International Journal of Information, Control and Computer Sciences
ISSN: 2517-9942
Vol:1, No:12, 2007

An Approach to Control Design for Nonlinear Systems via Two-stage Formal Linearization and Two-type LQ Controls

Kazuo Komatsu, and Hitoshi Takata

Abstract In this paper we consider a nonlinear control design for nonlinear systems by using two-stage formal linearization and two-type LQ controls. The ordinary LQ control is designed on almost linear region around the steady state point. On the other region, another control is derived as follows. This derivation is based on coordinate transformation twice with respect to linearization functions which are defined by polynomials. The linearized systems can be made up by using Taylor expansion considered up to the higher order. To the resulting formal linear system, the LQ control theory is applied to obtain another LQ control. Finally these two-type LQ controls are smoothly united to form a single nonlinear control. Numerical experiments indicate that this control show remarkable performances for a nonlinear system.

Keywords Formal Linearization, LQ Control, Nonlinear Control, Taylor Expansion, Zero Function.

I. Introduction

A wide range of nonlinear analysis tools have been provided so far (e.g. [1]–[8]), but it is usually uneasy to treat with nonlinear dynamical control systems. In a typical control problem, we may apply a feedback control theory to systems by linearization of Taylor expansion truncated at the first order about the desired equilibrium point(e.g. [7]). This approach is simple and easy to design, but clearly local; that is in general, it can only guarantee asymptotic stability. In order to extend the validity of the linearization approach, there are many studies provided like input-output linearization [6], extended linearization [8], exact linearization [6] and so on. Their conditions to linearize the systems are usually strict and may be difficult to design for real nonlinear systems. On the other hand there is a formal linearization method [9]–[13] to ease off linearizable conditions.

In this paper we propose a nonlinear control design for single input nonlinear systems using the formal linearization method. This approach is based on a combination of LQ controls. On almost linear region around the origin, the ordinary LQ control [14] is used. To the other nonlinear region, another LQ-type control is acquired by making use of formal linearization of two processes as follows. In the first process, we introduce a first stage linearization function which is composed of the polynomials of state variables for the system. By it, a given nonlinear system is transformed into bilinear one with respect to the first stage linearization function using Taylor expansion truncated up to higher order. In the second process, we introduce a second stage linearization function which is made by the linear combination of the first stage linearization function. A zero function is also introduced which is almost zero except for the neighborhood of the origin. From the bilinear system at the first stage, we obtain a formal linear system with respect to the second stage linearization function by using this zero function. Its inversion is easy to calculate because of including the original state itself within the second stage linearization function. To this formal linear system we apply the LQ control theory [14] to get another LQ control which is effective on the nonlinear region. Finally these LQ controls, which are obtained on two different regions, are smoothly united to design a single nonlinear feedback control by selecting functions of sigmoid type.

Numerical experiments of stabilization nonlinear problem are illustrated and indicate that this controller show remarkable performances.

Manuscript received September 10, 2007.

K. Komatsu is with Department of Information and Computer Sciences, Kumamoto National College of Technology, Koshi, Kumamoto 861-1102, Japan (Corresponding author to provide phone: +81-96-242-6103; fax: +81-96-242-6103; e-mail:komatsu@cs.knct.ac.jp).

H. Takata is with Department of Electrical and Electronics Engineering, Kagoshima University, Korimoto, Kagoshima 890-0065, Japan (e-mail: takata@eee.kagoshima-u.ac.jp).

II. Statement of Problem

For the sake of simplicity, we consider a nonlinear control problem using a formal linearization method for scalar systems. For vector systems, it is straightforward. We consider a class of nonlinear systems of the form

$$\Sigma : \dot{x}(t) = f\big(x(t)\big) + bu, \;\; x \in D, \tag{1}$$

where $t > 0$ denotes time, overdot represents derivative with respect to t, x is a state variable, D is domain, $f \in C^N$ is nonlinear function with $f(0) = 0$, b is a constant and u is a input. The D is divided into an almost linear region in the neighborhood of $x = 0$ and the other nonlinear region. For each region, a linearized system is obtained by applying the formal linearization approach so that the LQ control theory is applicable. These two-type LQ controls is smoothly united to form a single nonlinear control.

III. Nonlinear Control by Formal Linearization

A. Control on Almost Linear Region

We design the ordinary LQ control on the almost linear region in the neighborhood of $x = 0$. The given system (1) is linearized by Taylor expansion truncated at the first order about the origin

$$\Sigma_0 : \dot{x} = \bar{A}x + bu \tag{2}$$

where

$$\bar{A} = f'(0) = \frac{\partial}{\partial x} f(x)|_{x=0} .$$

Let a cost function be

$$J_0 = \int_0^\infty (Q_0 x^2 + R_0 u^2) dt \tag{3}$$

where $Q_0 \geq 0$ and $R_0 > 0$. An application of the LQ control theory to this linearized system (2) and (3) yields

$$u_0(x) = -R^{-1} b P_0 x \tag{4}$$

where P_0 satisfies the Riccati equation

$$2P_0\bar{A} + Q_0 - P_0^2 b^2 R_0^{-1} = 0. \tag{5}$$

B. Control on Nonlinear Region

We design another type LQ control on nonlinear region except for the neighborhood of $x = 0$. In this region we exploit a formal linearization method of polynomial type [9]–[13] using Taylor expansion truncating up to the N-th order. To linearize the system, we need two processes and define two types of formal linearization functions. At the first process, a first stage formal linearization function is defined as

$$\begin{aligned} \boldsymbol{\phi}(x) &= [x,\ x^2,\ x^3, \cdots,\ x^N]^T \\ &= [\phi_1(x),\ \phi_2(x),\ \phi_3(x), \cdots,\ \phi_N(x)]^T \end{aligned} \tag{6}$$

where T denotes transpose. The derivative of the element of the $\boldsymbol{\phi}$ is

$$\begin{aligned} \dot{\phi}_i(x) &= i x^{i-1} \dot{x} \\ &= i x^{i-1} \big(f(x) + bu \big) \qquad (i = 1, 2, \cdots, N). \end{aligned} \tag{7}$$

Applying Taylor expansion to the nonlinear function $f(x)$ about $x = 0$, (7) becomes

$$\dot{\phi}_i(x) = i x^{i-1} \Big(f'(0)x + \frac{1}{2!} f''(0)x^2 + \frac{1}{3!} f^{(3)}(0)x^3 + \cdots + bu \Big)$$

where

$$f^{(i)}(0) = \frac{\partial^i}{\partial x^i} f(x)|_{x=0} .$$

Truncating it up to the N-th order yields

$$\begin{aligned} \dot{\phi}_i(x) &\approx i f'(0) x^i + \frac{i}{2!} f''(0) x^{i+1} + \frac{i}{3!} f^{(3)}(0) x^{i+2} + \\ &\quad \cdots + \frac{i}{(N-i+1)!} f^{(N-i+1)}(0) x^N + i b x^{i-1} u \\ &= i f'(0)\phi_i + \frac{i}{2!} f''(0)\phi_{i+1} + \frac{i}{3!} f^{(3)}(0)\phi_{i+2} + \\ &\quad \cdots + \frac{i}{(N-i+1)!} f^{(N-i+1)}(0)\phi_N + i b \phi_{i-1} u. \end{aligned} \tag{8}$$

So a bilinear system with respect to the first stage linearization function

$$\dot{\boldsymbol{\phi}}(x) = A\boldsymbol{\phi}(x) + b \begin{pmatrix} 1 \\ 2\phi_1 \\ \vdots \\ N\phi_{N-1} \end{pmatrix} u \tag{9}$$

is derived where

$$[A_{ij}] = \begin{cases} \Big[\frac{i}{(j-i+1)!} f^{(j-i+1)}(0) \Big] & (i \leq j) \\ [0] & (i > j) \end{cases} ,$$

$$(i, j = 1, 2, \cdots, N).$$

In order to transform the bilinear system (9) into a formal linear system, the second stage linearization function $\boldsymbol{h}$ is defined by the linear combination of $\boldsymbol{\phi}$ as follows

$$\begin{aligned} \boldsymbol{h}(x) = \begin{pmatrix} h_1(x) \\ h_2(x) \\ h_3(x) \\ \vdots \\ h_N(x) \end{pmatrix} &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ C_{21} & C_{22} & \cdots & C_{2N} \\ C_{31} & C_{32} & \cdots & C_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{pmatrix} \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \phi_3(x) \\ \vdots \\ \phi_N(x) \end{pmatrix} \\ &= C\boldsymbol{\phi}(x). \end{aligned} \tag{10}$$

The derivative of $\boldsymbol{h}$ by (9) is

$$\begin{aligned} \dot{\boldsymbol{h}}(x) = C\dot{\boldsymbol{\phi}} &= CA\boldsymbol{\phi}(x) + bC \begin{pmatrix} 1 \\ 2\phi_1 \\ \vdots \\ N\phi_{N-1} \end{pmatrix} u \\ &= CA\boldsymbol{\phi} + b\boldsymbol{g}(x)u \end{aligned} \tag{11}$$

where

$$\boldsymbol{g}(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_N(x) \end{pmatrix} = C \begin{pmatrix} 1 \\ 2\phi_1 \\ \vdots \\ N\phi_{N-1} \end{pmatrix},$$

$$g_i(x) = \begin{cases} 1 & (i = 1) \\ C_{i1} + 2C_{i2}\phi_i + \cdots + NC_{iN}\phi_{N-1} & (2 \leq i \leq N) \end{cases} . \tag{12}$$

This coefficient C_{ij} $(j = 1, 2, \cdots, N)$ in (12) may be determined so that each $g_i(x)$ $(2 \leq i \leq N)$ is approximately zero by the following zero function.

We here introduce a zero function

$$G(M_i, x) = e^{-M_i \sqrt{x^2+\varepsilon}} \tag{13}$$

which is almost zero except for the neighborhood of $x = 0$. Here M_i is a natural number and ε is a small value $(\varepsilon \geq 0)$ (see Fig. 1). Expanding this zero function truncating up to the

International Journal of Information, Control and Computer Sciences
ISSN: 2517-9942
Vol:1, No:12, 2007

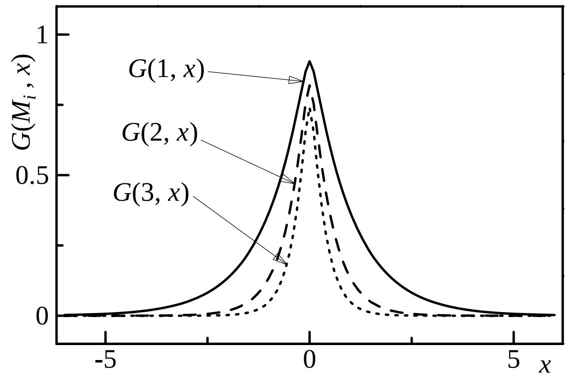

Fig. 1 Zero function

$(N-1)$-th order at $x = x_0$ yields

$$G(M_i, x) \approx G(M_i, x_0) + G'(M_i, x_0)(x - x_0) + \frac{G''(M_i, x_0)}{2!}(x - x_0)^2 + \cdots + \frac{G^{(N-1)}(M_i, x_0)}{(N-1)!}(x - x_0)^{N-1}. \quad (14)$$

Comparing $g_i(x)$ in (12) and this $G(M_i, x)$, C_{ij} $(i = 2, \cdots, N,\ j = 1, 2, \cdots, N)$ is so determined on condition

$$M_i \neq M_k \quad (g_i(x) \neq g_k(x))$$

that C is non-singular. Using this C in (10), the second stage linearization function in (11) is approximated by

$$\dot{\boldsymbol{h}}(x) \approx CA\boldsymbol{\phi}(x) + \begin{pmatrix} b \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u = CAC^{-1}\boldsymbol{h}(x) + \begin{pmatrix} b \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u \quad (15)$$

except for the neighborhood of $x = 0$. Thus, a formal linear system is design as

$$\Sigma_1 : \dot{\boldsymbol{h}}(x) = D\boldsymbol{h}(x) + Bu \quad (16)$$

where

$$D = CAC^{-1}, \ B = [b, \ 0, \cdots, \ 0]^T.$$

Its inversion is simply obtained from (6) and (10) by

$$\hat{x}(t) = [1 \ 0 \ 0 \cdots \ 0]\boldsymbol{h}(x(t)). \quad (17)$$

Let a cost function be

$$J_1 = \int_0^\infty (\boldsymbol{h}^T Q_1 \boldsymbol{h} + R_1 u^2) dt \quad (18)$$

where $Q_1 \geq 0$ and $R_1 > 0$. An application of the LQ control theory to this linearized system (16) and (18) yields

$$u_1(x) = -R^{-1}B^T P_1 \boldsymbol{h}(x) \quad (19)$$

where P_1 satisfies the Riccati equation

$$P_1 D + D^T P_1 + Q_1 - P_1 B R_1^{-1} B^T P_1 = 0. \quad (20)$$

C. Nonlinear Control

The two-type LQ controls of (4) and (19) are smoothly united as follows. We introduce selecting functions of sigmoid type

$$I_0(k, a, x) = 1 - \frac{1}{1 + e^{k(x+a)}} - \frac{1}{1 + e^{-k(x-a)}} \quad (21)$$

to select u_0 in the neighborhood region of $x = 0$, and

$$I_1(k, a, x) = \frac{1}{1 + e^{k(x+a)}} + \frac{1}{1 + e^{-k(x-a)}} \quad (22)$$

to select u_1 in the other region, where a is a proper separation point of the regions (see Fig. (2)).

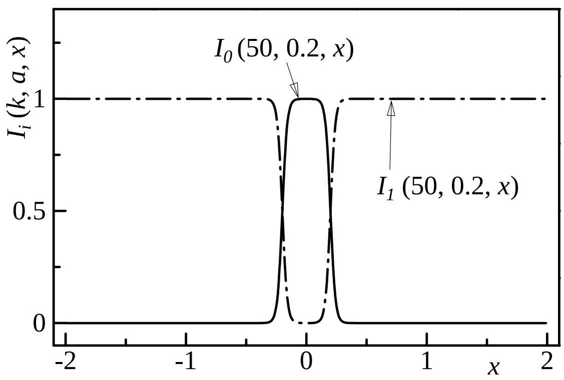

Fig. 2 Selecting functions

Finally we design a single nonlinear feedback control as

$$\hat{u}(x) = u_0(x)I_0(k, a, x) + u_1(x)I_1(k, a, x) \quad (23)$$

using (4) (19) (21) and (22). Thus the closed-loop system becomes

$$\dot{x}(t) = f\Big(x(t)\Big) + b\hat{u}(x). \quad (24)$$

IV. Numerical Experiments

In this section we illustrate numerical experiments of a stabilizing nonlinear feedback control problem. Consider a nonlinear system:

$$\dot{x}(t) = x^2(t) + bu. \quad (25)$$

This system is transformed into a bilinear system with respect to the first stage linearization function $\boldsymbol{\phi}$ of (6). When the order of $\boldsymbol{\phi}$ is $N = 3$, the system is

$$\dot{\boldsymbol{\phi}}(x) = \frac{\partial}{\partial t}\begin{pmatrix} x \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\boldsymbol{\phi}(x) + \begin{pmatrix} 1 \\ 2\phi_1 \\ 3\phi_2 \end{pmatrix} u \quad (26)$$

In order to investigate the accuracy of this first stage linearization, we show the trajectories of the state variable $\hat{x}$ for a free system $\dot{\boldsymbol{\phi}}(x) = A\boldsymbol{\phi}(x)$ when $u = 0$ and its approximated value $\hat{x}$ is obtained by inversion

$$\hat{x}(t) = [1 \ 0 \ 0 \cdots \ 0]\boldsymbol{\phi}(x(t)). \quad (27)$$

Fig. (3) shows true value $x(t)$ and $\hat{x}(t)$ when the order of $\boldsymbol{\phi}(x)$ is varied as $N = 1$ to 6 and an initial value is $x(0) = 0.1$.

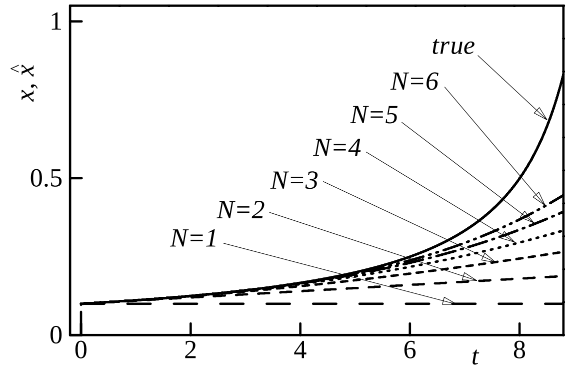

Fig. 3 x and $\hat{x}$ by the first stage linearization

In the second stage, a coefficient C in (15) is determined by approximating $g_i(x)$ $(i = 2, 3, \cdots, N)$ in (12) by the zero function $G(M_i, x)$ in (13). The order of the second stage linearization function $\boldsymbol{h}(x)$ is set $N = 3$ and the parameters are put $M_2 = 2,\ M_3 = 3,\ \varepsilon = 0.01$ and $x_0 = 1.5$ in the zero function. Then C becomes

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0.804 & -0.277 & 0.037 \\ 0.419 & -0.197 & 0.121 \end{pmatrix}$$

and the formal linear system (16) is

$$\dot{\boldsymbol{h}}(x) = \begin{pmatrix} 3.124 & -4.617 & 1.405 \\ 1.611 & 0.461 & -4.728 \\ 0.667 & 1.038 & -3.584 \end{pmatrix} \boldsymbol{h}(x) + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u.$$

Solving the Riccati equation (20) and P_1 is

$$P_1 = \begin{pmatrix} 3.662 & -4.699 & 3.499 \\ -4.699 & 15.27 & -17.516 \\ 3.499 & -17.516 & 22.905 \end{pmatrix}$$

when the parameters of a cost function (18) are

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ R_1 = 1.$$

The LQ control is

$$u_1(t) = -[3.662\ \ -4.699\ 3.499] \begin{pmatrix} h_1(x) \\ h_2(x) \\ h_3(x) \end{pmatrix}$$

$$= -[3.662\ \ -4.699\ 3.499] \begin{pmatrix} x \\ C_{21}x + C_{22}x^2 + C_{23}x^3 \\ C_{31}x + C_{32}x^2 + C_{33}x^3 \end{pmatrix}.$$

On the other hand, in the neighborhood of the origin, the linear system (2) is

$$\dot{x} = u$$

and the LQ control for this system is

$$u_0 = -x$$

when the parameters are

$$Q_0 = 1, R_0 = 1.$$

From the selecting functions (21) and (22), the closed-loop system of (24) becomes

$$\dot{x} = x^2 + u_0(x)I_0(k, a, x) + u_1(x)I_1(k, a, x). \qquad (28)$$

Fig. (4) shows results of time responses of the closed-loop system (28) at $x(0) = 2$ when the order of the formal linear system is varied as $N = 1$ to 6. In this case, parameters of the zero function are set

$$k = 50,\ \ a = 0.2.$$

When the order of the linearization functions is $N = 1$, the linearized system is the same as the ordinary LQ control system. It means that the proposed method can stabilize the system even in the region in which the conventional method can not stabilize. The performance is improved as N increases.

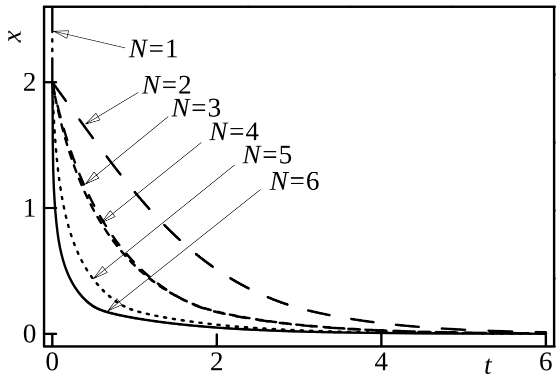

Fig. 4 Results for stabilizing nonlinear system

V. Conclusions

This paper has introduced a nonlinear control for nonlinear systems using two-stage formal linearization based on Taylor expansion truncated up to higher order. This approach is easily applicable to nonlinear systems and relax the linearizability conditions. Numerical experiments show that the proposed approach is effective in stabilizing nonlinear feedback control problem and can improve the performance of the nonlinear control as the order of the formal linear system is increased. Future study is required to clarify the way of selection of parameters such as N, M_i, k and a.

References

[1] R. W. Brockett, "Feedback Invariants for Nonlinear Systems," in *Proc. of IFAC Congress*, Helsinki, 1978, pp.1115–1120.
[2] B. Jakubczyk and W. Respondek, "On the Linearization of Control Systems," *Bull. Acad. Polon. Sci. Ser. Math.*, Vol.28, pp.517–522, 1980.
[3] A. J. Krener, "Approximate Linearization by State Feedback and Coordinate Change," *Systems and Control Letters*, Vol.5, pp.181–185, 1984.
[4] R. Marino, "On the Largest Feedback Linearizable Subsystem," *Systems and Control Letters*, Vol.6, pp.345–351, 1986.
[5] R. R. Kadiyala, "A Tool Box for Approximate Linearization on Nonlinear Systems," *IEEE Control Systems*, Vol.13, No.2, pp.47–57, 1993.
[6] A. Ishidori, *Nonlinear Control Systems, An Introduction*, 3rd ed., Berlin: Springer-Verlag, 1995.
[7] H. K. Khalil, *Nonlinear Systems*, 3rd ed., New Jersey: Prentice Hall, 2002.
[8] W. T. Baumann and W. J. Rugh, "Feedback Control of Nonlinear Systems by Extended Linearization," *IEEE Trans.*, AC-31, 1, pp.40–46, 1986.
[9] H. Takata," Transformation of a Nonlinear System into an Augmented Linear System," *IEEE Trans. on Automatic Control*, Vol.AC-24, No.5, pp.736–741, 1979.
[10] K. Komatsu and H. Takata, "A Computation Method of Formal Linearization for Time-Variant Nonlinear Systems via Chebyshev Interpolation," in *Proc. of the IEEE CDC*, Las Vegas, 2002, pp.4173–4178.
[11] K. Komatsu and H. Takata, "Computer Algorithms of Formal Linearization and Estimation for Time-Variant Nonlinear Systems via Chebyshev Expansion," *Journal of Signal Processing*, Vol.7, No.1, pp.23–29, 2003.
[12] K. Komatsu and H. Takata, "A Formal Linearization for a General Class of Time-Varying Nonlinear Systems and Its Applications," *IEICE Trans.* , Vol.E87-A, No.9, pp.2203–2209, 2004.
[13] K. Komatsu and H. Takata, "Design of Formal Linearization and Observer for Time-Delay Nonlinear Systems," in *Proc. of NOLTA*, Italy, 2006, pp.907–910.
[14] A. P. Sage and C. C. White III, *Optimum Systems Control*, 2nd ed., New Jersey:Prentice Hall, 1977.