The Game of Synchronized Quadromineering

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Abstract—In synchronized games players make their moves simultaneously rather than alternately. Synchronized Quadromineering is the synchronized version of Quadromineering, a variants of a classical two-player combinatorial game called Domineering. Experimental results for small $m \times n$ boards (with $m + n < 15$) and some theoretical results for general $k \times n$ boards (with $k = 4, 5, 6$) are presented. Moreover, some Synchronized Quadromineering variants are also investigated.

Keywords—Combinatorial games, Synchronized games, Quadromineering.

I. INTRODUCTION

THE game of Domineering is a typical two-player game with perfect information, proposed around 1973 by Göran Andersson [2], [10], [11]. The two players, usually denoted by Vertical and Horizontal, take turns in placing dominoes (2×1 tile) on a checkerboard. Vertical is only allowed to place its dominoes vertically and Horizontal is only allowed to place its dominoes horizontally on the board. Dominoes are not allowed to overlap and the first player that cannot find a place for one of its triominoes loses. After a time the remaining space may separate into several disconnected regions, and each player must choose into which region to place a domino. Berlekamp [1] solved the general problem for $2 \times n$ board for odd n. The 8×8 board and many other small boards were recently solved by Breuker, Uiterwijk and van den Herik [4] using a computer search with a good system of transposition tables. Subsequently, Lachmann, Moore, and Rapaport solved the problem for boards of width 2, 3, 5, and 7 and other specific cases [12]. Finally, Bullock solved the 10×10 board [5].

The game of Triomineering was proposed in 2004 by Blanco and Fraenkel [3]. In Triomineering Vertical and Horizontal alternate in tiling with a straight triomino (3×1 tile) on a checkerboard. Blanco and Fraenkel calculated Triomineering and values for boards up to 6 squares and small rectangular boards.

The game of Quadromineering is a further extension of Domineering where Vertical and Horizontal alternate in tiling with a straight quadromino (4×1 tile) on a checkerboard.

II. SYNCHRONIZED GAMES

For the sake of self containment, we recall the previous results concerning synchronized games. Initially, the concept of synchronism was introduced in the games of Cutcake [6], Maundy Cake [7], Domineering [8], and Triomineering [9] in order to study combinatorial games where players make their moves simultaneously.

TABLE I
THE POSSIBLE OUTCOMES IN SYNCHRONIZED QUADROMINEERING.

	Horizontal ls	Horizontal ds	Horizontal ws
Vertical ls	$G = VHD$	$G = HD$	$G = H$
Vertical ds	$G = VD$	$G = D$	-
Vertical ws	$G = V$	-	-

As a result, in the synchronized versions of these games there exist no zero-games (fuzzy-games), i.e., games where the winner depends exclusively on the player that makes the second (first) move. Moreover, there exists the possibility of a draw, which is impossible in a typical combinatorial game. In this work, we continue to investigate synchronized combinatorial games by focusing our attention on Quadromineering.

In the game of Synchronized Quadromineering, a general instance and the legal moves for Vertical and Horizontal are defined exactly in the same way as defined for the game of Quadromineering. There is only one difference: Vertical and Horizontal make their legal moves simultaneously, therefore, quadrominoes are allowed to overlap if they have a 1×1 tile in common. We note that 1×1 overlap is only possible within a simultaneous move. At the end, if both players cannot make a move, then the game ends in a draw, else if only one player can still make a move, then he/she is the winner.

For each player there exist 3 possible outcomes:

1) The player has a *winning strategy* (ws) independently of the opponent's strategy, or
2) The player has a *drawing strategy* (ds), i.e., he/she can always get a draw in the worst case, or
3) The player has a *losing strategy* (ls), i.e., he/she does not have a strategy for winning or for drawing.

Table I shows all the possible cases. It is clear that if one player has a winning strategy, then the other player has neither a winning strategy nor a drawing strategy. Therefore, the cases $ws-ws$, $ws-ds$, and $ds-ws$ never happen. As a consequence, if G is an instance of Synchronized Quadromineering, then we have 6 possible legal cases:

1) $G = D$ if both players have a drawing strategy, and the game will always end in a draw under perfect play, or
2) $G = V$ if Vertical has a winning strategy, or
3) $G = H$ if Horizontal has a winning strategy, or
4) $G = VD$ if Vertical can always get a draw in the worst case, but he/she could be able to win if Horizontal makes a wrong move, or
5) $G = HD$ if Horizontal can always get a draw in the worst case, but he/she could be able to win if Vertical makes a wrong move, or
6) $G = VHD$ if both players have a losing strategy and the outcome is totally unpredictable.
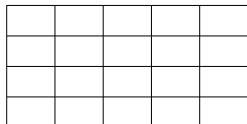
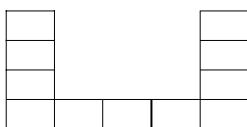## III. EXAMPLES OF SYNCHRONIZED QUADROMINEERING

The game
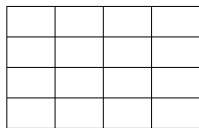
always ends in a draw, therefore $G = D$.
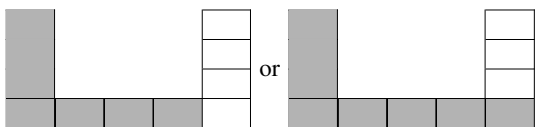
In the game

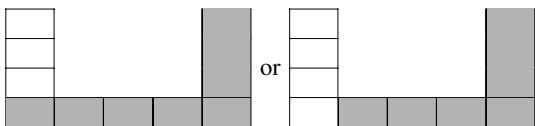Vertical has a winning strategy moving in the central column, therefore $G = V$.

In the game

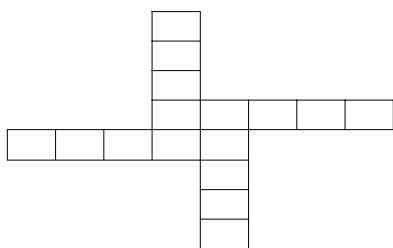if Vertical moves in the first column we have two possibilities

or

therefore, either Vertical wins or the game ends in a draw. Symmetrically, if Vertical moves in the third column we have two possibilities

or

therefore, either Vertical wins or the game ends in a draw. It follows $G = VD$.

In the game

each player has 4 possible moves. For every move of Vertical, Horizontal can win or draw (and sometimes lose); likewise, for every move by Horizontal, Vertical can win or draw (and sometimes lose). As a result it follows that $G = VHD$.

IV. RESULTS FOR SYNCHRONIZED QUADROMINEERING

Table II shows the results obtained using an exhaustive search algorithm for $n \times m$ boards with $n + m \leq 14$.

Theorem 1: Let $G = [n, 5]$ be a rectangle of Synchronized Quadromineering with $n \geq 20$. Then Vertical has a winning strategy.

TABLE II
OUTCOMES FOR RECTANGLES IN SYNCHRONIZED QUADROMINEERING.

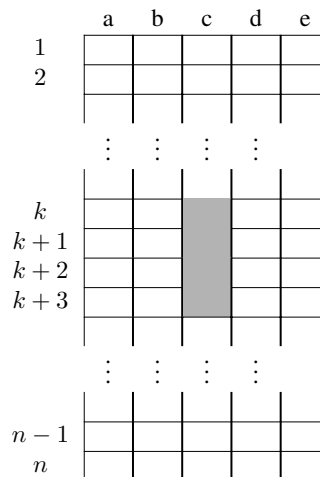|	4	5	6	7	8	9	10
4	D	V	V	V	D	V	V
5	H	D	V	V	H	H	
6	H	H	D	V	H		
7	H	H	H	D			
8	D	V	V				
9	H	V					
10	H						

Fig. 1. Synchronized Quadromineering played on $n \times 5$ rectangular board.

Proof: In the beginning, Vertical will always move into the central column of the board, i.e., $(k, c), (k + 1, c), (k + 2, c), (k + 3, c)$ where $k \equiv 1 \pmod 4$, as shown in Fig. 1. When Vertical cannot move anymore into the central column, let us imagine that we divide the main rectangle into 4×5 sub-rectangles starting from the top of the board (by using horizontal cuts). Of course, if $n \not\equiv 0 \pmod 4$, then the last sub-rectangle will be of size 1×5, 2×5 or 3×5, and Horizontal will be able to make respectively one more move, two more moves or three more moves. We can classify all these sub-rectangles into 6 different classes:

- Class A. Vertical is able to make four more moves in each sub-rectangle of this class.
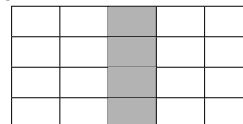
- Class B. Vertical is able to make one more move in each sub-rectangle of this class. For example
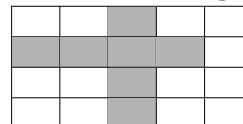
- Class C. Horizontal is able to make three more moves in each sub-rectangle and Vertical is able to make at least $\lceil |C|/2 \rceil$ moves where $|C|$ is the number of sub-rectangles

belonging to this class. The last statement is true under the assumption that Vertical moves into the sub-rectangles of this class as long as they exist before to move into the sub-rectangles of the other classes. For example

- Class D. Horizontal is able to make two more moves in each sub-rectangle of this class. For example

- Class E. Horizontal is able to make one more move in each sub-rectangle of this class. For example

- Class F. Neither Vertical nor Horizontal are able to make a move in the sub-rectangles of this class. For example

We show that when Vertical cannot move anymore in the central column, he/she can make a greater number of moves than Horizontal, i.e., $moves(H) < moves(V)$. We denote with $|A|$ the number of sub-rectangles in the A class, with $|B|$ the number of sub-rectangles in the B class, and so on. Both Vertical and Horizontal have placed the same number of quadrominoes, therefore

$$|A| = |C| + 2|D| + 3|E| + 4|F|$$

It follows that

$$
\begin{aligned}
moves(H) &\leq 3|C| + 2|D| + |E| + 3 \\
&= 3|A| - 4|D| - 8|E| - 12|F| + 3 \\
&< 4|A| + |B| + \lceil |C|/2 \rceil \\
&\leq moves(V)
\end{aligned}
$$

The condition

$$3|A| - 4|D| - 8|E| - 12|F| + 3 < 4|A| + |B| + \lceil |C|/2 \rceil$$

is always true, as shown below:

- If $|A| = 0$ then $|C| = 0$, $|D| = 0$, $|E| = 0$, $|F| = 0$ and $|B| \geq 5$ because by hypothesis $n \geq 20$,
- If $|A| = 1$ then $|C| = 1$, $|D| = 0$, $|E| = 0$, $|F| = 0$ and $|B| \geq 3$ because by hypothesis $n \geq 20$,
- If $|A| = 2$ then either $|B| \geq 1$, $|C| = 2$, $|D| = 0$, $|E| = 0$, and $|F| = 0$ or $|B| \geq 2$, $|C| = 0$, $|D| = 1$, $|E| = 0$, and $|F| = 0$,
- If $|A| = 3$ then either $|C| = 3$, $|D| = 0$, $|E| = 0$, and $|F| = 0$ or $|C| = 1$, $|D| = 1$, $|E| = 0$, and $|F| = 0$,

Fig. 2. Synchronized Quadromineering played on $n \times 6$ rectangular board.

- If $|A| \geq 4$ then $3|A| + 3 < 4|A|$.

$\blacksquare$

Theorem 2: Let $G = [n, 6]$ be a rectangle of Synchronized Quadromineering with $n \geq 12$. Then Vertical has a winning strategy.
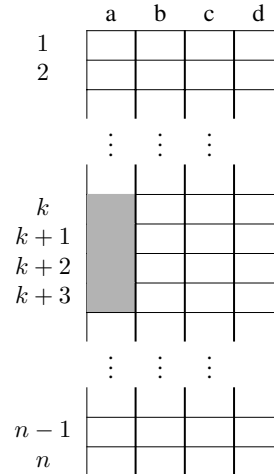
Proof: In the beginning, Vertical will always move into the third column of the board, i.e., (k, c), $(k+1, c)$, $(k+2, c)$ and $(k+3, c)$ where $k \equiv 1 \pmod 4$, as shown in Fig. 2. When Vertical cannot move anymore into the third column, let us imagine that we divide the main rectangle into 4×6 sub-rectangles starting from the top of the board (by using horizontal cuts). Of course, if $n \not\equiv 0 \pmod 4$, then the last sub-rectangle will be of size either 1×6, 2×6 or 3×6, and Horizontal will be able to make respectively one more move, two more moves, or three more moves.

We can classify all these sub-rectangles into 6 different classes according to:

- The number of vertical quadrominoes already placed in the sub-rectangle (vq),
- The number of horizontal quadrominoes already placed in the sub-rectangle (hq),
- The number of moves that Vertical is able to make in the worst case, in all the sub-rectangles of that class (vm),
- The number of moves that Horizontal is able to make in the best case, in all the sub-rectangles of that class (hm),

as shown in Table III. We denote with $|A|$ the number of sub-rectangles in the A class, with $|B|$ the number of sub-rectangles in the B class, and so on. In the C class, Vertical is able to make $|C|$ moves under the assumption that he/she moves into the sub-rectangles of this class as long as they exist before to move into the sub-rectangles of the other classes.

When Vertical cannot move anymore into the third column, both Vertical and Horizontal have placed the same number of quadrominoes, therefore

$$|A| = |C| + 2|D| + 3|E| + 4|F| \tag{1}$$

Let us prove by contradiction that Vertical can make a larger number of moves than Horizontal. Assume therefore

TABLE III
THE 6 CLASSES FOR 4×6 SUB-RECTANGLES.

Class	vq	hq	vm	hm	Example
A	1	0	$5|A|$	0	
B	1	1	$2|B|$	0	
C	0	1	$|C|$	$3|C|$	
D	0	2	0	$2|D|$	
E	0	3	0	$|E|$	
F	0	4	0	0	

$moves(V) \leq moves(H)$ using the data in Table III

$$5|A| + 2|B| + |C| \quad \leq \quad 3|C| + 2|D| + |E| + 3$$

and applying Equation 1

$$3|A| + 2|B| + 2|D| + 5|E| + 8|F| \quad \leq \quad 3$$

which is false because

$$|A| + |B| + |C| + |D| + |E| + |F| = \lfloor n/4 \rfloor$$

and by hypothesis $n \geq 12$. We note that if $|A| = 0$ then $|C| = 0$, $|D| = 0$, $|E| = 0$, $|F| = 0$, and $|B| \geq 3$. Moreover, if $|A| = 1$ then $|C| = 1$, $|D| = 0$, $|E| = 0$, $|F| = 0$, and $|B| \geq 1$. So $moves(V) \leq moves(H)$ does not hold and consequently $moves(V) > moves(H)$. ∎

By symmetry the following two theorems hold.

Theorem 3: Let $G = [5, n]$ be a rectangle of Synchronized Quadromineering with $n \geq 20$. Then Horizontal has a winning strategy.

Theorem 4: Let $G = [6, n]$ be a rectangle of Synchronized Quadromineering with $n \geq 12$. Then Horizontal has a winning strategy.

Fig. 3. Synchronized Quadromineering played on $n \times 4$ rectangular board.

Theorem 5: Let $G = [n, 4]$ be a rectangle of Synchronized Quadromineering. If $n \equiv 0 \pmod 4$, then Vertical has a drawing strategy.

Proof: In the beginning, Vertical will always move into the first column of the board, i.e., (k, b), $(k+1, b)$, $(k+2, b)$, and $(k+3, b)$ where $k \equiv 1 \pmod 4$, as shown in Fig. 3. When Vertical cannot move anymore into the central column, let us imagine that we divide the main rectangle into 4×4 sub-rectangles starting from the top of the board (by using horizontal cuts). We can classify all these sub-rectangles into 6 different classes:

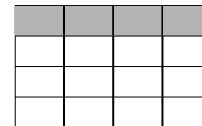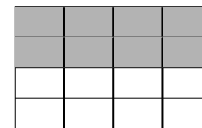- Class A. Vertical is able to make three more moves in each sub-rectangle of this class.
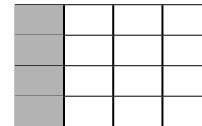
- Class B. Neither Vertical nor Horizontal are able to make another move in the sub-rectangles of this class. For example
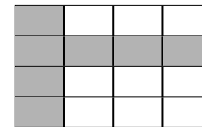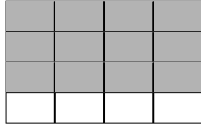
- Class C. Horizontal is able to make three more moves in each sub-rectangle of this class. For example
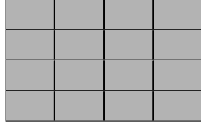
- Class D. Horizontal is able to make two more moves in each sub-rectangle of this class. For example

- Class E. Horizontal is able to make one more move in each sub-rectangle of this class. For example

- Class F. In each sub-rectangle of this class Horizontal has already made four moves.

We show that when Vertical cannot move anymore into the first column, he/she can make a number of moves greater or equal to Horizontal, i.e., $moves(H) \leq moves(V)$. We denote with $|A|$ the number of sub-rectangles in the class A, with $|B|$ the number of sub-rectangles in the class B, and so on We observe that $|A| = |C| + 2|D| + 3|E| + 4|F|$ because both Vertical and Horizontal have placed the same number of triominoes.

$$
\begin{aligned}
moves(H) &= 3|C| + 2|D| + |E| \\
&= 3|A| - 4|D| - 8|E| - 12|F| \\
&\leq 3|A| \\
&= moves(V)
\end{aligned}
$$

■

By symmetry the following theorem holds.

Theorem 6: Let $G = [4, n]$ be a rectangle of Synchronized Quadromineering. If $n \equiv 0 \pmod 4$, then Horizontal has a drawing strategy.

V. SYNCHRONIZED QUADROMINEERING VARIANTS

In this section, we present results obtained using an exhaustive search algorithm for two Synchronized Quadromineering variants where quadrominoes have different shape from 1×4 and 4×1 tile.

A. First variant

In this variant, Vertical can place
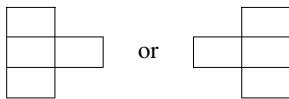
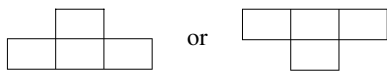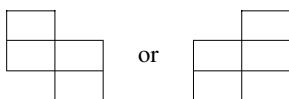and Horizontal can place

Table IV shows the results for $n \times m$ boards with $n + m \leq 12$.

B. Second variant

In this variant, Vertical can place

and Horizontal can place

TABLE IV
OUTCOMES FOR RECTANGLES IN SYNCHRONIZED QUADROMINEERING
(FIRST VARIANT).

	3	4	5	6	7	8	9
3	D	D	D	VD	D	D	D
4	D	VHD	D	HD	HD	HD	
5	D	D	VHD	HD	HD		
6	HD	VD	VD	VHD			
7	D	VD	VD				
8	D	VD					
9	D						

TABLE V
OUTCOMES FOR RECTANGLES IN SYNCHRONIZED QUADROMINEERING
(SECOND VARIANT).

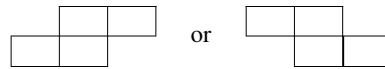|	3	4	5	6	7	8	9
3	D	D	D	D	D	D	D
4	D	VHD	D	HD	HD	HD	
5	D	D	D	HD	HD		
6	D	VD	VD	D			
7	D	VD	VD				
8	D	VD					
9	D						

Table V shows the results for $n \times m$ boards with $n + m \leq 12$.

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