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Inferences on Compound Rayleigh Parameters with Progressively Type-II Censored Samples

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***Abstract*—**This paper considers inference under progressive type II censoring with a compound Rayleigh failure time distribution. The maximum likelihood (ML), and Bayes methods are used for estimating the unknown parameters as well as some lifetime parameters, namely reliability and hazard functions. We obtained Bayes estimators using the conjugate priors for two shape and scale parameters. When the two parameters are unknown, the closed-form expressions of the Bayes estimators cannot be obtained. We use Lindley.s approximation to compute the Bayes estimates. Another Bayes estimator has been obtained based on continuous-discrete joint prior for the unknown parameters. An example with the real data is discussed to illustrate the proposed method. Finally, we made comparisons between these estimators and the maximum likelihood estimators using a Monte Carlo simulation study.

***Keywords*—**Progressive type II censoring; Compound Rayleigh failure time distribution; Maximum likelihood estimation; Bayes estimation; Lindley's approximation method; Monte Carlo simulation.

I. Introduction

In the past several decades, censoring is very common in reliability data analysis. It is usually applies when the exact lifetimes are known for only a portion of the products and the remainder of the lifetimes has only partial information. The most common censoring schemes are type I and type II censoring. One important characteristic of these two censoring schemes is that they do not allow for units to be removed from the test at any other point other than the final termination point. However, if an experimenter desires to remove surviving units at points other than the final termination point of the life test, these two traditional censoring schemes will not be of use to the experimenter. The allowance of removing surviving units from the test before the final termination point is desirable, as in the case of studies of wear, in which the study of the actual aging process requires units to be fully disassembled at different stages of the experiment. In addition, when a compromise between the reduced time of experimentation and the observation of at least some extreme lifetimes is sought, such an allowance is also desirable. These reasons lead us into the area of progressive censoring. The scheme of progressive type-II right censoring arises naturally in life-testing experimentation, as it is often desirable to remove live items from experimentation at points other than the final termination point. In this scheme, we begin the test at time zero with n independent live items on test. Immediately following the first observed failure, a fixed number R_1 of surviving items are removed at random from the test. Immediately following the next observed failure, a fixed number R_2 of surviving items are removed at random from the test. This process continues until, immediately following the time of the m^{th} observed failure, the remaining $R_1 = n - R_1 - R_2 - \cdots - R_{m-1}$ items are removed from the test. We will denote the ordered observed failure times by $X_{i;;m;n}^{(R_1,R_2,.....,R_m)}, i = 1,.....,m$ and cal1 them the progressive type-II right censored order statistics of size m from a sample of size n with progressive censoring scheme. It is clear that $n = m + \sum_{i=1}^{m} R_i$ the special case when $R_1 = R_2 = ... = R_{m-1} = 0,$ so that $R_m = n - m$ is the case of conventional type-II right censored sampling. Also when $R_1 = R_2 = ... = R_{m-1} = 0,$ so that $m = n,$ the progressively type-II right censoring scheme reduces to the case of no censoring (ordinary order statistics). Many authors have discussed inference under progressive type-II censored using different lifetime distributions, see for example [5] – [7], [9] and [12]. A thorough overview of the subject of progressive censoring is given in [4], and in the excellent review article by [3].

The two-parameter compound Rayleigh distribution (which is denoted by CRD (α, β)) provides a population model which is useful in several areas of statistics, including life testing and reliability. The probability density function (pdf), and the cumulative distribution function (cdf) of the CRD (α, β) are given, respectively, by

$$f(x, \alpha, \beta) = (2\alpha)\beta^{\alpha} x(\beta + x^2)^{-(\alpha+1)}, x > 0, \alpha, \beta > 0 \tag{1}$$

and

$$F(x, \alpha, \beta) = 1 - (1 + \frac{x^2}{\beta})^{-\alpha}, x > 0, \alpha, \beta > 0 \tag{2}$$

where α and β are the shape and the scale parameters respectively. The compound Rayleigh distribution (CRD) is a special case of the 3-parameter Burr type XII distribution. The two-parameter version of this distribution was studied by several authors, such as [1], [2] among others.

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The reliability function $S(t)$ and hazard (instantaneous failure rate) function $H(t)$ at mission time t for the CRD (α,β) are given respectively by

$$S(t) = (1+\frac{t^2}{\beta})^{-\alpha} \tag{3}$$

$$H(t) = \frac{2\alpha t}{\beta + t^2} \tag{4}$$

In this paper, we assume that the lifetimes have a two-parameter CRD. Based on progressively type-II censoring order statistics arising from it, we obtain and discuss MLE's and Bayesian estimation for the parameters, and some lifetime parameters such as reliability and hazard functions. The remaining of this paper is organized as follows. In Section II, progressively censored samples, the corresponding likelihood function, estimation of the parameters, reliability and hazard functions based on the maximum likelihood method are obtained and discussed. We also, drive the expression for the observed Fisher information of parameters based on standard normal approximation of the distribution of the MLE's. Section III provides Bayes estimation using two types of prior distributions, the first one is the informative continues bivariate prior and using Lindley.s approximation form. The second one is the continuous-discrete prior for the two parameters. In Section IV, for illustrative purposes, we performed a real data analysis. Comparisons among estimators are investigated through Monte Carlo simulations and presented in Section V. Finally, we conclude the paper in Section VI.

II. Maximum Likelihood Estimators (MLE)

If the failure times of the items originally on test with progressive censoring scheme $(R_1, R_2, ...R_m)$ are from a continuous population with cumulative distribution function $F(x)$ and probability density function $f(x)$, then the joint probability density function of a progressively type-II censored sample $\underline{x} = (X_{1:m:n}^{R_1,R_2,R_m}, X_{2:m:n}^{R_1,R_2,R_m}, ..., X_{m:m:n}^{R_1,R_2,R_m})$ is given by

$$f_{1,2,.....m}(x_1, x_2, ..., x_m) = c\prod_{i=1}^{m}\left[f(x_i,\theta)(1-F(x_i,\theta))^{R_i}\right]; \quad -\infty < x_1 < x_2 < .... < x_m < \infty \tag{5}$$

where x_i is used instead of $X_{i:m:n}^{(R_1,R_2,R_m)}$, $R_i \ge 0, (i = 1,2,..., m)$ and

$c = n(n-1-R_1)(n-2-R_1-R_2)...(n-m+1-R_1-...-R_{m-1})$

In this paper, we assume that the underlying failure times follow a two-parameter compound Rayleigh distribution, with pdf and cdf given by (1) and (2) respectively. Substituting (1) and (2) in (5), the likelihood function can be written as

$$L(\alpha,\beta \,|\, \underline{x}) = C(2\alpha)^m u \exp(-\alpha T) \tag{6}$$

where

$$u = \prod_{i=1}^{m}(\frac{x_i}{\beta + x_i^2}). \quad \text{and} \quad T = \sum_{i=1}^{m}(R_i+1)\, In(1+\frac{x_i^2}{\beta}). \tag{7}$$

The log-likelihood function may then be written as

$$\ell(\alpha,\beta \,|\, \underline{x}) = \ln L(\alpha,\beta \,|\, \underline{x}) \propto m\ln(\alpha)$$
$$+\sum_{i=1}^{m}\ln(x_i) - \sum_{i=1}^{m}\ln(\beta + x_i^2) - \alpha\sum_{i=1}^{m}(R_i+1)\ln(1+\frac{x_i^2}{\beta}). \tag{8}$$

Assuming that the parameters α and β are unknown, the MLE $\hat{\alpha}_{ML}$ and $\hat{\beta}_{ML}$ of α and β can be obtained respectively by solving the following likelihood equations;

$$\frac{\partial\ell(\alpha,\beta \,|\, \underline{x})}{\partial\alpha} = \frac{m}{\alpha} - \sum_{i=1}^{m}(R_i+1)\ln(1+\frac{x_i^2}{\beta}) = 0 \tag{9}$$

$$\frac{\partial\ell(\alpha,\beta \,|\, \underline{x})}{\partial\beta} = -\sum_{i=1}^{m}\frac{1}{\beta + x_i^2} + \alpha\sum_{i=1}^{m}(R_i+1)\frac{x_i^2/\beta^2}{1+(x_i^2/\beta^2)} = 0 \tag{10}$$

from (9) we obtain the MLE $\hat{\alpha}_{ML}$ as

$$\hat{\alpha}_{ML} = \frac{m}{\sum_{i=1}^{m}(R_i+1)\ln(1+\frac{x_i^2}{\hat{\beta}_{ML}})} \tag{11}$$

Where $\hat{\beta}_{ML}$ can be obtained by eliminating α between (9) and (10) , and solve the following resulting equation numeric-ally

$$\frac{m}{\sum_{i=1}^{m}(R_i+1)\ln(1+\frac{x_i^2}{\hat{\beta}_{ML}})} - \frac{\sum_{i=1}^{m}\frac{1}{\hat{\beta}_{ML}+x_i^2}}{\sum_{i=1}^{m}(R_i+1)\frac{x_i^2}{\hat{\beta}_{ML}(\hat{\beta}_{ML}+x_i^2)}} = 0 \tag{12}$$

The solution of (12) can be obtained by using the Newton-Raphson method.

For a given t, the MLE of the reliability and hazard functions $\hat{R}_{ML}(t)$ and $\hat{R}_{ML}(t)$ can be obtained by replacing α and β by $\hat{\alpha}_{ML}$ and $\hat{\beta}_{ML}$ in (3) and (4), respectively.

The asymptotic variance-covariance matrix of the MLE for parameters α and β is given by the elements of the Fisher information matrix

$$I_{i,j} = -E\left(\frac{\partial^2\ell(\alpha,\beta|\underline{x})}{\partial\alpha\partial\beta}\right), \quad i,j = 1,2. \tag{13}$$

But, the exact mathematical expressions for the above expectation in (12) do not have exact forms. Therefore, we take the approximate asymptotic variance-covariance matrix for the MLE.

The asymptotic variance-covariance matrix is given by

$$\Sigma=\begin{bmatrix}-\frac{\partial^2\ell(\alpha,\beta)}{\partial\alpha^2} & -\frac{\partial^2\ell(\alpha,\beta)}{\partial\alpha\,\partial\beta}\\ -\frac{\partial^2\ell(\alpha,\beta)}{\partial\beta\,\partial\alpha} & -\frac{\partial^2\ell(\alpha,\beta)}{\partial\beta^2}\end{bmatrix}^{-1}_{[(\alpha=\hat{\alpha}_{ML},\beta=\hat{\beta}_{ML})]}$$

$$=\begin{bmatrix}Var(\alpha) & Cov(\alpha,\beta)\\ Cov(\beta,\alpha) & Var(\beta)\end{bmatrix} \quad (14)$$

With

$$-\frac{\partial^2\ell(\alpha,\beta)}{\partial\alpha^2}=-\frac{m}{\alpha^2} \quad (15)$$

$$-\frac{\partial^2\ell(\alpha,\beta)}{\partial\alpha\partial\beta}=-\sum_{i=1}^{m}(R_i+1)\frac{x_i^2/\beta^2}{1+(x_i^2/\beta)} \quad (16)$$

$$-\frac{\partial^2\ell(\alpha,\beta)}{\partial\beta^2}=-\sum_{i=1}^{m}\frac{1}{\beta+x_i^2}$$

$$-\frac{\alpha}{\beta^3}\sum_{i=1}^{m}x_i^2(R_i+1)[\frac{x_i^2/\beta}{(1+(x_i^2/\beta))^2}-\frac{2}{1+(x_i^2/\beta)}] \quad (17)$$

The asymptotic normality of the MLE can be used to compute the approximate confidence intervals for the parameters α and β. The $100(1-\gamma)$ approximate confidence intervals for α and β are, respectively

$$\hat{\alpha}_{ML}\pm z_{\frac{\gamma}{2}}\sqrt{Var(\alpha)} \quad and \quad \hat{\beta}_{ML}\pm z_{\frac{\gamma}{2}}\sqrt{Var(\beta)} \quad (18)$$

where $z_{\frac{\gamma}{2}}$ is a standard normal variate.

III. Bayes Estimation

In this section, we deal with the problem of Bayes estimation for the parameters α,β reliability and hazard functions under squared error loss (SEL) function. For the prior believes about the scale and shape parameters of the model we consider two cases. The first is the informative continues bivariate prior for the two parameters and the second is the continuous-discrete prior for the two parameters.

A. Informative Continues Bivariate Prior

The prior distribution for the parameters of the model has been taken as a natural conjugate prior. Since the parameter α and β are assumed to be unknown, we suggested a bivariate prior density as the following forms

$$\pi_1(\alpha,\beta)=g_1(\alpha\mid\beta)\,.\,g_1(\beta) \quad (19)$$

where

$$g_1(\alpha\mid\beta)=\frac{\beta^{-\varsigma}\ \alpha^{\varsigma-1}\ \exp(-\alpha/\beta)}{\Gamma(\zeta)}, \qquad \alpha,\beta,\varsigma>0$$

is a gamma prior density function when β is known and

$$g_2(\beta)=\frac{1}{\delta}\exp(-\beta/\delta),\beta,\delta>0$$

is an exponential density function. Here, ζ and δ are assumed to be known and are chosen to reflect prior knowledge about α and β. Therefore, the bivariate prior density function of α and β in (19) can be written as

$$\pi_1(\alpha,\beta)=A^*\beta^{-\zeta}\alpha^{\zeta-1}\exp[-(\frac{\alpha}{\beta}+\frac{\beta}{\delta})], \quad (20)$$

where $A^*=1/\partial\Gamma(\zeta)$

It follows from (6) and (20) that the joint posterior density function of α and β given $\underline{x}$ is proportion to

$$\pi_1^*(\alpha,\beta|\underline{x})\propto L(\alpha,\beta|\underline{x})\cdot\pi_1(\alpha,\beta)$$

$$\propto u\beta^{-\zeta}\alpha^{(m+\zeta)-1}\exp[-\alpha(T+\frac{1}{\beta})-\frac{\beta}{\delta}], \quad (21)$$

where T and u are given in (7).

Under the squared error loss function (SEL), the Bayes estimate of a function $g=g(\alpha,\beta)$ denoted by $\hat{g}_{BS}$ is the posterior mean of g given by

$$\hat{g}_{BS}=E(g|\underline{x})\frac{\int_0^\infty\int_0^\infty g(\alpha,\beta)L(\alpha,\beta|\underline{x})\cdot\pi_1(\alpha,\beta)d\alpha d\beta}{\int_0^\infty\int_0^\infty l(\alpha,\beta|\underline{x})\cdot\pi_1(\alpha,\beta)d\alpha d\beta} \quad (22)$$

In general, the ratio of integrals in (22) can be written in another form as follows

$$E(g|\underline{x})=\frac{\int_0^\infty\int_0^\infty g(\alpha,\beta)\exp[l(\alpha,\beta)+\rho(\alpha,\beta)]d\alpha d\beta}{\int_0^\infty\int_0^\infty \exp[l(\alpha,\beta)+\rho(\alpha,\beta)]d\alpha d\beta} \quad (23)$$

where $l(\alpha,\beta)=\ln[L(\underline{x}|\alpha,\beta)]$ and $\rho(\alpha,\beta)=\ln(\pi_1(\alpha,\beta))$.

The ratio of the two integrals given by (23) cannot be obtained in a closed form. Therefore, we resort to use of a numeric integration technique such as Lindley's approximation. Reference [10] developed approximate procedures for the evaluation of the ratio such that (23). For the two-parameter case (α,β) Lindley's approximation form can be written as

$$\hat{g}(\alpha,\beta)=g(\hat{\alpha},\hat{\beta})+\frac{1}{2}[V+L_{30}W_{12}+L_{21}Z_{12}$$
$$+L_{12}Z_{21}+L_{03}W_{21}]+\rho_1V_{12}+\rho_2V_{21} \quad (24)$$

where

$$V=\sum_{i=1}^{2}\sum_{i=1}^{2}g_{ij}\ \sigma_{ij},\ L_{ds}=\frac{\partial^{d+s}\ l(\alpha,\beta)}{\partial\alpha^d\ \partial\beta^s},$$

$d,s=0,1,2,3$, $d+s=3$

$$\rho_1=\frac{\partial\rho}{\partial\alpha}\ ,\ \rho_2=\frac{\partial\rho}{\partial\beta},\ g_1=\frac{\partial g}{\partial\alpha}\ ,\ g_2=\frac{\partial g}{\partial\beta}$$

$$g_{11}=\frac{\partial^2 g}{\partial\alpha^2}\ ,\ g_{22}=\frac{\partial^2 g}{\partial\beta^2}\ ,\ g_{12}=g_{21}=\frac{\partial^2 g}{\partial\alpha\,\partial\beta}$$

$$V_{ij}=g_i\sigma_{ii}+g_j\sigma_{ji},\quad W_{ij}=(g_i\sigma_{ii}+g_j\sigma_{ij})\sigma_{ii},$$

$i\neq j,\quad i,j=1,2$

and

$$Z_{ij} = 3g_i\, \sigma_{ii}\, \sigma_{ij} + g_j\,(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2), \quad i \neq j, \quad i, j = 1,2\,.$$

where σ_{ij} is the $(i, j)th$ element of the inverse of the Fisher information matrix. Moreover, $\hat{a}$ and $\hat{\beta}$ are the MLEs of α and β and all of the quantities in (24) are evaluated at $(\hat{a}, \hat{\beta})$. Therefore, the elements σ_{ij} can be obtained as following

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \frac{\eta_2}{\Psi} & \frac{O_1}{\Psi} \\ \frac{O_1}{\Psi} & \frac{m/\alpha}{\Psi} \end{bmatrix} \tag{25}$$

where

$$\eta_1 = \frac{2}{\beta} O_1 - O_2, \quad \eta_2 = \alpha\eta_1 - S_1, \quad \Psi = \frac{m}{\alpha^2}\eta_2 - O_1{}^2,$$

$$\varpi_i = \sum_{i=1}^{m} \frac{x_i^2}{\beta(\beta + x_i^2)}, \quad O_1 = \sum_{i=1}^{m}(1+R_i)\,\varpi_i\,,$$

$$O_2 = \sum_{i=1}^{m}(1+R_i)\,\varpi_i^2, \quad O_3 = \sum_{i=1}^{m}(1+R_i)\,\varpi_i^3,$$

$$S_1 = \sum_{i=1}^{m} \frac{1}{(\beta + x_i^2)^2}, \quad S_2 = \sum_{i=1}^{m} \frac{1}{(\beta + x_i^2)^3}\,.$$

In our case of CRD (α, β) using the prior density (20), we obtain

$$\rho(\alpha, \beta) = \ln(\pi_1(\alpha, \beta)) = \ln(A^*) - \zeta\, In(\beta) + (\zeta - 1)\ln(\alpha) - \frac{\alpha}{\beta} - \frac{\beta}{\delta} \tag{26}$$

and then we get

$$\rho_1 = \frac{\zeta - 1}{\alpha} - \frac{1}{\beta}, \text{ and } \quad \rho_2 = \frac{\alpha}{\beta^2} - \frac{\zeta}{\beta} - \frac{1}{\delta}$$

Also, the values of $L_{\eta\zeta} = \frac{\partial^{\eta+\varsigma} l(\alpha, \beta)}{\partial\alpha^{\eta}\partial\beta^{\zeta}}$, $\eta, \zeta = 0,1,2,3,$ $\eta + \zeta = 3,$ can be obtained as follows

$$L_{30} = \frac{2m}{\alpha^3}, \quad L_{21} = 0, \text{ and } L_{12} = O_2 - \frac{2\,O_1}{\beta},$$

$$L_{03} = \frac{6\alpha}{\beta^2} O_1 - \frac{6\alpha}{\beta} O_2 + 2\alpha\, O_3 - 2S_2.$$

It follows from (24) that the Bayes estimate of $g(\alpha, \beta)$ relative to the SEL is

$$\hat{g} = E\big(g(\alpha, \beta)\big) = g(\alpha, \beta) + \Phi + \Omega_1 + \Omega_2 \tag{27}$$

Where

$$\Phi = \frac{1}{2\Psi}[\eta_2 g_{11} + 2O_1 g_{12} + \frac{m}{\alpha^2} g_{22}],$$

$$\Omega_1 = \frac{1}{\Psi}\left[\rho_1(\eta_2\, g_1 + O_1\, g_2) + \rho_2\,(O_1\, g_1 + \frac{m}{\alpha^2} g_2)\right],$$

$$\Omega_2 = \frac{1}{2\Psi^2}\Big[\frac{2m}{\alpha^3}(\eta_2^2\, g_1 + \eta_2\, O_1\, g_2) - \eta_3\,((2O_1^2 + \frac{m}{\alpha^2}\eta_2)\, g_1 + \frac{3m}{\alpha^2} O_1 g_2) + \eta_4\,(\frac{m}{\alpha^2}\, O_1\, g_1 + (\frac{m}{\alpha^2})^2\, g_2)\Big].$$

and

$$\eta_3 = \frac{6}{\beta^2}\, O_1 - \frac{6}{\beta}\, O_2 + 2O_3, \quad \eta_4 = \alpha\eta_3 - 2S_2\,.$$

All of the functions of the right-hand side of Eq. (24) are to be evaluated at $(\hat{a}, \hat{\beta})$. From Eq. (24), we can deduce the values of the Bayes estimates under SEL of various parameters in what follows.

(I) If $g(\alpha, \beta) = \alpha,$ then

$$\hat{\alpha}_{BS} = \alpha + \frac{1}{\Psi}\left[(\eta_2\,\rho_1 + O_1\,\rho_2)\right] + \frac{1}{2\Psi^2}\Big[\frac{m}{\alpha^2}(\frac{2}{\alpha}\eta_2^2 + O_1\eta_4 - \eta_3(\frac{2\alpha^2}{m} O_1^2 + \eta_2))\Big] \tag{28}$$

(II) If $g(\alpha, \beta) = \beta,$ then

$$\hat{\beta}_{BS} = \beta + \frac{1}{\Psi}\left[O_1\,\rho_1 + \frac{m}{\alpha^2}\rho_2)\right] + \frac{1}{2\Psi^2}\Big[\frac{m}{\alpha^2}(\frac{m}{\alpha^2}\eta_4 + \frac{2}{\alpha} O_1\eta_2 - 3O_1\eta_1))\Big] \tag{29}$$

(III) If $g(\alpha, \beta) = S(t) = (1 + \frac{t^2}{\beta})^{-\alpha},$ then

$$\hat{S}_{BS} = S\,(t)\,\{1 + \frac{1}{\Psi}[\rho_1\,(\alpha\, O_1\,\omega_2 - \eta_2\omega_1) + \rho_2\,(\frac{m}{\alpha}\omega_2 - O_1\omega_1)] + \frac{m}{2\alpha}(\alpha - 1)\,\omega_2^2 - 2\,\omega_2\omega_3) + \eta_2\,\omega_1 + O_1\,\omega_2\,(1 - \alpha\omega_1)]$$
$$+ \frac{1}{2\Psi^2}\Big[\frac{-m}{\alpha^2}(\frac{m}{\alpha}\omega_2 - O_1\,\omega_1)\,\eta_4 + \frac{2m}{\alpha^3}(\alpha\, O_1\,\omega_2 - \eta_2\,\omega_1)\,\eta_2 - \eta_1\,\{\frac{3m}{\alpha}\, O_1\,\omega_1 - (2O_1^2 - \frac{m}{\alpha^2}\eta_2)\,\omega_1\}\Big]\} \tag{30}$$

(IV) If $g(\alpha, \beta) = H(t) = \frac{2\alpha t}{\beta + t^2},$ then

$$\hat{H}_{BS} = H\,(t) + \{1 + \frac{1}{\Psi}\left[\rho_1\,(\frac{\eta_2}{\alpha} - O_1\,\omega_3) + \rho_2\,(\frac{O_1}{\alpha} - \frac{m}{\alpha^2}\omega_3)\right]$$

$$+\frac{\omega_3}{\alpha}(\frac{m}{\alpha}\omega_3-O_1)+\frac{1}{\Psi^2}[\frac{-m}{2\alpha^3})(O_1-\frac{m}{\alpha}\omega_3)\eta_4-\frac{m}{\alpha^3}(O_1\omega_3$$

$$-\frac{\eta_2}{\alpha})\eta_2+\eta_1\{\frac{3m}{2\alpha^2}O_1\omega_3-\frac{1}{2\alpha}(2O_1^2+\frac{m}{\alpha^2}\eta_2)\}]\} \tag{31}$$

where

$$\omega_1=\log[1+(t^2/\beta)],\quad \omega_2=\frac{(t/\beta)^2}{1+(t^2/\beta)} \text{ and } \omega_3=\frac{1}{\beta+t^2}.$$

B. Continuous-Discrete Prior

It is clear from the previous section that specifying a general joint prior for α and β leads to computational complexities. In trying to solve this problem and simplify the Bayesian analysis, we assume that the scale parameter β has a discrete prior, while the conditional distribution of α given $\beta=\beta_j$ has a conjugate gamma prior. Now, we suppose that the parameter is β restricted to a .finite number of values, say $\beta_1,\beta_2,...\beta_v$, i.e.,

$$P_r(\beta=\beta_k)=\pi(\beta_k)=l_k,\qquad k=1,2,....v$$

where $\sum_{k=1}^{v} l_k=1, 0\le l_k\le 1$.

Further, suppose that conditional upon $\beta=\beta_k$, $k=1,2,....v$, α has a natural conjugate gamma (a_k,b_k) prior, with density

$$\pi_2(\alpha|a_k,b_k)=\frac{b_k^{ak}}{\Gamma(a_k)}\alpha^{a_k-1}\exp[-b_k\alpha],\alpha>0 \tag{32}$$

Combining the likelihood function in (6), and prior density (32), we obtain the marginal posterior probability of α conditional on $\beta=\beta_k$

$$\pi_2^*(\alpha|\beta=\beta_k)=\frac{\pi_2(\alpha|a_k,b_k)\cdot L(\alpha,\beta_k)}{\int_0^\infty \pi_2(\alpha|a_k,b_k)\cdot L(\alpha,\beta_k)d\alpha}$$
$$=\frac{B_k^{A_k}}{\Gamma(A_k)}\alpha^{A_k-1}\exp[-\alpha B_k],\qquad \alpha>0 \tag{33}$$

where

$$B_k=b_k+T_k,\ A_k=a_k+m,\quad T_k=\sum_{i=1}^{m}(1+R_i)\ln[1+\frac{x_i^2}{\beta_k}]$$

On applying the discrete version of Bayes theorem, the marginal posterior probability distribution of β is given by

$$P_k=\pi^*(\beta_k|T_k)\propto\int_0^\infty l_k(C2^m)u_k\frac{b_k^{a_k}}{\Gamma(a_k)}\alpha^{a_k+m-1}\exp[-\alpha(b_k+T_k)]d\alpha,$$

then

$$P_k=\frac{b_k^{a_k}\Gamma(A_k)}{B_k^{A_k}\Gamma(a_k)}(C2^m)u_k l_k Q_1^{-1} \tag{34}$$

where

$$u_k=\prod_{i=1}^{m}(\frac{x_i}{\beta_k+x_i^2}),\quad Q_1=\sum_{k=1}^{v}\frac{b_k^{a_k}\Gamma(A_k)}{B_k^{A_k}\Gamma(a_k)}(C2^m)u_k l_k$$

The joint posterior density of α and β is

$$\pi^*(\alpha,\beta|\underline{x})=P_k\frac{B_k^{ak}}{\Gamma(A_k)}\alpha^{A_k-1}\exp[-\alpha B_k],\quad \alpha>0 \tag{35}$$

Using the fact that the Bayes estimate of the parameter relative to SEL function $(\cdot)_{BS}$ is the posterior mean, we obtain the Bayes estimates for different parameters as follows:

$$\hat{\alpha}_{BS}=E[\alpha|\underline{x}]=\sum_{k=1}^{v}\int_0^\infty \alpha P_k\pi_2^*(\alpha|\beta=\beta_k)d\alpha$$
$$=\sum_{k=1}^{v}P_k\frac{B_k^{A_k}}{\Gamma(A_k)}\int_0^\infty\alpha^{A_k}\exp[-\alpha B_k]d\alpha$$
$$=\sum_{k=1}^{v}P_k\frac{A_k}{B_k}, \tag{36}$$

$$\hat{\beta}_{BS}=\sum_{k=1}^{v}P_k\,\beta_k, \tag{37}$$

$$\hat{R}_{BS}(t)=\sum_{k=1}^{v}P_k\frac{B_k^{A_k}}{\Gamma(A_k)}\int_0^\infty\alpha^{A_{k-1}}$$
$$\times\exp[-\alpha(B_k+\ln(1+\frac{t^2}{\beta_k}))]d\alpha$$
$$=\sum_{k=1}^{v}P_k\left[1+\frac{\ln(1+\frac{t^2}{\beta_k})}{\beta_k}\right]^{-A_k} \tag{38}$$

and

$$\hat{H}_{BS}(t)=\sum_{k=1}^{v}P_k\frac{B_k^{A_k}}{\Gamma(A_k+t^2)}(\frac{2t}{\beta_k+t^2})\int_0^\infty\alpha^{A_k}\exp[-\alpha B_k]d\alpha$$
$$=\sum_{k=1}^{v}P_k\frac{2tA_k}{B_k(\beta_k+t^2)} \tag{39}$$

IV. Data Analysis and Discussion

To illustrate and to compare the above different estimation procedures, we present the analysis of one real data set represent the survival times of a group of patients given chemotherapy treatments. The computations are performed using Mathematica (ver. 8.0).

Example: In this example, the original data is a subset of data which was reported by [8] and represents the survival times in years of a group of patients given chemotherapy treatment. The data consisting of 46 survival times (in years) for 46 patients are: 0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501,0.507, 0.529, 0.534, 0.540, 0.570, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

Reference [8] shows that the Compound Rayleigh model is acceptable for these data. To illustrate the use of the proposed methods, we have a previous data consisting of 46 survival times from the Compound Rayleigh distribution. Suppose that the predetermined progressively type-II censoring scheme is given by $(R_1 = 20, \quad R_2 = R_3 \quad = ... = R_{25} = 0)$, for simplicity we denoted to this censoring scheme (C.S) by $(20, 24^{\circ})$. Then a progressively type-II censored sample of size 25 out of 46 survival times is obtained as $(X_1, ..., X_{25}) =$ 0.047, 0.121, 0.132, 0.260, 0.282, 0.334, 0.395, 0.458, 0.540, 0.570, 0.641, 0.644, 0.863, 1.099, 1.326, 1.485, 1.553, 2.178, 2.343, 2.416, 3.578, 3.658, 3.743, 3.978, 4.033. For this example, 21 patient's survival times are censored, and 25 times are observed.

Maximum likelihood estimates: The MLE's $\hat{\alpha}_{ML}, \hat{\beta}_{ML}$ of the parameters α and β using the Newton-Raphson method when solving (11) and (12) are obtained. The MLE's $\hat{R}_{ML}(t)$ and $\hat{H}_{ML}(t)$ of the reliability and hazard functions are obtained at $(t = 1.5)$ by substituted the resulting MLE's of the parameters into (3) and (4). We also compute the local estimate of the variance-covariance matrix by inverting the observed Fisher information matrix (14), and used that to construct two-sided approximate 95% confidence intervals (C.I) for the parameters using (17). The results are obtained to be:

$\hat{\alpha}_{ML} = 0.563, \quad 95\% C.I = (0.334, 0829); \quad \hat{\beta}_{ML} = 0.266,$
$95\% C.I = (0.114, 0.428), \ \hat{R}_{ML}(t) = 0.282, \ \hat{H}_{ML}(t) = 0.672$

Bayes estimates: For the Bayesian approach we consider two case:

(i) *Informative continuous bivariate prior.* We assume that $\zeta = 1$ and $\delta = 2$, The Bayes point estimates under SEL $(.)_{BS}$ of $\alpha, \beta, S(t)$ and $H(t)$ are calculated using Lindley's approximation forms in (28-31) and to be.

$\hat{\alpha}_{BS} = 0.638, \ \hat{\beta}_{BS} = 0.322, \ \hat{R}_{BS}(t) = 0.271, \ \hat{H}_{BS}(t) = 0.743$

(ii) Continuous-discrete prior. To implement the calculations in this case, it is first necessary to elicit the values of the hyperparameters (a_k, b_k) in the prior (32), for $k = 1,2,....,v$, It is necessary to condition prior beliefs about α on each β_k in turn, and this can be difficult in practice. An alternative method for obtaining the values (a_k, b_k) can be based on the expected value of the reliability function $S(t)$ conditional on $\beta = \beta_k, \quad k = 1,2,..v$, which is given using (3) and (32) by

$$E_{|\alpha|\beta=\beta_k} = \beta_k[S(t)] = \int_0^{\infty} (1 + \frac{t^2}{\beta_k})^{-\alpha} \pi_2(\alpha | a_k, b_k) d\alpha$$

$$= \frac{b_k^{a_k}}{\Gamma(a_k)} \int_0^{\infty} \alpha^{a_k - 1} \exp[-\alpha(\ln(1 + \frac{t^2}{\beta_k}) + b_k)] d\alpha$$

$$= \left[\frac{\ln(1 + \frac{t^2}{\beta_k}) + b_k)}{b_k} \right]^{-\alpha_k} , \quad k = 1,2,....., v \qquad (40)$$

Now, suppose that prior beliefs about the lifetime distribution enable one to specify values $(S(t_1), t_1), (S(t_2), t_2)$. Thus, for the two prior values $S(t = t_1)$ and $S(t = t_2)$ the values of a_k and b_k for each value β_k, can be obtained numerically from (51). If there is no prior beliefs, a nonparametric procedure can be use to estimate the corresponding two different values of $S(t)$, see [11]. In this example, a nonparametric procedure can be used as follows

1. based on the above 46 survival times, we estimate two values of the reliability function using a nonparametric procedure $S(t_i) = 1 - i/(n+1), \quad i = 1,2,....46,$ as follows, see [11].

$$S(t_1 = 0.047) = 1 - i/(n+1) = 1 - (1/47) = 0.979, \text{ and}$$
$$S(t_2 = 1.219) = 1 - (27/47) = 0.426. \qquad (41)$$

2. concerning the value of the MLE of the parameter $\beta, (\hat{\beta}_{ML} = 0.266),$, we assume that $\beta_k (k = 1,2,....,5)$ takes the values, 0.1, 0.2, 0.28, 0.35, 0.40, with equal probability (0.2) for each.
3. the two prior values obtained in step 1 are substituted into (40), where a_k and b_k are solved numerically for each given $k, k = 1,2,...5$, using the Newton-Raphson method. Table I gives the values of the hyperparameters and the posterior probabilities derived for each β_k .The Bayes estimators under SEL $(.)_{BS}$ for the parameters α and β reliability function $S(t)$ and hazard function $H(t)$ are computed using results outlined in subsection (4.2). The results are

$\hat{\alpha}_{BS} = 0.612, \ \hat{\beta}_{BS} = 0.306, \ \hat{R}_{BS}(t) = 0.289, \ \hat{H}_{BS}(t) = 0.717$

TABLE I
PRIOR INFORMATION, HYPER PARAMETER VALUES AND THE POSTERIOR PROBABILITIES

k	1	2	3	4	5
l_k	0.2	0.2	0.2	0.2	0.2
β_k	0.10	0.20	0.28	0.35	0.40
a_k	6.536	2.881	6.032	5.094	5.637
b_k	5.536	2.221	6.835	5.447	5.447
T_k	102.175	80.456	70.581	65.117	61.762
u_k	6.6×10^{-5}	4.8×10^{-9}	1.7×10^{-11}	$2\ .9 \times\ 10$	2.3×10^{-14}
P_k	0.012	0.309	0.309	0.281	0.289

V. Simulation Study

To see how the MLEs and the Bayes estimators compare, we carried out a Monte Carlo simulation. We compared the Bayes estimators to the MLEs in terms of bias and means squared error (MSE), for different sample sizes and censoring schemes. For a particular n, m and a censoring scheme R, we generate a progressively censored sample from the C.R distribution with $\alpha = 000$ and $\beta = 000$ using the algorithm presented in [4] according to the following steps:

1. Generate m independent Uniform $(0,1)$ observations $W_1, W_2, ..., W_m$.
2. Determine the values of the censored scheme R_i for $i = 1,2,..., m$.
3. Set $E_i = 1/(i + \sum_{j=m-i+1}^{m} R_j)$ for $i = 1,2,..., m$.
4. Set $V_i = W_i^{E_i}$ for $i = 1,2,..., m$.
5. Set $U_{i,m,n} \equiv U_i = 1 - V_m . V_{m-1} ... V_{m-i+1}$ for $i = 1,2,..., m$. Then, $U_1, U_2, ... U_m$. is the required progressively type-II right censored sample from the Uniform $(0,1)$ distribution.
6. Finally, for $(\alpha = 3,\ \beta = 0.5)$,. we set $X_{i,m,n} \equiv X_i = F^{-1}(U_i)$ $= [\beta(1-(1-U_i)^{-1/\alpha})]^{\frac{1}{2}}$, for $i = 1,2,..., m$.

The resulting sample $X_1, X_2, ..., X_m$ is the required progressively type-II right censored sample from the CRD.

Using the algorithm described above, random progressively type-II censored samples of various sizes & censoring schemes are generated from the CRD with $(\alpha = 2.5, \beta = 0.5)$. In each case, we compute the MLEs and the Bayes estimators of the parameters α and β reliability function $S(t)$ and hazard function $H(t)$. We replicate the process 1000 times and compute the estimated risks (ER) computed by averaging the squared deviations over the repetitions. The results, up to three decimal places, are reported in Tables II and III.

TABLE II
The ER of the Estimates for α and β with $n = 50$

m	Scheme	α	β	α	β	α	β
		ML		Bayes			
				Lindley		Continuous-discrete prior	
30	$(20, 29^{*0})$	0.089	0.082	0.051	0.044	0.037	0.022
30	$(29^{*0}, 20)$	0.100	0.094	0.077	0.056	0.042	0.036
30	$(15^{*0}, 20, 14^{*0})$	0.101	0.096	0.083	0.066	0.063	0.054
20	$(30, 19^{*0})$	0.118	0.110	0.104	0.095	0.078	0.066
20	$(19^{*0}, 30)$	0.126	0.121	0.111	0.100	0.096	0.072
20	$(10^{*0}, 30, 9^{*0})$	0.137	0.129	0.117	0.112	0.106	0.091

TABLE III
The ER of the Estimates for $S(t)$ and $H(t)$ with t=1.5 and n=50

m	Scheme	$S(t)$	$H(t)$	$S(t)$	$H(t)$	$S(t)$	$H(t)$
		ML		Bayes			
				Lindley		Continuous-discrete prior	
30	$(20, 29^{*0})$	0.088	0.012	0.007	0.009	0.007	0.008
30	$(29^{*0}, 20)$	0.016	0.024	0.022	0.013	0.021	0.011
30	$(15^{*0}, 20, 14^{*0})$	0.021	0.046	0.031	0.018	0.028	0.012
20	$(30, 19^{*0})$	0.025	0.110	0.017	0.018	0.014	0.016
20	$(19^{*0}, 30)$	0.041	0.121	0.024	0.027	0.020	0.021
20	$(10^{*0}, 30, 9^{*0})$	0.059	0.129	0.036	0.040	0.031	0.035

VI. Conclusions

Censoring is a common phenomenon in life-testing, and reliability studies. The subject of progressive censoring has received considerable attention in the past few years, due in part to the availability of high speed computing resources, which make it both a feasible topic for simulation studies for researchers, and a feasible method of gathering lifetime data for practitioners. It has been illustrated by [13], that the inference is feasible, and practical when the sample data are gathered according to a type-II progressively censored experimental scheme. In this article, we have considered the maximum likelihood (ML), and Bayes estimates for some survival time parameters, reliability function, and hazard function, as well as the parameters of the CRD using progressively type-II censored data. MLEs, and the corresponding variance-covariance matrix, are obtained. We have also proposed a Bayesian approach to estimating the model parameters. Compared the MLEs and Bayes estimates obtained by numerical simulation in terms of the estimated risks (ER) for different censoring schemes. It is observed that overall, the Bayes estimators perform better, when compared with the MLEs. From the results, we observe the following:

i. All of the results obtained in this article can be specialized to both the complete sample case by taking $(m = n, r_i = 0, i = 1,2,3,...m)$, and the type-II right censored sample for $(r_i = 0, i = 1,2,3,..., m-1, r_m = n - m)$.
ii. The use of a discrete distribution for parameter c resulted in a closed form expression for the posterior pdf, and the equal probabilities in the discrete distribution cased an element of uncertainly, which can be desirable in some cases.

iii. From Tables II and III, as the effective sample proportion m/n increases, the estimated risk of the estimators, reduce significantly. For a fixed n and m, we can determine the censoring scheme which is most efficient; for example, we observe that the censoring scheme $r_1 = n - m$, $r_2 = ... = r_m = 0$, seems to provide the smallest variance for the estimate of the reliability, and hazard functions.

iv. The type-II progressive censoring scheme described in this paper can be generalized to accommodate censoring on the left as well. We may assume that the observation of failures begins at the time of the $(s+1)th$ failure, at which time r_{s+1} surviving units are removed from the sample. The exact failure times of the s units which are known to have failed before this starting time are unknown.

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