 U.$$

Therefore, we finally obtain from (6) and the condition (ii) that

$$\dot{V}(.) + 2\alpha V(.) < x^T(t)J_i x(t), \quad \forall i = 1, 2, ..., N, \quad t \in R^+.$$

We now apply the condition (i) and Proposition 2.1., the system J_i is strictly complete, and the sets α_i and $\bar{\alpha}_i$ by (2) are well defined such that

$$\bigcup_{i=1}^{N} \alpha_i = R^n \backslash \{0\},$$

$$\bigcup_{i=1}^{N} \bar{\alpha}_i = R^n \backslash \{0\}, \quad \bar{\alpha}_i \cap \bar{\alpha}_j = \emptyset, i \neq j.$$

Therefore, for any $x(t) \in R^n$, $t \in R^+$, there exists $i \in \{1, 2, ..., N\}$ such that $x(t) \in \bar{\alpha}_i$. By choosing switching rule as $\gamma(x(t)) = i$ whenever $\gamma(x(t)) \in \bar{\alpha}_i$, from (6) we have

$$\dot{V}(.) + 2\alpha V(.) \leq x^T(t)J_i x(t) < 0, \quad t \in R^+,$$

and hence

$$\dot{V}(t, x_t) \leq -2\alpha V(t, x_t), \quad \forall t \in R^+. \tag{7}$$

Integrating both sides of (7) from 0 to t, we obtain

$$V(t, x_t) \leq V(\phi)e^{-2\alpha t}, \quad \forall t \in R^+.$$

Furthermore, taking condition (3) into account, we have

$$\lambda_1 \parallel x(t, \phi) \parallel^2 \leq V(x_t) \leq V(\phi)e^{-2\alpha t} \leq \lambda_2 e^{-2\alpha t} \parallel \phi \parallel^2,$$

then

$$\parallel x(t, \phi) \parallel \leq \sqrt{\frac{\lambda_2}{\lambda_1}} e^{-\alpha t} \parallel \phi \parallel, \quad t \in R^+,$$

which concludes the proof by the Lyapunov stability theorem [21]. ∎

Based on Theorem 3.1, we have the following result for switched linear control systems with interval time-varying :

$$\dot{x}(t) = A_\gamma x(t) + B_\gamma u(t), \quad t \in R^+,$$
$$x(t) = \phi(t), \quad t \in [-h_2, 0], \tag{8}$$

where $u(t) \in R^m$ is the control input, $A_i, B_i, i = 1, 2, ..., N$, are given constant matrices with appropriate dimensions. We consider a delayed feedback control law

$$u(t) = F_\gamma x(t - h(t)), \tag{9}$$

and F_γ is the controller gain to be determined. Applying the feedback controller (9) to the system (8), the closed-loop time-delay system is

$$\dot{x}(t) = A_\gamma x(t) + B_\gamma F_\gamma x(t - h(t)), \quad t \in R^+,$$
$$x(t) = \phi(t), \quad t \in [-h_2, 0], \tag{10}$$

The time-varying delay function $h(t)$ satisfies

$$0 \leq h_1 \leq h(t) \leq h_2, \quad t \in R^+.$$

The Stabilization problem for switched linear control systems (8) is to construct a switching rule that makes the system exponentially stablilizable.

Definition 3.1. The switched linear control systems (8) is stablilizable if there is a delayed feedback control (9) such that the switched linear systems (10) is exponentially stable.

Let us set

$$\mathcal{M}_i = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ * & M_{22} & 0 & M_{24} & S_2 \\ * & * & M_{33} & M_{34} & S_3 \\ * & * & * & M_{44} & M_{45} \\ * & * & * & * & M_{55} \end{bmatrix},$$

$$J_i = -S_1 A_i - A_i^T S_1^T, \tag{11}$$
$$\alpha_i = \{x \in R^n : \quad x^T J_i x < 0\}, \quad i = 1, 2, ..., N,$$
$$\bar{\alpha}_1 = \alpha_1, \quad \bar{\alpha}_i = \alpha_i \backslash \bigcup_{j=1}^{i-1} \bar{\alpha}_j, \quad i = 2, 3, ..., N,$$
$$\lambda_1 = \lambda_{\min}(P),$$
$$\lambda_2 = \lambda_{\max}(P) + (h_2 - h_1)^2 \lambda_{\max}(U),$$

$$M_{11} = A_i^T P + P A_i + 2\alpha P,$$
$$M_{12} = -S_2 A_i, \quad M_{13} = -S_3 A_i,$$
$$M_{14} = P - S_1 - S_4 A_i, \quad M_{15} = S_1 - S_5 A_i,$$
$$M_{22} = -e^{-2\alpha h_2} U, \quad M_{24} = e^{-2\alpha h_2} U - S_2,$$
$$M_{33} = -e^{-2\alpha h_2} U, \quad M_{34} = e^{-2\alpha h_2} U - S_3,$$
$$M_{44} = -S_4 - 2e^{-2\alpha h_2} U, \quad M_{45} = S_4 - S_5,$$
$$M_{55} = S_5 + S_5^T + (h_2 - h_1)^2 U.$$

Theorem 3.2. *Given* $\alpha > 0$. *The zero solution of the switched linear control systems (8) is* $\alpha-$*exponentially stablilizable by the delayed feedback control (9), where*

$$F_i = B_i^T [B_i B_i^T]^{-1}, \quad i = 1, 2, ..., N,$$

if there exist symmetric positive definite matrices P, U, *and matrices* $S_i, i = 1, 2, ..., 5$ *such that satisfying the following conditions*

(i) $\exists \delta_i \geq 0, i = 1, 2, ..., N, \quad \sum_{i=1}^{N} \delta_i > 0 : \sum_{i=1}^{N} \delta_i J_i < 0.$
(ii) $\mathcal{M}_i < 0, \quad i = 1, 2, ..., N.$

Moreover, the solution $x(t, \phi)$ *of the system satisfies*

$$\parallel x(t, \phi) \parallel \leq \sqrt{\frac{\lambda_2}{\lambda_1}} e^{-\alpha t} \parallel \phi \parallel, \quad \forall t \in R^+.$$

Proof: We consider the following Lyapunov-Krasovskii functional for the system (10)

$$V(t, x_t) = V_1 + V_2,$$

where

$$V_1 = x^T(t) P x(t),$$
$$V_2 = (h_2 - h_1) \int_{t-h_2}^{t-h_1} \int_{t+s}^{t} e^{2\alpha(\tau-t)} \dot{x}^T(\tau) U \dot{x}(\tau) \, d\tau \, ds.$$

It easy to check that

$$\lambda_1 \parallel x(t) \parallel^2 \leq V(t, x_t) \leq \lambda_2 \parallel x_t \parallel^2, \quad \forall t \geq 0, \qquad (12)$$

Taking the derivative of V_1 along the solution of system (10) we have

$$\dot{V}_1 = 2x^T(t)P\dot{x}(t)$$
$$= x^T(t)[A_i^T P + A_i P]x(t) + 2x^T(t)Px(t - h(t));$$
$$\dot{V}_2 = (h_2 - h_1)^2 \dot{x}^T(t)U\dot{x}(t)$$
$$- (h_2 - h_1)e^{-2\alpha h_2} \int_{t-h_2}^{t-h_1} \dot{x}^T(s)U\dot{x}(s)\,ds - 2\alpha V_2.$$

Using Proposition 2.2 gives

$$[h_2 - h(t)] \int_{t-h_2}^{t-h(t)} \dot{x}^T(s)U\dot{x}(s)ds \geq$$
$$\Big[\int_{t-h_2}^{t-h(t)} \dot{x}(s)ds\Big]^T U \Big[\int_{t-h_2}^{t-h(t)} \dot{x}(s)ds\Big] \geq$$
$$[x(t - h(t)) - x(t - h_2)]^T U[x(t - h(t)) - x(t - h_2)]$$

Since $h_2 - h(t) \leq h_2 - h_1$, we have

$$[h_2 - h_1] \int_{t-h_2}^{t-h(t)} \dot{x}^T(s)U\dot{x}(s)ds \geq$$
$$[x(t - h(t)) - x(t - h_2)]^T U[x(t - h(t)) - x(t - h_2)],$$

then

$$-(h_2 - h_1) \int_{t-h_2}^{t-h(t)} \dot{x}^T(s)U\dot{x}(s)ds \leq$$
$$-[x(t - h(t)) - x(t - h_2)]^T U[x(t - h(t)) - x(t - h_2)].$$

Similarly, we have

$$-(h_2 - h_1) \int_{t-h(t)}^{t-h_1} \dot{x}^T(s)U\dot{x}(s)\,ds \leq$$
$$-[x(t - h_1) - x(t - h(t))]^T U[x(t - h_1) - x(t - h(t))].$$

Therefore, we have

$$\dot{V}(.) + 2\alpha V(.) \leq x^T(t)[A_i^T P + A_i P + 2\alpha P)]x(t)$$
$$+ 2x^T(t)Px(t - h(t))$$
$$+ \dot{x}^T(t)[(h_2 - h_1)^2 U]\dot{x}(t)$$
$$- e^{-2\alpha h_2}[x(t - h(t)) - x(t - h_2)]^T U$$
$$[x(t - h(t)) - x(t - h_2)]$$
$$- e^{-2\alpha h_2}[x(t - h_1) - x(t - h(t))]^T U$$
$$[x(t - h_1) - x(t - h(t))].$$
$$(13)$$

By using the following identity relation

$$\dot{x}(t) - A_i x(t) - I x(t - h(t)) = 0,$$

we have

$$2x^T(t)S_1\dot{x}(t) - 2x^T(t)S_1 A_i x(t)$$
$$-2x^T(t)S_1 x(t - h(t)) = 0$$
$$2x^T(t - h_1)S_2\dot{x}(t) - 2x^T(t - h_1)S_2 A_i x(t)$$
$$-2x^T(t - h_1)S_2 x(t - h(t)) = 0$$
$$2x^T(t - h_2)S_3\dot{x}(t) - 2x^T(t - h_2)S_3 A_i x(t)$$
$$-2x^T(t - h_2)S_3 x(t - h(t)) = 0 \qquad (14)$$
$$2x^T(t - h(t))S_4\dot{x}(t) - 2x^T(t - h(t))S_4 A_i x(t)$$
$$-2x^T(t - h(t))S_4 x(t - h(t)) = 0$$
$$2\dot{x}^T(t)S_5\dot{x}(t) - 2\dot{x}^T(t)S_5 A_i x(t)$$
$$-2\dot{x}^T(t)S_5 x(t - h(t)) = 0$$

Adding all the zero items of (14) into (13), we obtain

$$\dot{V}(.) + 2\alpha V(.) \leq x^T(t)[A_i^T P + PA_i + 2\alpha P$$
$$- S_1 A_i - A_i^T S_1^T]x(t)$$
$$+ 2x^T(t)[-S_2 A_i]x(t - h_1)$$
$$+ 2x^T(t)[-S_3 A_i]x(t - h_2)$$
$$+ 2x^T(t)[P - S_1 - S_4 A_i]x(t - h(t))$$
$$+ 2x^T(t)[S_1 - S_5 A_i]\dot{x}(t)$$
$$+ x^T(t - h_1)[-e^{-2\alpha h_2}U]x(t - h_1)$$
$$+ 2x^T(t - h_1)[e^{-2\alpha h_2}U - S_2]x(t - h(t))$$
$$+ 2x^T(t - h_1)S_2\dot{x}(t)$$
$$+ x^T(t - h_2)[-e^{-2\alpha h_2}U]x(t - h_2)$$
$$+ x^T(t - h_2)[e^{-2\alpha h_2}U - S_3]x(t - h(t))$$
$$+ 2x^T(t - h_2)S_3\dot{x}(t)$$
$$+ x^T(t - h(t))[-S_4 - 2e^{-2\alpha h_2}U]x(t - h(t))$$
$$+ 2x^T(t - h(t))[S_4 - S_5]\dot{x}(t)$$
$$+ \dot{x}^T(t)[S_5 + S_5^T + (h_2 - h_1)^2 U]\dot{x}(t)$$
$$= x^T(t)J_i x(t) + \zeta^T(t)\mathcal{M}_i\zeta(t),$$
$$(15)$$

where

$$\zeta(t) = [x(t), x(t - h_1), x(t - h_2), x(t - h(t)), \dot{x}(t)],$$
$$J_i = -S_1 A_i - A_i^T S_1^T,$$

$$\mathcal{M}_i = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ * & M_{22} & 0 & M_{24} & S_2 \\ * & * & M_{33} & M_{34} & S_3 \\ * & * & * & M_{44} & M_{45} \\ * & * & * & * & M_{55} \end{bmatrix},$$

$$M_{11} = A_i^T P + PA_i + 2\alpha P,$$
$$M_{12} = -S_2 A_i, \quad M_{13} = -S_3 A_i,$$
$$M_{14} = P - S_1 - S_4 A_i, \quad M_{15} = S_1 - S_5 A_i,$$
$$M_{22} = -e^{-2\alpha h_2}U, \quad M_{24} = e^{-2\alpha h_2}U - S_2,$$
$$M_{33} = -e^{-2\alpha h_2}U, \quad M_{34} = e^{-2\alpha h_2}U - S_3,$$
$$M_{44} = -S_4 - 2e^{-2\alpha h_2}U, \quad M_{45} = S_4 - S_5,$$
$$M_{55} = S_5 + S_5^T + (h_2 - h_1)^2 U.$$

Therefore, we finally obtain from (15) and the condition (ii) that

$$\dot{V}(.) + 2\alpha V(.) < x^T(t)J_i x(t), \quad \forall i = 1, 2, ..., N, \quad t \in R^+.$$

We now apply the condition (i) and Proposition 2.1., the system J_i is strictly complete, and the sets α_i and $\bar{\alpha}_i$ by (11) are well defined such that

$$\bigcup_{i=1}^{N} \alpha_i = R^n \backslash \{0\},$$

$$\bigcup_{i=1}^{N} \bar{\alpha}_i = R^n \backslash \{0\}, \quad \bar{\alpha}_i \cap \bar{\alpha}_j = \emptyset, i \neq j.$$

Therefore, for any $x(t) \in R^n$, $t \in R^+$, there exists $i \in \{1, 2, \ldots, N\}$ such that $x(t) \in \bar{\alpha}_i$. By choosing switching rule as $\gamma(x(t)) = i$ whenever $\gamma(x(t)) \in \bar{\alpha}_i$, from (15) we have

$$\dot{V}(.) + 2\alpha V(.) \leq x^T(t)J_i x(t) < 0, \quad t \in R^+,$$

and hence

$$\dot{V}(t, x_t) \leq -2\alpha V(t, x_t), \quad \forall t \in R^+. \tag{16}$$

Integrating both sides of (16) from 0 to t, we obtain

$$V(t, x_t) \leq V(\phi)e^{-2\alpha t}, \quad \forall t \in R^+.$$

Furthermore, taking condition (12) into account, we have

$$\lambda_1 \parallel x(t, \phi) \parallel^2 \leq V(x_t) \leq V(\phi)e^{-2\alpha t} \leq \lambda_2 e^{-2\alpha t} \parallel \phi \parallel^2,$$

then

$$\parallel x(t, \phi) \parallel \leq \sqrt{\frac{\lambda_2}{\lambda_1}} e^{-\alpha t} \parallel \phi \parallel, \quad t \in R^+,$$

which concludes the proof by the Lyapunov stability theorem [21]. ∎

To illustrate the obtained result, let us give the following numerical examples.

Example 3.1. Consider the following the switched linear systems with interval time-varying delay (1), where the delay function $h(t)$ is given by

$$h(t) = 0.1 + 0.5749 sin^2 t,$$

and

$$A_1 = \begin{pmatrix} -1 & 0.01 \\ 0.02 & -2 \end{pmatrix}, A_2 = \begin{pmatrix} -1.1 & 0.02 \\ 0.01 & -2 \end{pmatrix},$$

$$D_1 = \begin{pmatrix} -0.1 & 0.01 \\ 0.02 & -0.3 \end{pmatrix}, D_2 = \begin{pmatrix} -0.1 & 0.02 \\ 0.01 & -0.2 \end{pmatrix}.$$

It is worth noting that, the delay function $h(t)$ is non-differentiable. Therefore, the methods used is in $[3, 4, 7]$ are not applicable to this system. By LMI toolbox of Matlab, we find that the conditions (i), (ii) of Theorem 3.1 are satisfied with $h_1 = 0.1, h_2 = 0.6749, \delta_1 = 0.1, \delta_2 = 0.1, \alpha = 0.1$ and

$$P = \begin{pmatrix} 0.9222 & -0.0008 \\ -0.0008 & 0.6540 \end{pmatrix}, U = \begin{pmatrix} 0.5688 & -0.0024 \\ -0.0024 & 0.5201 \end{pmatrix},$$

$$S_1 = \begin{pmatrix} -0.3099 & -0.0099 \\ 0.0121 & -0.1821 \end{pmatrix}, S_2 = \begin{pmatrix} -0.1219 & 0.0080 \\ 0.0135 & -0.1703 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} -0.1219 & 0.0080 \\ 0.0135 & -0.1703 \end{pmatrix}, S_4 = \begin{pmatrix} 0.2300 & -0.0178 \\ -0.0235 & 0.3329 \end{pmatrix},$$

$$S_5 = \begin{pmatrix} -0.5631 & 0.0057 \\ -0.0010 & -0.3965 \end{pmatrix}.$$

In this case, we have

$$(J_1, J_2) = \left(\begin{bmatrix} -0.6195 & -0.0009 \\ -0.0009 & -0.7285 \end{bmatrix}, \begin{bmatrix} -0.6816 & 0.0015 \\ 0.0015 & -0.7288 \end{bmatrix} \right).$$

Moreover, the sum

$$\delta_1 J_1(R, Q) + \delta_2 J_2(R, Q) = \begin{bmatrix} -0.1301 & 0.0001 \\ 0.0001 & -0.1457 \end{bmatrix}$$

is negative definite; i.e. the first entry in the first row and the first column $-0.1301 < 0$ is negative and the determinant of the matrix is positive. The sets α_1 and α_2 are given as

$$\alpha_1 = \{(x_1, x_2) : -0.6195x_1^2 - 0.0018x_1x_2 - 0.7285x_2^2 < 0\},$$
$$\alpha_2 = \{(x_1, x_2) : 0.6816x_1^2 - 0.0030x_1x_2 + 0.7288x_2^2 > 0\}.$$

Obviously, the union of these sets is equal to $R^2 \backslash \{0\}$. The switching regions are defined as

$$\bar{\alpha}_1 = \{(x_1, x_2) : -0.6195x_1^2 - 0.0018x_1x_2 - 0.7285x_2^2 < 0\},$$
$$\bar{\alpha}_2 = \alpha_2 \backslash \bar{\alpha}_1.$$

By Theorem 3.1 the switched linear systems (1) is 0.1−exponentially stable and the switching rule is chosen as $\gamma(x(t)) = i$ whenever $x(t) \in \bar{\alpha}_i$. Moreover, the solution $x(t, \phi)$ of the system satisfies

$$\parallel x(t, \phi) \parallel \leq 1.7315e^{-0.1t} \parallel \phi \parallel, \quad \forall t \in R^+.$$

Example 3.2. Consider the following the switched linear control systems with interval time-varying delay (8), where the delay function $h(t)$ is given by

$$h(t) = 0.1 + 0.7011 sin^2 3t,$$

and

$$A_1 = \begin{pmatrix} -1 & 0.01 \\ 0.02 & -2 \end{pmatrix}, A_2 = \begin{pmatrix} -1.1 & 0.02 \\ 0.01 & -2 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} -0.1 & 0.01 \\ 0.02 & -0.3 \end{pmatrix}, B_2 = \begin{pmatrix} -0.1 & 0.02 \\ 0.01 & -0.2 \end{pmatrix}.$$

It is worth noting that, the delay function $h(t)$ is non-differentiable. Therefore, the methods used is in $[3, 4, 7]$ are not applicable to this system. By LMI toolbox of Matlab, we find that the conditions (i), (ii) of Theorem 3.2 are satisfied with $h_1 = 0.1, h_2 = 0.8011, \delta_1 = 0.1, \delta_2 = 0.1, \alpha = 0.1$ and

$$P = \begin{pmatrix} 0.2151 & 0.0012 \\ 0.0012 & 0.1404 \end{pmatrix}, U = \begin{pmatrix} 0.2376 & 0.0013 \\ 0.0013 & 0.1563 \end{pmatrix},$$

$$S_1 = \begin{pmatrix} -0.0507 & -0.0001 \\ -0.0002 & -0.0412 \end{pmatrix}, S_2 = \begin{pmatrix} 0.1147 & 0.0011 \\ 0.0011 & 0.0458 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 0.1147 & 0.0011 \\ 0.0011 & 0.0458 \end{pmatrix}, S_4 = \begin{pmatrix} -0.2286 & -0.0021 \\ -0.0021 & -0.0943 \end{pmatrix},$$

$$S_5 = \begin{pmatrix} -0.2042 & -0.0018 \\ -0.0018 & -0.0893 \end{pmatrix}.$$

In this case, we have

$$(J_1, J_2) = \left(\begin{bmatrix} -0.1013 & 0.0009 \\ 0.0009 & -0.1650 \end{bmatrix}, \begin{bmatrix} -0.1115 & 0.0010 \\ 0.0010 & -0.1650 \end{bmatrix} \right).$$

Moreover, the sum

$$\delta_1 J_1(R, Q) + \delta_2 J_2(R, Q) = \begin{bmatrix} -0.0213 & 0.0002 \\ 0.0002 & -0.0330 \end{bmatrix}$$

is negative definite; i.e. the first entry in the first row and the first column $-0.0213 < 0$ is negative and the determinant of the matrix is positive. The sets α_1 and α_2 are given as

$$\alpha_1 = \{(x_1, x_2) : -0.1013x_1^2 + 0.0018x_1x_2 - 0.1650x_2^2 < 0\},$$
$$\alpha_2 = \{(x_1, x_2) : 0.1115x_1^2 - 0.0020x_1x_2 + 0.1650x_2^2 > 0\}.$$

Obviously, the union of these sets is equal to $R^2 \setminus \{0\}$. The switching regions are defined as

$$\overline{\alpha}_1 = \{(x_1, x_2) : -0.1013x_1^2 + 0.0018x_1x_2 - 0.1650x_2^2 < 0\},$$
$$\overline{\alpha}_2 = \alpha_2 \setminus \overline{\alpha}_1.$$

By Theorem 3.2 the switched linear systems (8) is $0.1-$exponentially stabilizable and the switching rule is chosen as $\gamma(x(t)) = i$ whenever $x(t) \in \bar{\alpha}_i$, whenever $x(t) \in \bar{\Omega}_i$, the delayed feedback control is:

$$u_1(t) = \begin{bmatrix} -10.0671x_1^1(k - h(t)) - 0.3356x_1^2(k - h(t)) \\ -0.6711x_1^1(k - h(t)) - 3.3557x_1^2(k - h(t)) \end{bmatrix},$$

$$u_2(t) = \begin{bmatrix} -10.1010x_2^1(k - h(t)) - 1.0101x_2^2(k - h(t)) \\ -0.5051x_2^1(k - h(t)) - 5.0505x_2^2(k - h(t)) \end{bmatrix}.$$

Moreover, the solution $x(t, \phi)$ of the system satisfies

$$\| x(t, \phi) \| \le 1.8726e^{-0.1t} \| \phi \|, \quad \forall t \in R^+.$$

IV. CONCLUSION

This paper has proposed a switching design for the exponential stability and stabilization of switched linear systems with interval time-varying delays. Based on the improved Lyapunov-Krasovskii functionals, a switching rule for the exponential stability and stabilization for the system is designed via linear matrix inequalities.

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