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The Vertex and Edge Irregular Total Labeling of an Amalgamation of Two Isomorphic Cycles

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***Abstract*—**Suppose $\boldsymbol{G(V,E)}$ is a graph, a function $\boldsymbol{f: V \cup E \rightarrow \{1,2,3,\cdots,k\}}$ is called the total edge(vertex) irregular $\boldsymbol{k}$-labeling for $\boldsymbol{G}$ such that for each two edges are different having distinct weights. The total edge(vertex) irregularity strength of $\boldsymbol{G}$, denoted by $\boldsymbol{tes(G)(tvs(G)}$, is the smallest $\boldsymbol{k}$ positive integers such that $\boldsymbol{G}$ has a total edge(vertex) irregular $\boldsymbol{k}$-labelling. In this paper, we determined the total edge(vertex) irregularity strength of an amalgamation of two isomorphic cycles. The total edge irregularity strength and the total vertex irregularity strength of two isomorphic cycles on $\boldsymbol{n}$ vertices are $\lceil \boldsymbol{(2n+2)/3} \rceil$ and $\lceil \boldsymbol{2n/3} \rceil$ for $\boldsymbol{n \geq 3}$, respectively.

***Keywords*—**Amalgamation of graphs, irregular labelling, irregularity strength.

I. Introduction

GRAPHS labeling is a topic in graph theory are interesting to study. Object of study in graph labeling in general represented by vertices, edges, and subset of natural number, called *label*. Graphs labeling was first introduced by Sadlacek in 1964 [10], then Stewart in 1966 [11], and Kotzig and Rossa in 1970 [6].

In many cases, the number of all labels associated with an element on a graph called a weight of the element. In the edge labeling, the weight of a vertex is defined as the sum of all the labels associated with that vertex. A type of labeling associated with this is irregular labeling.

The irregular labeling was first introduced by Chartrand et al. in 1986. In formally, the irregular labeling defined as follows. Suppose $G(V,E)$ is a graph. The function

$$f: E \rightarrow \{1,2,3,\cdots,k\}$$

is called *irregular k-labeling* of G, if every two different vertices, x and u in V have distinct weights, that is

$$\sum_{y\in V} f(xy) \neq \sum_{v\in V} f(uv).$$

The *irregularity strength* of G, denoted by $s(G)$, is the smallest positive natural number k such that G have a irregular k-labellings [3].

In 2007, Baca et al. introduced the other type of irregular labeling based on total labeling. For a $G(V,E)$, the function

$$f: V \cup E \rightarrow \{1,2,3,\cdots,k\}$$

is called the *vertex irregular total k-labeling* of G, if the weight of every vertices are distinct, i.e.

$$f(x) + \sum_{y\in V} f(xy)$$

are distinct for every vertex $x \in V$. The *total vertex irregularity strength* or G, denoted by $tvs(G)$, is the smallest positive natural number k such that G have a total vertex irregular k-labeling [1].

There are not many graphs of which their total vertex irregularity strengths are known. Baca *et al*. [1] have determined the total vertex irregularity strengths for some classes of graphs, namely cycles, stars, and prisms. Besides that, Wijaya *et al*. [12] have determined the total vertex irregularity strengths of a complete bipartite graph.

Baca *et al*. [1] derived lower and upper bounds of the total vertex irregularity strength of any tree T with no vertices of degree 2 as described in Theorem A.

Theorem A. *Let T be a tree with t pendant vertices and no vertex of degree* 2. *Then*, $\left\lceil \frac{t+1}{2} \right\rceil \leq tvs(T) \leq t$.

Recently, Nurdin *et al*. [9] determined the total vertex irregularity strength for several types of trees containing vertices of degree 2, namely a subdivision of a star and a subdivision of a particular caterpillar. This paper also derived the total irregularity strength for a complete $k-ary$ tree.

Besides that, Baca *et al*. also introduced the total edge irregularity strength of graph. For a $G(V,E)$, the function

$$f: V \cup E \rightarrow \{1,2,3,\cdots,k\}$$

is called the *edge irregular total k-labeling* of G, if the weight of every edges are distinct, i.e.

$$f(x) + f(xy) + f(y)$$

are distinct for every vertex $x \in V$. The *total edge irregularity strength* of G, denoted by $tes(G)$, is the smallest positive natural number k such that G have a total edge irregular k-labelling [1].

They also derived a lower bound and an upper bound of the total edge irregularity strength for any graph. These bounds are mentioned in the following theorem.

Theorem B. *Let $G = (V,E)$ be a graph with a vertex set V set E, then* $\left\lceil \frac{|E|+2}{3} \right\rceil \leq tes(G) \leq |E|$.

By investigating the maximum degree of any graphs, Baca

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et al. proved the next theorem.

Theorem C. *Let* $G = (V, E)$ *be a graph with the maximum degree* Δ, then

i. $tes\,(G) \geq \left\lceil \frac{\Delta+1}{2} \right\rceil$, *and*

ii. $tes(G) \leq |E| - \Delta, if\ \Delta \leq \frac{|E|-1}{2}$.

In 2007 Ivanco and Jendrol [4] gave a conjecture about the total edge irregularity strength, as follows.

Conjecture. *Let* G *be a graph different from* K_5, *then* $tes(G) = max\left\{\left\lceil \frac{\Delta+1}{2} \right\rceil, \left\lceil \frac{|E(G)|+2}{3} \right\rceil\right\}$.

This conjecture is true for some graphs, i.e.: cycles, paths, stars, wheels, and friendships [1], graphs of linear size [2], trees [4], complete graphs and complete bipartite graphs [5], and the corona of paths with paths, wheels, cycles, stars, gears, or friendships [8].

Some classes of graph have been determined its the total vertex irregularity strength and the total edge irregularity strength. Baca et al. have been determined the total vertex irregularity strength and the total edge irregularity strength of cycle [1]. But the total vertex irregularity strength and the total edge irregularity strength of an amalgamation of cycle not yet found.

II. Amalgamation of a Graph

The formal definition of an amalgamation is as follows. Let G and H be two graphs. Let $u \in V(G)$ and $v \in V(H)$. Then the amalgamation of $G(V, u)$ with $H(V, v)$ is the graph obtained by forming the disjoint union of G and H and then identifying u and v. It is denoted as $Amal(G, H, (u, v))$ [7].

Amalgamation of isomorphic of m cycles graph C_n, denoted by $C_{n,m}$. In this paper we study about irregular labelling of C_n^2.

Suppose the vertex set of C_n^2 is

$$V(C_n^2) = \{x_{i,j} | i = 1,2 \text{ and } j = 1,2,\cdots,n-1\} \cup \{x_n\},$$

and the edge set of C_n^2 is

$$E(C_n^2) = \{x_{i,j}\, x_{i,j+1} | i = 1,2 \text{ and } j = 1,2,\cdots,n-2\} \cup \{x_{i,1}x_n, x_{i,n-1}\, x_n | \; i = 1,2\}.$$

III. Main Results

In this section will be determined the total vertex irregularity strength and the total edge irregularity strength of an amalgamation of two isomorphic cycles. The total vertex irregularity strength of an amalgamation of two isomorphic cycles, denoted by $tvs(C_n^2)$, is

$$tvs(C_n^2) = \left\lceil \frac{2n}{3} \right\rceil,$$

and the total edge irregularity strength of an amalgamation of two isomorphic cycles, denoted by $tes(C_n^2)$, is

$$tes(C_n^2) = \left\lceil \frac{2n+2}{3} \right\rceil$$

for $n \geq 3$.

The proof of two equations above, in Appendix A and Appendix B, respectively.

Appendix A. Proof That $tvs(C_n^2) = \left\lceil \frac{2n}{3} \right\rceil$

Since C_n^2 have $2(n-1)$ vertices of degree two, the largest weight of the vertex at least $2n$.. Since the weight of all vertices is the number of three positive integer number, the largest label used is at least $\left\lceil \frac{2n}{3} \right\rceil$.

Therefore, $tvs(C_n^2) \geq \left\lceil \frac{2n}{3} \right\rceil$.

Next step, we will show that $tvs(C_n^2) \leq \left\lceil \frac{2n}{3} \right\rceil$. Will be construction a total labeling λ on C_n^2 as follows:

for $j = 1,2,\cdots, n-1$,

$\lambda(x_{1,j}) = 1$

$$\lambda(v_2^j) = \begin{cases} 1 & \text{for } j = 1,2,\cdots,n - \left\lceil \frac{2n}{3} \right\rceil - 1 \\ \left\lceil \frac{2n}{3} \right\rceil + j + 1 - n & \text{for others,} \end{cases}$$

$\lambda(x_n) = \left\lceil \frac{2n}{3} \right\rceil$,

$\lambda(x_n x_{1,1}) = 1$,

for $j = 1,2, \cdots, n-2$,

$\lambda(x_{1,j}x_{1,j+1}) = \left\lceil \frac{j+1}{2} \right\rceil$,

$\lambda(x_{1,n-1}x_n) = \left\lceil \frac{n}{2} \right\rceil$,

for $j = 1,2, \cdots, n - \left\lceil \frac{2n}{3} \right\rceil - 1$,

$\lambda(x_{2,j}x_{2,j+1}) = \left\lceil \frac{n+j}{2} \right\rceil$,

and for $j = n - \left\lceil \frac{2n}{3} \right\rceil,\ n - \left\lceil \frac{2n}{3} \right\rceil + 1, \cdots, n-2$, the total labeling λ divides in to 4 cases as follows:

1. $\lambda(x_{2,j}x_{2,j+1}) = \left\lceil \frac{2n}{3} \right\rceil$ for $n = 0\ mod(3)$,
2. $\lambda(x_{2,j}x_{2,j+1}) = \left\lceil \frac{2n}{3} \right\rceil$ for $j = n-t+a,\ n-t+a+1,\cdots,n-2$, where a is a positive even natural number,
3. $\lambda(x_{2,j}x_{2,j+1}) = \left\lceil \frac{2n}{3} \right\rceil - 1$ for $j = n - \left\lceil \frac{2n}{3} \right\rceil + a,\ n - \left\lceil \frac{2n}{3} \right\rceil + a + 1, \cdots, n-2$, where a is a positive odd natural number,
4. $\lambda(x_{2,j}x_{2,j+1}) = \left\lceil \frac{2n}{3} \right\rceil - 1$ for $n = 2\ mod(3)$,

and for edges $x_n x_{2,1}$ and $x_{2,n-1}x_n$, will be defined $\lambda(x_n x_{2,1}) = \left\lceil \frac{n}{2} \right\rceil$ and $\lambda(x_{2,n-1}x_n) = \left\lceil \frac{2n}{3} \right\rceil$.

By definition of λ. Suppose $t = \left\lceil \frac{2n}{3} \right\rceil$, we can show that the weight of all vertices of C_n^2 are

1. $wt(x_{1,1}) = 3$,
2. for $2 \leq j \leq n-1$

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$wt(x_{1,j}) = 1 + \left\lceil \frac{j}{2} \right\rceil + \left\lceil \frac{j+1}{2} \right\rceil$,

3. $wt(x_{2,1}) = 1 + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n+1}{2} \right\rceil$,
4. for $2 \le j \le n - \left\lceil \frac{2n}{3} \right\rceil - 1$,
   $wt(x_{2,j}) = 1 + \left\lceil \frac{n+j-1}{2} \right\rceil + \left\lceil \frac{n+j}{2} \right\rceil$,
5. $wt(x_{2,n-t}) = \left\lceil \frac{2n}{3} \right\rceil + 1 + \left\lceil \frac{2n-t-1}{2} \right\rceil$, $n \equiv 0, 1 \ mod(3)$,
6. $wt(x_{2,n-t}) = \left\lceil \frac{2n}{3} \right\rceil + \left\lceil \frac{2n-t-1}{2} \right\rceil$ with $n \equiv 2 \ mod(3)$,
7. for $n - \left\lceil \frac{2n}{3} \right\rceil + 1 \le j \le n-2$,
   $wt(x_{2,j}) = 3 \left\lceil \frac{2n}{3} \right\rceil + j - n + 1$, with $n \equiv 0 \ mod(3)$,
8. for $n - \left\lceil \frac{2n}{3} \right\rceil + 1 \le j \le n-2$,
   $wt(x_{2,j}) = 3 \left\lceil \frac{2n}{3} \right\rceil + j - n$, with $n \equiv 1 \ mod(3)$,
9. for $n - \left\lceil \frac{2n}{3} \right\rceil + 1 \le j \le n-2$,
   $wt(x_{2,j}) = 3 \left\lceil \frac{2n}{3} \right\rceil + j - n - 1$, with $n \equiv 2 \ mod(3)$,
10. $wt(x_{2,n-1}) = 3 \left\lceil \frac{2n}{3} \right\rceil$ with $n \equiv 0 \ mod(3)$,
11. $wt(x_{2,n-1}) = 3 \left\lceil \frac{2n}{3} \right\rceil - 1$ with $n \equiv 1, 2 \ mod(3)$,
12. $wt(x_n) = 2 \left\lceil \frac{2n}{3} \right\rceil + 1 + \left\lceil \frac{n}{2} \right\rceil$.

By definition of weight of vertices, we showed that the weights of all vertices are distinct.

Next step, we will show that

$$\lambda: V \cup E \to \left\{1, 2, 3, \cdots, \left\lceil \frac{2n}{3} \right\rceil \right\}.$$

By definition of λ, we can showed that:

1. For $1 \le j \le n-1$, $\lambda(x_{1,j}) = 1 < \left\lceil \frac{2n}{3} \right\rceil$.
2. For $1 \le j \le n-1$, $\lambda(x_{2,j}) = 1 < \left\lceil \frac{2n}{3} \right\rceil$.
3. For $n - \left\lceil \frac{2n}{3} \right\rceil \le j \le n-1$, $\lambda(x_{2,j}) < \left\lceil \frac{2n}{3} \right\rceil$.
4. $\lambda(x_n) = \left\lceil \frac{2n}{3} \right\rceil$.
5. $\lambda(x_n x_{1,1}) = 1 < \left\lceil \frac{2n}{3} \right\rceil$.
6. For $1 \le j \le n-2$, $\lambda(x_{1,j} x_{1,j+1}) < \left\lceil \frac{2n}{3} \right\rceil$.
7. $\lambda(x_{1,n-1} x_n) = \left\lceil \frac{n}{2} \right\rceil < \left\lceil \frac{2n}{3} \right\rceil$.
8. For $1 \le j \le n - \left\lceil \frac{2n}{3} \right\rceil - 1$, $\lambda(x_{2,j} x_{2,j+1}) < \left\lceil \frac{2n}{3} \right\rceil$.
9. For $n - \left\lceil \frac{2n}{3} \right\rceil \le j \le n-2$, $\lambda(x_{2,j} x_{2,j+1}) = \left\lceil \frac{2n}{3} \right\rceil$.
10. For $n - \left\lceil \frac{2n}{3} \right\rceil \le j \le n-2$, $n \equiv 1 \ mod(3)$, and $n - \left\lceil \frac{2n}{3} \right\rceil + a \le j \le n-2$ where a is a nonnegative even natural number,
    $\lambda(x_{2,j} x_{2,j+1}) = \left\lceil \frac{2n}{3} \right\rceil$.
11. For $n - \left\lceil \frac{2n}{3} \right\rceil \le j \le n-2$, $n \equiv 1 \ mod(3)$, and $n - \left\lceil \frac{2n}{3} \right\rceil + a \le j \le n-2$ where a is a positive odd natural number,
    $\lambda(x_{2,j} x_{2,j+1}) = \left\lceil \frac{2n}{3} \right\rceil - 1 < \left\lceil \frac{2n}{3} \right\rceil$.
12. For $n - \left\lceil \frac{2n}{3} \right\rceil \le j \le n-2$, $n \equiv 2 \ mod(3)$
    $\lambda(x_{2,j} x_{2,j+1}) = \left\lceil \frac{2n}{3} \right\rceil - 1 < \left\lceil \frac{2n}{3} \right\rceil$.
13. $\lambda(x_n x_{2,1}) = \left\lceil \frac{n}{2} \right\rceil < \left\lceil \frac{2n}{3} \right\rceil$.
14. $\lambda(x_{2,n-1} x_{2,n}) = \left\lceil \frac{2n}{3} \right\rceil$.

Therefore, we find that

$$\lambda: V \cup E \to \left\{1, 2, 3, \cdots, \left\lceil \frac{2n}{3} \right\rceil \right\}.$$

So that, $tvs(C_n^2) \le \left\lceil \frac{2n}{3} \right\rceil$.

Appendix B. Proof That $tes(C_n^2) = \left\lceil \frac{2n+2}{3} \right\rceil$

Since C_n^2 have $2n$ edges, the largest weight of the vertex at least $2n+2$. Since the weight of all edges is the number of three positive integer number, the largest label used is at least $\left\lceil \frac{2n+2}{3} \right\rceil$. Therefore, $tes(C_n^2) \ge \left\lceil \frac{2n+2}{3} \right\rceil$.

Next step, we will show that $tes(C_n^2) \le \left\lceil \frac{2n+2}{3} \right\rceil$. will be construction a total labelling γ on C_n^2 as follows:

$\gamma(x_{1,j}) = \left\lceil \frac{j}{2} \right\rceil$ for $j = 1, 2, 3, \cdots, n-1$,

$\gamma(x_{2,j}) = \gamma(x_{2,n-j}) = \left\lceil \frac{n-1}{2} \right\rceil + j$
for $j = 1, 2, 3, \cdots, \left\lceil \frac{2n+2}{3} \right\rceil - \left\lceil \frac{n-1}{2} \right\rceil$,

$\gamma(x_{2,j}) = \left\lceil \frac{2n+2}{3} \right\rceil$
for $j = \left\lceil \frac{2n+2}{3} \right\rceil + 1 - \left\lceil \frac{n-1}{2} \right\rceil, \cdots, n - \left\lceil \frac{2n+2}{3} \right\rceil + \left\lceil \frac{n-1}{2} \right\rceil + 1$,

$\gamma(x_n) = 2$,

$\gamma(x_n x_{1,1}) = \gamma(x_{1,1} x_{1,2}) = 1$,

$\gamma(x_{1,j} x_{1,j+1}) = 3 + j - \left\lceil \frac{j}{2} \right\rceil - \left\lceil \frac{j+1}{2} \right\rceil$
for $j = 2, 3, \cdots, n-2$,

$\gamma(x_{1,n-1} x_n) = \gamma(x_{n-1} x_{2,1}) = n - \left\lceil \frac{n-1}{2} \right\rceil$,

$\gamma(x_n x_{2,n-1}) = n + 1 - \left\lceil \frac{n-1}{2} \right\rceil$,

$$\gamma(v_2^j) = \begin{cases} n + 2 + 2j - 2\left\lceil \frac{n-1}{2} \right\rceil - 2j \\ \quad \text{for } j = 1, 2, \cdots, \left\lceil \frac{2n+2}{3} \right\rceil - \left\lceil \frac{n-1}{2} \right\rceil - 1 \\ n + 3 + j - \left\lceil \frac{n-1}{2} \right\rceil - \left\lceil \frac{2n+2}{3} \right\rceil \\ \quad \text{for } j = \left\lceil \frac{2n+2}{3} \right\rceil - \left\lceil \frac{n-1}{2} \right\rceil \\ n + j - 2\left\lceil \frac{n-1}{2} \right\rceil \\ \quad \text{for others} \end{cases}$$

$\gamma(x_{2,n-1} x_{2,n-j-1}) = n + 3 - 2\left\lceil \frac{n-1}{2} \right\rceil$
for $j = 1, 2, 3, \cdots, \left\lceil \frac{2n+2}{3} \right\rceil - \left\lceil \frac{n-1}{2} \right\rceil - 1$.

By definition of γ, we found that the weight of all edges of C_n^2 are:

1. $wt(x_{1,1} x_{1,2}) = 3$.
2. $wt(x_n x_{1,1}) = 4$.

3. For $2 \leq j \leq n-2$,
   $wt(x_{1,j}x_{1,j+1}) = 3 + j$.
4. $wt(x_{1,n-1}x_n) = 2 + n$.
5. $wt(x_n x_{2,1}) = 3 + n$.
6. $wt(x_n x_{2,n-2}) = 4 + n$.
7. For $1 \leq j \leq \left\lceil\frac{2n+2}{3}\right\rceil - \left\lceil\frac{n-1}{2}\right\rceil$,
   $wt(x_{2,j}x_{2,j+1}) = 2j + 3 + n$.
8. For $\left\lceil\frac{2n+2}{3}\right\rceil - \left\lceil\frac{n-1}{2}\right\rceil + 1 \leq j \leq n - \left\lceil\frac{2n+2}{3}\right\rceil + \left\lceil\frac{n-1}{2}\right\rceil - 1$,
   $wt(x_{2,j}x_{2,j+1}) = 2\left\lceil\frac{2n+2}{3}\right\rceil + n + j - 2\left\lceil\frac{n-1}{2}\right\rceil$.
9. for $1 \leq j \leq \left\lceil\frac{2n+2}{3}\right\rceil - \left\lceil\frac{n-1}{2}\right\rceil - 1$,
   $wt(x_{2,j}x_{2,j+1}) = 2j + 4 + n$.

By definition of weight of edges, we showed that the weights of all edges are distinct.

Next step, we will show that

$$\gamma: V \cup E \to \left\{1,2,3,\cdots,\left\lceil\frac{2n+2}{3}\right\rceil\right\}.$$

By definition of λ, we can showed that:

1. For $1 \leq j \leq n-1$, $\gamma(x_{1,j}) = \left\lceil\frac{j}{2}\right\rceil \leq \left\lceil\frac{2n+2}{3}\right\rceil$.
2. For $1 \leq j \leq \left\lceil\frac{2n+2}{3}\right\rceil - \left\lceil\frac{n-1}{2}\right\rceil$,
   $$\gamma(x_{2,j}) = \gamma(x_{2,n-j}) = \left\lceil\frac{n-1}{2}\right\rceil + j \leq \left\lceil\frac{2n+2}{3}\right\rceil.$$
3. $\gamma(x_n) = 2 < \left\lceil\frac{2n+2}{3}\right\rceil$.
4. $\gamma(x_n x_{1,1}) = \gamma(x_{1,1}x_{1,2}) = 1 < \left\lceil\frac{2n+2}{3}\right\rceil$.
5. For $2 \leq j \leq n-2$,
   $$\gamma(x_{1,j}x_{1,j+1}) = 3 + j - \left\lceil\frac{j}{2}\right\rceil - \left\lceil\frac{j+1}{2}\right\rceil \leq \left\lceil\frac{2n+2}{3}\right\rceil.$$
6. $\gamma(x_{1,n-1}x_n) = \gamma(x_n x_{2,1}) = n - \left\lceil\frac{n-1}{2}\right\rceil \leq \left\lceil\frac{2n+2}{3}\right\rceil$.
7. $\gamma(x_n x_{2,n-1}) = n + 1 - \left\lceil\frac{n-1}{2}\right\rceil \leq \left\lceil\frac{2n+2}{3}\right\rceil$.
8. For $1 \leq j \leq \left\lceil\frac{2n+2}{3}\right\rceil - \left\lceil\frac{n-1}{2}\right\rceil - 1$,
   $$\gamma(x_{2,j}x_{2,j+1}) = n + 2 - 2\left\lceil\frac{n-1}{2}\right\rceil \leq \left\lceil\frac{2n+2}{3}\right\rceil.$$
9. For $j = \left\lceil\frac{2n+2}{3}\right\rceil - \left\lceil\frac{n-1}{2}\right\rceil$,
   $$\gamma(x_{2,j}x_{2,j+1}) = n + 3 + j - \left\lceil\frac{n-1}{2}\right\rceil - \left\lceil\frac{2n+2}{3}\right\rceil \leq \left\lceil\frac{2n+2}{3}\right\rceil.$$
10. For $\left\lceil\frac{2n+2}{3}\right\rceil - \left\lceil\frac{n-1}{2}\right\rceil + 1 \leq j \leq n - \left\lceil\frac{2n+2}{3}\right\rceil + \left\lceil\frac{n-1}{2}\right\rceil - 1$,
    $$\gamma(x_{2,j}x_{2,j+1}) = n + j - 2\left\lceil\frac{n-1}{2}\right\rceil \leq \left\lceil\frac{2n+2}{3}\right\rceil.$$
11. For $1 \leq j \leq \left\lceil\frac{2n+2}{3}\right\rceil - \left\lceil\frac{n-1}{2}\right\rceil - 1$,
    $$\gamma(x_{2,n-j}x_{2,n-j-1}) = n + 3 - 2\left\lceil\frac{n-1}{2}\right\rceil \leq \left\lceil\frac{2n+2}{3}\right\rceil.$$

Therefore, we find that

$$\gamma: V \cup E \to \left\{1,2,3,\cdots,\left\lceil\frac{2n+2}{3}\right\rceil\right\}.$$

So that, $tes(C_n^2) \leq \left\lceil\frac{2n+2}{3}\right\rceil$.

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