

An adaptive least-squares mixed finite element method for pseudo-parabolic integro-differential equations

Zilong Feng, Hong Li, Yang Liu, Siriguleng He

Abstract—In this article, an adaptive least-squares mixed finite element method is studied for pseudo-parabolic integro-differential equations. The solutions of least-squares mixed weak formulation and mixed finite element are proved. A posteriori error estimator is constructed based on the least-squares functional and the posteriori errors are obtained.

Keywords—Pseudo-parabolic integro-differential equation; least-squares mixed finite element method; Adaptive method; A posteriori error estimates.

I. INTRODUCTION

IN this article, we consider the following pseudo-parabolic integro-differential equation

$$\begin{cases} u_t - \nabla \cdot (a(x)\nabla u_t + b(x)\nabla u \\ + d(x) \int_0^t \nabla u(x, \tau) d\tau) + qu = f(x, t), & x \in \Omega, \quad t \in J, \\ u(x, t) = 0, & x \in \partial\Omega, \quad t \in J, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (1)$$

where Ω is a bounded convex polygonal domain in R^d ($d = 1, 2, 3$) with Lipschitz continuous boundary $\partial\Omega$, $J = (0, T]$ is the time interval with $0 < T < \infty$. $q = q(x) \geq 0$ and $f = f(x, t)$ in (1) are given functions. We shall make the following assumptions on the coefficients c, a, b and d : there exist some positive constants $c_*, c^*, a_*, a^*, b_*, b^*$ and d_*, d^* such that $0 < c_* \leq c(x) \leq c^*, 0 < a_* \leq a(x) \leq a^*, 0 < b_* \leq b(x) \leq b^*, 0 < d_* \leq d(x) \leq d^*$.

Evolution integro-differential equations are a class of important evolution partial differential equations, and have a lot of applications in many physical problems, such as the transport of reactive and passive contaminants in aquifers. Ewing et al. [1] studied finite volume element approximations for two-dimensional parabolic integro-differential equations. Lin et al. [2] proposed the Ritz-Volterra projection onto finite element spaces for integro-differential and related equations. H^1 -Galerkin mixed element methods were studied for parabolic, pseudo-parabolic, and pseudo-hyperbolic integro-differential equations in [3], [4], [5], [6]. Refs.[7], [8] analysed the space-time finite element methods for second-order and fourth-order parabolic partial integro-differential equations.

Refs.[9], [10] studied the least-squares mixed finite element method for parabolic partial integro-differential equations. Cui [11] studied pseudo-parabolic, and pseudo-hyperbolic integro-differential equations, proposed a new Sobolev-Volterra projection. Zhang [12] investigated the superconvergence properties of finite element approximations to parabolic and hyperbolic integro-differential equations.

In recent years, a lot of researchers have studied the least-squares mixed finite element methods for partial differential equations. Luo et al. [13] proposed a nonlinear Galerkin/Petrov-least squares mixed element method for the stationary Navier-Stokes equations. Yang [14] studied the least-square mixed finite element methods for nonlinear parabolic problems. Chen et al. [15] studied least-squares mixed finite element method for degenerate elliptic problems. Duan and Lin [17] studied least-squares mixed finite elements for elasticity. Cai et al. [16] proposed and analysed an adaptive least-squares mixed finite element method for the stress-displacement formulation of linear elasticity. Gu and Li [18] studied an adaptive least-squares mixed finite element method for nonlinear parabolic problems. In this article, we study an adaptive least-squares mixed finite element method for the pseudo-parabolic integro-differential equations, and obtain a posteriori error estimates.

II. A LEAST-SQUARES FORMULATION

Introduce the auxiliary variable $\sigma = -(a(x)\nabla u_t + b(x)\nabla u + d(x) \int_0^t \nabla u(x, \tau) d\tau)$ to have

$$\begin{cases} u_t + \operatorname{div} \sigma + qu = f, & x \in \Omega, \quad t \in J, \\ \sigma + a(x)\nabla u_t + b(x)\nabla u \\ + d(x) \int_0^t \nabla u(x, \tau) d\tau = 0, & x \in \Omega, \quad t \in J, \\ u = 0, & x \in \partial\Omega, \quad t \in J, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (2)$$

We differentiate the second equation of system (2) with respect to time t to obtain

$$\sigma_t = -a(x)\nabla u_{tt} - b(x)\nabla u_t - d(x)\nabla u. \quad (3)$$

Taking $\omega = u_t$, we can derive the following equivalent

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system of (2)

$$\begin{cases} \omega + \operatorname{div} \sigma + qu = f, & x \in \Omega, \quad t \in J, \\ \sigma_t + a(x) \nabla \omega_t + b(x) \nabla \omega + d(x) \nabla u = 0, & x \in \Omega, \quad t \in J, \\ \omega - u_t = 0, & x \in \Omega, \quad t \in J, \\ u = 0, & x \in \partial\Omega, \quad t \in J, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (4)$$

Using the finite difference to discretize (4) in t . Let $\tau > 0, \tau^n = n\tau, n = 0, 1, \dots, N = T/\tau$. Let $\omega^n = (u^n - u^{n-1})/\tau, n \geq 1$. We can rewrite the system (4) :

$$\begin{cases} \omega^n + \operatorname{div} \sigma^n + qu^n = f^n, & x \in \Omega, \quad t \in J, \\ \frac{\sigma^n - \sigma^{n-1}}{\tau} + a(x) \frac{\nabla \omega^n - \nabla \omega^{n-1}}{\tau} \\ + b(x) \nabla \omega^n + d(x) \nabla u^n = 0, & x \in \Omega, \quad t \in J, \\ \omega^n = \frac{u^n - u^{n-1}}{\tau}, & x \in \Omega, \quad t \in J, \\ u^n = 0, & x \in \partial\Omega, \quad t \in J, \\ u^n(x, 0) = u_0^n(x), & x \in \Omega, \end{cases} \quad (5)$$

Introduce Sobolev spaces:

$$L \doteq H^1(\Omega) = \{p \in L^2(\Omega) : \nabla p \in L^2(\Omega)^2\}, \\ H \doteq H(\operatorname{div}, \Omega) = \{\mathbf{w} \in L^2(\Omega)^2 : \operatorname{div} \mathbf{w} \in L^2(\Omega)\}.$$

We can use the least-squares mixed finite element method for (5). The least-squares minimization problem is: find $(u^n, \sigma^n, \omega^n) \in X := L \times H \times L$ such that

$$I(u^n, \sigma^n, \omega^n) = \inf_{(\nu, \xi, \eta) \in X} I(\nu, \xi, \eta), \quad (6)$$

where I is the quadratic functional defined by

$$I(\nu^n, \xi^n, \eta^n) = \|\eta^n + \operatorname{div} \xi^n + q\nu^n - f^n\|_{0,\Omega}^2 \\ + \left\| \frac{\xi^n - \xi^{n-1}}{\tau} + a(x) \frac{\nabla \eta^n - \nabla \eta^{n-1}}{\tau} \right. \\ \left. + b(x) \nabla \eta^n + d(x) \nabla \nu^n \right\|_{0,\Omega}^2.$$

Set $J^n = [t^{n-1}, t^n]$. The corresponding variational statement is: find $(u^n, \sigma^n, \omega^n) : J^n \rightarrow X$

$$B(u^n, \sigma^n, \omega^n; \nu, \xi, \eta) = 0, \quad \forall (\nu, \xi, \eta) \in X, \quad (7)$$

where

$$B(u^n, \sigma^n, \omega^n; \nu, \xi, \eta) \\ = \tau(\omega^n + \operatorname{div} \sigma^n + qu^n, \eta + \operatorname{div} \xi + q\nu)_{0,\Omega} \\ + (\sigma^n - \sigma^{n-1} + a(x)(\nabla \omega^n - \nabla \omega^{n-1}) + \tau b(x) \nabla \omega^n \\ + \tau d(x) \nabla u^n, \xi + a(x) \nabla \eta + \tau b(x) \nabla \eta + \tau d(x) \nabla \nu)_{0,\Omega}. \quad (8)$$

Theorem 2.1: The bilinear form $B(\cdot, \cdot, \cdot; \cdot, \cdot, \cdot)$ is continuous and coercive. That is that there exist positive constants α and β such that

$$B(u^n, \sigma^n, \omega^n; \nu, \xi, \eta) \\ \leq \alpha(\|\omega^n\|_{0,\Omega}^2 + \|\operatorname{div} \sigma^n\|_{0,\Omega}^2 + \|u^n\|_{0,\Omega}^2 + \|\sigma^n\|_{0,\Omega}^2 \\ + \|\nabla u^n\|_{0,\Omega}^2 + \|\nabla \omega^n\|_{0,\Omega}^2)^{\frac{1}{2}} (\|\eta\|_{0,\Omega}^2 + \|\operatorname{div} \xi\|_{0,\Omega}^2 \\ + \|v\|_{0,\Omega}^2 + \|\xi\|_{0,\Omega}^2 + \|\nabla \eta\|_{0,\Omega}^2 + \|\nabla \nu\|_{0,\Omega}^2)^{\frac{1}{2}}, \quad (9)$$

and

$$B(\nu, \xi, \eta; \nu, \xi, \eta) \geq \beta(\|\eta\|_{0,\Omega}^2 + \|\operatorname{div} \xi\|_{0,\Omega}^2 + \|v\|_{0,\Omega}^2 \\ + \|\xi\|_{0,\Omega}^2 + \|\nabla \eta\|_{0,\Omega}^2 + \|\nabla \nu\|_{0,\Omega}^2). \quad (10)$$

Proof: (I). For the upper bound

Noting that the functions $a(x), b(x), d(x)$ and $q(x)$ are bounded, we have

$$B(\nu, \xi, \eta; \nu, \xi, \eta) \\ = \tau(\eta + \operatorname{div} \xi + q\nu, \eta + \operatorname{div} \xi + q\nu)_{0,\Omega} \\ + (\xi + a(x) \nabla \eta + \tau b(x) \nabla \eta + \tau d(x) \nabla \nu, \\ \xi + a(x) \nabla \eta + \tau b(x) \nabla \eta + \tau d(x) \nabla \nu)_{0,\Omega} \\ = \tau\|\eta + \operatorname{div} \xi + q\nu\|_{0,\Omega}^2 + \|\xi + a(x) \nabla \eta \\ + \tau b(x) \nabla \eta + \tau d(x) \nabla \nu\|_{0,\Omega}^2 \\ \leq c(\|\eta\|_{0,\Omega}^2 + \|\operatorname{div} \xi\|_{0,\Omega}^2 + \|v\|_{0,\Omega}^2 + \|\xi\|_{0,\Omega}^2 + \|\nabla \eta\|_{0,\Omega}^2 \\ + \|\tau b(x) \nabla \eta\|_{0,\Omega}^2 + \|\tau d(x) \nabla \nu\|_{0,\Omega}^2) \\ \leq c(\|\eta\|_{0,\Omega}^2 + \|\operatorname{div} \xi\|_{0,\Omega}^2 + \|v\|_{0,\Omega}^2 \\ + \|\xi\|_{0,\Omega}^2 + \|\nabla \eta\|_{0,\Omega}^2 + \|\nabla \nu\|_{0,\Omega}^2).$$

Since the bilinear form is symmetric, this is sufficient for the upper bound in Theorem 2.1.

(II). For the lower bound

Note that the bounded functions $a(x), b(x), d(x)$ and $q(x)$ and use the Holder inequality, the Cauchy inequality to get

$$B(\nu, \xi, \eta; \nu, \xi, \eta) \\ = \tau(\eta + \operatorname{div} \xi + q\nu, \eta + \operatorname{div} \xi + q\nu)_{0,\Omega} \\ + (\xi + a(x) \nabla \eta + \tau b(x) \nabla \eta + \tau d(x) \nabla \nu, \\ \xi + a(x) \nabla \eta + \tau b(x) \nabla \eta + \tau d(x) \nabla \nu)_{0,\Omega} \\ = \tau(\eta, \eta)_{0,\Omega} + \tau(\operatorname{div} \xi, \operatorname{div} \xi)_{0,\Omega} + \tau(q\nu, q\nu)_{0,\Omega} \\ + 2\tau(\eta, \operatorname{div} \xi)_{0,\Omega} + 2\tau(\eta, q\nu)_{0,\Omega} + 2\tau(\operatorname{div} \xi, q\nu)_{0,\Omega} \\ + (\xi, \xi)_{0,\Omega} + a^2(x)(\nabla \eta, \nabla \eta)_{0,\Omega} + \tau^2 b^2(x)(\nabla \eta, \nabla \eta)_{0,\Omega} \\ + \tau^2 d^2(x)(\nabla \nu, \nabla \nu)_{0,\Omega} + 2(\xi, a(x) \nabla \eta)_{0,\Omega} + 2\tau(\xi, b(x) \nabla \eta)_{0,\Omega} \\ + 2\tau(\xi, d(x) \nabla \nu)_{0,\Omega} + 2\tau(a(x) \nabla \eta, b(x) \nabla \eta)_{0,\Omega} \\ + 2\tau(a(x) \nabla \eta, d(x) \nabla \nu)_{0,\Omega} + 2\tau^2(b(x) \nabla \eta, d(x) \nabla \nu)_{0,\Omega}. \\ \geq \tau\|\eta\|_{0,\Omega}^2 + \tau\|\operatorname{div} \xi\|_{0,\Omega}^2 + c_1\|v\|_{0,\Omega}^2 + \|\xi\|_{0,\Omega}^2 + c_2\|\nabla \eta\|_{0,\Omega}^2 \\ + c_3\|\nabla \nu\|_{0,\Omega}^2 - \epsilon_1(\|\eta\|_{0,\Omega}^2 + \|\operatorname{div} \xi\|_{0,\Omega}^2) - \epsilon_2(\|\eta\|_{0,\Omega}^2 \\ + \|\nu\|_{0,\Omega}^2) - \epsilon_3(\|\operatorname{div} \xi\|_{0,\Omega}^2 + \|\nu\|_{0,\Omega}^2) - \epsilon_4(\|\xi\|_{0,\Omega}^2 + \|\nabla \eta\|_{0,\Omega}^2) \\ - \epsilon_5(\|\xi\|_{0,\Omega}^2 + \|\nabla \eta\|_{0,\Omega}^2) - \epsilon_6(\|\xi\|_{0,\Omega}^2 + \|\nabla \nu\|_{0,\Omega}^2) \\ - \epsilon_7(\|\nabla \eta\|_{0,\Omega}^2 + \|\nabla \nu\|_{0,\Omega}^2) - \epsilon_8(\|\nabla \eta\|_{0,\Omega}^2 + \|\nabla \nu\|_{0,\Omega}^2) \\ = (\tau - \epsilon_1 - \epsilon_2)\|\eta\|_{0,\Omega}^2 + (\tau - \epsilon_1 - \epsilon_3)\|\operatorname{div} \xi\|_{0,\Omega}^2 \\ + (c_1 - \epsilon_2 - \epsilon_3)\|\nu\|_{0,\Omega}^2 + (1 - \epsilon_4 - \epsilon_5 - \epsilon_6)\|\xi\|_{0,\Omega}^2 \\ + (c_2 - \epsilon_5 - \epsilon_7 - \epsilon_8)\|\nabla \eta\|_{0,\Omega}^2 + (c_3 - \epsilon_6 - \epsilon_7 - \epsilon_8)\|\nabla \nu\|_{0,\Omega}^2.$$

So we can select constants $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8, c_1, c_2, c_3$ such that

$$B(\nu, \xi, \eta; \nu, \xi, \eta) \geq c\|\eta\|_{0,\Omega}^2, \\ B(\nu, \xi, \eta; \nu, \xi, \eta) \geq c\|\operatorname{div} \xi\|_{0,\Omega}^2, \\ B(\nu, \xi, \eta; \nu, \xi, \eta) \geq c\|\nu\|_{0,\Omega}^2,$$

$$\begin{aligned} B(\nu, \xi, \eta; \nu, \xi, \eta) &\geq c\|\xi\|_{0,\Omega}^2, \\ B(\nu, \xi, \eta; \nu, \xi, \eta) &\geq c\|\nabla\eta\|_{0,\Omega}^2, \\ B(\nu, \xi, \eta; \nu, \xi, \eta) &\geq c\|\nabla\nu\|_{0,\Omega}^2, \end{aligned}$$

So we obtain

$$B(\nu, \xi, \eta; \nu, \xi, \eta) \geq \beta(\|\eta\|_{0,\Omega}^2 + \|\operatorname{div}\xi\|_{0,\Omega}^2 + \|\nu\|_{0,\Omega}^2 + \|\xi\|_{0,\Omega}^2 + \|\nabla\eta\|_{0,\Omega}^2 + \|\nabla\nu\|_{0,\Omega}^2).$$

From the above inequality, we can get the proof of Theorem 2.1. ■

Theorem 2.2: We obtain that Eq.(7) have a unique solution $(u^n, \sigma^n, \omega^n) \in X := L \times H \times L$.

Proof: Use the similar proof to Ref [18] to obtain the conclusion of Theorem 2.2.

III. FINITE ELEMENT APPROXIMATION

Assume the spaces $L_h \subset L, H_h \subset H$. The completely discrete least-squares mixed finite method approximation is: find $(u_h^n, \sigma_h^n, \omega_h^n) \in X_h := L_h \times H_h \times L_h$ such that

$$I(u_h^n, \sigma_h^n, \omega_h^n) = \inf_{(\nu_h, \xi_h, \eta_h) \in X} I(\nu_h, \xi_h, \eta_h). \quad (11)$$

where

$$\begin{aligned} I(\nu_h^n, \xi_h^n, \eta_h^n) &= \sum_{T \in T_h} (\|\eta_h^n + \operatorname{div}\xi_h^n + q\nu_h^n - f^n\|_{0,T}^2 \\ &\quad + \|\frac{\xi_h^n - \xi_h^{n-1}}{\tau} + a(x)\frac{\nabla\eta_h^n - \nabla\eta_h^{n-1}}{\tau} \\ &\quad + b(x)\nabla\eta_h^n + d(x)\nabla\nu_h^n\|_{0,T}^2). \end{aligned}$$

We defined the bilinear form $B(\cdot, \cdot; \cdot, \cdot; \cdot, \cdot)$ as follows

$$\begin{aligned} B(u_h^n, \sigma_h^n, \omega_h^n; \nu_h, \xi_h, \eta_h) &= \sum_{T \in T_h} (\tau(\omega_h^n + \operatorname{div}\sigma_h^n + q\nu_h^n, \eta_h + \operatorname{div}\xi_h + q\nu_h)_{0,T} \\ &\quad + (\sigma_h^n - \sigma_h^{n-1} + a(x)(\nabla\omega_h^n - \nabla\omega_h^{n-1}) \\ &\quad + \tau b(x)\nabla\omega_h^n + \tau d(x)\nabla\nu_h^n, \\ &\quad \xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h)_{0,T}). \end{aligned} \quad (12)$$

Theorem 3.1: The bilinear $B(\cdot, \cdot; \cdot, \cdot; \cdot, \cdot)$ is continuous and coercive, that is that there exist positive constants α_h and β_h such that

$$\begin{aligned} B(u_h^n, \sigma_h^n, \omega_h^n; \nu_h, \xi_h, \eta_h) &\leq \alpha_h (\sum_{T \in T_h} \|\omega_h^n\|_{0,T}^2 + \|\operatorname{div}\sigma_h^n\|_{0,T}^2 + \|u_h^n\|_{0,T}^2 + \|\sigma_h^n\|_{0,T}^2 \\ &\quad + \|\nabla u_h^n\|_{0,T}^2 + \|\nabla\omega_h^n\|_{0,T}^2)^{\frac{1}{2}} (\sum_{T \in T_h} \|\eta_h\|_{0,T}^2 + \|\operatorname{div}\xi_h\|_{0,T}^2 \\ &\quad + \|v_h\|_{0,T}^2 + \|\xi_h\|_{0,T}^2 + \|\nabla\eta_h\|_{0,T}^2 + \|\nabla\nu_h\|_{0,T}^2)^{\frac{1}{2}}, \end{aligned} \quad (13)$$

and

$$\begin{aligned} B(\nu_h, \xi_h, \eta_h; \nu_h, \xi_h, \eta_h) &\geq \beta_h \sum_{T \in T_h} (\|\eta_h\|_{0,\Omega}^2 + \|\operatorname{div}\xi_h\|_{0,\Omega}^2 + \|\nu_h\|_{0,\Omega}^2 \\ &\quad + \|\xi_h\|_{0,\Omega}^2 + \|\nabla\eta_h\|_{0,\Omega}^2 + \|\nabla\nu_h\|_{0,\Omega}^2). \end{aligned} \quad (14)$$

which holds for all $(u_h^n, \sigma_h^n, \omega_h^n) \in X_h := L_h \times H_h \times L_h$, $(\nu_h, \xi_h, \eta_h) \in X_h := L_h \times H_h \times L_h$.

Proof: (I). For the upper bound

Since the functions $a(x), b(x), d(x)$ and $q(x)$ are bounded, we have

$$\begin{aligned} B(\nu_h, \xi_h, \eta_h; \nu_h, \xi_h, \eta_h) &= \sum_{T \in T_h} (\tau(\eta_h + \operatorname{div}\xi_h + q\nu_h, \eta_h + \operatorname{div}\xi_h + q\nu_h)_{0,T} \\ &\quad + (\xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h, \\ &\quad \xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h)_{0,T}) \\ &= \sum_{T \in T_h} (\tau\|\eta_h + \operatorname{div}\xi_h + q\nu_h\|_{0,T}^2 + \|\xi_h + a(x)\nabla\eta_h \\ &\quad + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h\|_{0,T}^2) \\ &\leq c \sum_{T \in T_h} (\|\eta_h\|_{0,T}^2 + \|\operatorname{div}\xi_h\|_{0,T}^2 + \|\nu_h\|_{0,T}^2 + \|\xi_h\|_{0,T}^2 \\ &\quad + \|\nabla\eta_h\|_{0,T}^2 + \|\tau b(x)\nabla\eta_h\|_{0,T}^2 + \|\tau d(x)\nabla\nu_h\|_{0,T}^2) \\ &\leq c \sum_{T \in T_h} (\|\eta_h\|_{0,T}^2 + \|\operatorname{div}\xi_h\|_{0,T}^2 + \|\nu_h\|_{0,T}^2 \\ &\quad + \|\xi_h\|_{0,T}^2 + \|\nabla\eta_h\|_{0,T}^2 + \|\nabla\nu_h\|_{0,T}^2). \end{aligned}$$

Since the bilinear form is symmetric, this is sufficient for the upper bound in Theorem 3.1.

(II). For the lower bound

Note that the bounded functions $a(x), b(x), d(x)$ and $q(x)$ and use the Holder inequality, the Cauchy inequality to get

$$\begin{aligned} B(\nu_h, \xi_h, \eta_h; \nu_h, \xi_h, \eta_h) &= \sum_{T \in T_h} (\tau(\eta_h + \operatorname{div}\xi_h + q\nu_h, \eta_h + \operatorname{div}\xi_h + q\nu_h)_{0,T} \\ &\quad + (\xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h, \\ &\quad \xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h)_{0,T}) \\ &= \sum_{T \in T_h} (\tau(\eta_h, \eta_h)_{0,T} + \tau(\operatorname{div}\xi_h, \operatorname{div}\xi_h)_{0,T} + \tau(q\nu_h, q\nu_h)_{0,T} \\ &\quad + 2\tau(\eta_h, \operatorname{div}\xi_h)_{0,T} + 2\tau(\eta_h, q\nu_h)_{0,T} + 2\tau(\operatorname{div}\xi_h, q\nu_h)_{0,T} \\ &\quad + (\xi_h, \xi_h)_{0,T} + a^2(x)(\nabla\eta_h, \nabla\eta_h)_{0,T} \\ &\quad + \tau^2 b^2(x)(\nabla\eta_h, \nabla\eta_h)_{0,T} + \tau^2 d^2(x)(\nabla\nu_h, \nabla\nu_h)_{0,T} \\ &\quad + 2(\xi_h, a(x)\nabla\eta_h)_{0,T} + 2\tau(\xi_h, b(x)\nabla\eta_h)_{0,T} \\ &\quad + 2\tau(\xi_h, d(x)\nabla\nu_h)_{0,T} + 2\tau(a(x)\nabla\eta_h, b(x)\nabla\eta_h)_{0,T} \\ &\quad + 2\tau(a(x)\nabla\eta_h, d(x)\nabla\nu_h)_{0,T} + 2\tau^2(b(x)\nabla\eta_h, d(x)\nabla\nu_h)_{0,T}) \\ &\geq \sum_{T \in T_h} (\tau\|\eta_h\|_{0,T}^2 + \tau\|\operatorname{div}\xi_h\|_{0,T}^2 + \tilde{c}_1\|\nu_h\|_{0,T}^2 + \|\xi_h\|_{0,T}^2 \\ &\quad + \tilde{c}_2\|\nabla\eta_h\|_{0,T}^2 + \tilde{c}_3\|\nabla\nu_h\|_{0,T}^2 - \tilde{c}_1(\|\eta_h\|_{0,T}^2 + \|\operatorname{div}\xi_h\|_{0,T}^2) \\ &\quad - \tilde{c}_2(\|\eta_h\|_{0,T}^2 + \|\nu_h\|_{0,T}^2) - \tilde{c}_3(\|\operatorname{div}\xi_h\|_{0,T}^2 + \|\nu_h\|_{0,T}^2)) \end{aligned}$$

$$\begin{aligned}
& -\tilde{\epsilon}_4(\|\xi_h\|_{0,T}^2 + \|\nabla\eta_h\|_{0,T}^2) - \tilde{\epsilon}_5(\|\xi_h\|_{0,T}^2 + \|\nabla\eta_h\|_{0,T}^2) \\
& -\tilde{\epsilon}_6(\|\xi_h\|_{0,T}^2 + \|\nabla\nu_h\|_{0,T}^2) - \tilde{\epsilon}_7(\|\nabla\eta_h\|_{0,T}^2 + \|\nabla\nu_h\|_{0,T}^2) \\
& -\tilde{\epsilon}_8(\|\nabla\eta_h\|_{0,T}^2 + \|\nabla\nu_h\|_{0,T}^2) \\
= & \sum_{T \in T_h} ((\tau - \tilde{\epsilon}_1 - \tilde{\epsilon}_2)\|\eta_h\|_{0,T}^2 + (\tau - \tilde{\epsilon}_1 - \tilde{\epsilon}_3)\|div\xi_h\|_{0,T}^2 \\
& + (\tilde{c}_1 - \tilde{\epsilon}_2 - \tilde{\epsilon}_3)\|\nu_h\|_{0,T}^2 + (1 - \tilde{\epsilon}_4 - \tilde{\epsilon}_5 - \tilde{\epsilon}_6)\|\xi_h\|_{0,T}^2 \\
& + (\tilde{c}_2 - \tilde{\epsilon}_5 - \tilde{\epsilon}_7 - \tilde{\epsilon}_8)\|\nabla\eta_h\|_{0,T}^2 \\
& + (\tilde{c}_3 - \tilde{\epsilon}_6 - \tilde{\epsilon}_7 - \tilde{\epsilon}_8)\|\nabla\nu_h\|_{0,T}^2).
\end{aligned}$$

So we can select constants $\tilde{\epsilon}_1, \tilde{\epsilon}_2, \tilde{\epsilon}_3, \tilde{\epsilon}_4, \tilde{\epsilon}_5, \tilde{\epsilon}_6, \tilde{\epsilon}_7, \tilde{\epsilon}_8, \tilde{c}_1, \tilde{c}_2, \tilde{c}_3$ such that

$$\begin{aligned}
B(\nu, \xi, \eta; \nu, \xi, \eta) & \geq c \sum_{T \in T_h} \|\eta_h\|_{0,T}^2, \\
B(\nu, \xi, \eta; \nu, \xi, \eta) & \geq c \sum_{T \in T_h} \|div\xi_h\|_{0,T}^2, \\
B(\nu, \xi, \eta; \nu, \xi, \eta) & \geq c \sum_{T \in T_h} \|\nu_h\|_{0,T}^2, \\
B(\nu, \xi, \eta; \nu, \xi, \eta) & \geq c \sum_{T \in T_h} \|\xi_h\|_{0,T}^2, \\
B(\nu, \xi, \eta; \nu, \xi, \eta) & \geq c \sum_{T \in T_h} \|\nabla\eta_h\|_{0,T}^2, \\
B(\nu, \xi, \eta; \nu, \xi, \eta) & \geq c \sum_{T \in T_h} \|\nabla\nu_h\|_{0,T}^2,
\end{aligned}$$

So we can obtain

$$\begin{aligned}
& B(\nu_h, \xi_h, \eta_h; \nu_h, \xi_h, \eta_h) \\
\geq & \beta_h \sum_{T \in T_h} (\|\eta_h\|_{0,T}^2 + \|div\xi_h\|_{0,T}^2 + \|\nu_h\|_{0,T}^2 \\
& + \|\xi_h\|_{0,T}^2 + \|\nabla\eta_h\|_{0,T}^2 + \|\nabla\nu_h\|_{0,T}^2).
\end{aligned}$$

We can obtain the proof for Theorem 3.1. ■

Theorem 3.2: It is easy to show that Eq.(12) have a unique solution $(u_h^n, \sigma_h^n, \omega_h^n) \in X_h := L_h \times H_h \times L_h$.

IV. THE LEAST-SQUARES FUNCTIONAL OF AN A POSTERIORI ERROR ESTIMATOR

In this section, we use the similar method to Ref [18]. From the least-squares functional

$$\begin{aligned}
I(u_h^n, \sigma_h^n, \omega_h^n) & = \sum_{T \in T_h} (\|\omega_h^n + div\sigma_h^n + qu_h^n - f^n\|_{0,T}^2 \\
& + \|\frac{\sigma_h^n - \sigma_h^{n-1}}{\tau} + a(x)\frac{\nabla\omega_h^n - \nabla\omega_h^{n-1}}{\tau} \\
& + b(x)\nabla\omega_h^n + d(x)\nabla u_h^n\|_{0,T}^2).
\end{aligned}$$

So we obtain

$$\begin{aligned}
& B(u_h^n - u^n, \sigma_h^n - \sigma^n, \omega_h^n - \omega^n; u_h^n - u^n, \sigma_h^n - \sigma^n, \omega_h^n - \omega^n) \\
= & \sum_{T \in T_h} (\|\omega_h^n - \omega^n + div(\sigma_h^n - \sigma^n) + q(u_h^n - u^n)\|_{0,T}^2 \\
& + \|\frac{\sigma_h^n - \sigma^n - (\sigma_h^{n-1} - \sigma^{n-1})}{\tau} \\
& + a(x)\frac{\nabla(\omega_h^n - \omega^n) - \nabla(\omega_h^{n-1} - \omega^{n-1})}{\tau} \\
& + b(x)\nabla(\omega_h^n - \omega^n) + d(x)\nabla(u_h^n - u^n)\|_{0,T}^2).
\end{aligned}$$

For obtaining the definition of the a posteriori estimator, we can see that the least-squares functional is the sum of its element-wise contributions

$$\begin{aligned}
I(u_h^n, \sigma_h^n, \omega_h^n) & = \sum_{T \in T_h} (\|\omega_h^n + div\sigma_h^n + qu_h^n - f^n\|_{0,T}^2 \\
& + \|\frac{\sigma_h^n - \sigma_h^{n-1}}{\tau} + a(x)\frac{\nabla\omega_h^n - \nabla\omega_h^{n-1}}{\tau} \\
& + b(x)\nabla\omega_h^n + d(x)\nabla u_h^n\|_{0,T}^2) =: \sum_{T \in T_h} \theta_T^2.
\end{aligned}$$

So we can obtain the following theorem.

Theorem 4.1: The element-wise computation of the least-squares functional constitutes an a posteriori error estimator. That is, for

$$\begin{aligned}
& \|\omega_h^n + div\sigma_h^n + qu_h^n - f^n\|_{0,T}^2 + \|\frac{\sigma_h^n - \sigma_h^{n-1}}{\tau} \\
& + a(x)\frac{\nabla\omega_h^n - \nabla\omega_h^{n-1}}{\tau} + b(x)\nabla\omega_h^n + d(x)\nabla u_h^n\|_{0,T}^2 = \theta_T^2.
\end{aligned}$$

there exist positive constants α_T and β_T such that

$$\begin{aligned}
\sum_{T \in T_h} \theta_T^2 & \geq \beta_T \sum_{T \in T_h} (\|\omega_h^n - \omega^n\|_{0,T}^2 + \|div(\sigma_h^n - \sigma^n)\|_{0,T}^2 \\
& + \|u_h^n - u^n\|_{0,T}^2 + \|\sigma_h^n - \sigma^n\|_{0,T}^2 \\
& + \|\nabla(\omega_h^n - \omega^n)\|_{0,T}^2 + \|\nabla(u_h^n - u^n)\|_{0,T}^2),
\end{aligned}$$

$$\begin{aligned}
\sum_{T \in T_h} \theta_T^2 & \leq \alpha_T \sum_{T \in T_h} (\|\omega_h^n - \omega^n\|_{0,T}^2 + \|div(\sigma_h^n - \sigma^n)\|_{0,T}^2 \\
& + \|u_h^n - u^n\|_{0,T}^2 + \|\sigma_h^n - \sigma^n\|_{0,T}^2 \\
& + \|\nabla(\omega_h^n - \omega^n)\|_{0,T}^2 + \|\nabla(u_h^n - u^n)\|_{0,T}^2).
\end{aligned}$$

where α_T and β_T are the same constants as in Theorem 3.1.

V. ERROR ESTIMATES

In this section, we assume that the spaces L_h, H_h can be the spaces of Raviart-Thomas for $r \geq 0$. The completely discrete least-squares mixed finite element approximation is: find $(u_h^n, \sigma_h^n, \omega_h^n) \in X_h := L_h \times H_h \times L_h$ such that

$$\begin{aligned}
& B(u_h^n, \sigma_h^n, \omega_h^n; \nu_h, \xi_h, \eta_h) \\
= & \tau(\omega_h^n + div\sigma_h^n + qu_h^n, \eta_h + div\xi_h + q\nu_h)_{0,\Omega} \\
& + (\sigma_h^n - \sigma_h^{n-1} + a(x)(\nabla\omega_h^n - \nabla\omega_h^{n-1}) + \tau b(x)\nabla\omega_h^n \\
& + \tau d(x)\nabla u_h^n, \xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h)_{0,\Omega}.
\end{aligned}$$

We shall split the error as a sum of two terms,

$$u - u_h = (u - \tilde{u}) + (\tilde{u} - u_h).$$

where $\tilde{u}$ is an elliptic projection in L_h of the exact solution u . The projection $\tilde{u} = \tilde{u}(t)$ is defined by

$$(d(x)\nabla(u - \tilde{u}), \nabla\chi) = 0, \forall \chi \in L_h. \quad (15)$$

It is known that the projection $(\tilde{u}, \tilde{\sigma}, \tilde{\omega}) \in X_h := L_h \times H_h \times L_h$ exists and has the following properties [16], [18], for $1 \leq r \leq k+1$

$$\|u - \tilde{u}\| \leq ch^r \|u\|_r,$$

$$\|(u - \tilde{u})_t\| \leq ch^r(\|u\|_r + \|u_t\|_r),$$

$$\|\sigma - \tilde{\sigma}\| \leq ch^r\|\sigma\|_r,$$

$$\|\operatorname{div}(\sigma - \tilde{\sigma})\| \leq ch^r\|\sigma\|_{r+1},$$

$$\|(\sigma - \tilde{\sigma})_t\| \leq ch^r(\|\sigma\|_r + \|\sigma_t\|_r),$$

From the bilinear form $B(\cdot, \cdot; \cdot, \cdot; \cdot, \cdot)$ in section 2, we have

$$\begin{aligned} & B(u^n, \sigma^n, \omega^n; \nu, \xi, \eta) \\ &= \tau(\omega^n + \operatorname{div}\sigma^n + qu^n, \eta + \operatorname{div}\xi + q\nu)_{0,\Omega} \\ &+ (\sigma^n - \sigma^{n-1} + a(x)(\nabla\omega^n - \nabla\omega^{n-1}) \\ &+ \tau b(x)\nabla\omega^n + \tau d(x)\nabla u^n, \xi + a(x)\nabla\eta \\ &+ \tau b(x)\nabla\eta + \tau d(x)\nabla\nu)_{0,\Omega}. \end{aligned}$$

Thus,

$$\begin{aligned} & B(u_h^n - u^n, \sigma_h^n - \sigma^n, \omega_h^n - \omega^n; \nu_h, \xi_h, \eta_h) \\ &= \tau(\omega_h^n - \omega_h^n + \operatorname{div}(\sigma_h^n - \sigma_h^n) + q(u_h^n - u_h^n), \\ &\eta_h + \operatorname{div}\xi_h + q\nu_h)_{0,\Omega} + (\sigma_h^n - \sigma_h^n - (\sigma_h^{n-1} - \sigma_h^{n-1}) \\ &+ a(x)(\nabla(\omega_h^n - \omega_h^n) - \nabla(\omega_h^{n-1} - \omega_h^{n-1})) \\ &+ \tau b(x)\nabla(\omega_h^n - \omega_h^n) + \tau d(x)\nabla(u_h^n - u_h^n), \\ &\xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h)_{0,\Omega} = 0. \end{aligned}$$

Let $\theta = \sigma - \tilde{\sigma}$, $\pi = \sigma_h - \tilde{\sigma}$, $\rho = u - \tilde{u}$, $\varepsilon = u_h - \tilde{u}$, $\rho_t = \omega - \tilde{\omega}$, $\varepsilon_t = \omega_h - \tilde{\omega}$. We shall use the following notation for the difference operator:

$$\bar{\partial}_t f^n = \frac{f^n - f^{n-1}}{\Delta t},$$

We then have the following estimate:

$$\|\bar{\partial}_t \rho^n\|^2 \leq \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \|\rho_t\|^2 dt,$$

Lemma 5.1: With $\tilde{u}$ defined in (15), we have

$$\|\nabla \tilde{u}\|_{L^\infty} \leq c, \forall t \in J$$

$$\|\nabla \tilde{u}_{tt}\| \leq c, \forall t \in J$$

Please see [16], [18].

From the above equation $B(u^n - u_h^n, \sigma^n - \sigma_h^n, \omega^n - \omega_h^n; \nu_h, \xi_h, \eta_h)$ and the boundedness of $a(x)$, $b(x)$, $d(x)$, $q(x)$, we can get

$$\begin{aligned} & B(u^n - u_h^n, \sigma^n - \sigma_h^n, \omega^n - \omega_h^n; \nu_h, \xi_h, \eta_h) \\ &= \tau(\rho_t^n - \varepsilon_t^n + \operatorname{div}(\theta^n - \pi^n) + q(\rho^n - \varepsilon^n), \\ &\eta_h + \operatorname{div}\xi_h + q\nu_h)_{0,\Omega} + (\theta^n - \pi^n - (\theta^{n-1} - \pi^{n-1}) \\ &+ a(x)(\nabla(\rho_t^n - \varepsilon_t^n) - \nabla(\rho_t^{n-1} - \varepsilon_t^{n-1})) \\ &+ \tau b(x)\nabla(\rho_t^n - \varepsilon_t^n) + \tau d(x)\nabla(\rho^n - \varepsilon^n), \\ &\xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h)_{0,\Omega} \\ &= 0. \end{aligned}$$

Take $\xi_h = \pi^n$, $\nu_h = \varepsilon^n$, $\eta_h = \varepsilon_t^n$ to get

$$\begin{aligned} & B(u^n - u_h^n, \sigma^n - \sigma_h^n, \omega^n - \omega_h^n; \varepsilon^n, \pi^n, \varepsilon_t^n) \\ &= \tau(\rho_t^n - \varepsilon_t^n + \operatorname{div}(\theta^n - \pi^n) + q(\rho^n - \varepsilon^n), \\ &\varepsilon_t^n + \operatorname{div}\pi^n + q\varepsilon^n)_{0,\Omega} + (\theta^n - \pi^n - (\theta^{n-1} - \pi^{n-1}) \\ &+ a(x)(\nabla(\rho_t^n - \varepsilon_t^n) - \nabla(\rho_t^{n-1} - \varepsilon_t^{n-1})) \\ &+ \tau b(x)\nabla(\rho_t^n - \varepsilon_t^n) + \tau d(x)\nabla(\rho^n - \varepsilon^n), \\ &\pi^n + a(x)\nabla\varepsilon_t^n + \tau b(x)\nabla\varepsilon_t^n + \tau d(x)\nabla\varepsilon^n)_{0,\Omega} \\ &= 0. \end{aligned}$$

The left side of the above equation is

$$\begin{aligned} I_3 = & \tau(\rho_t^n + \operatorname{div}\theta^n + q\rho^n, \varepsilon_t^n + \operatorname{div}\pi^n + q\varepsilon^n)_{0,\Omega} \\ &+ (\theta^n - \theta^{n-1} + a(x)\nabla(\rho_t^n - \rho_t^{n-1}) \\ &+ \tau b(x)\nabla\rho_t^n + \tau d(x)\nabla\rho^n, \pi^n + a(x)\nabla\varepsilon_t^n \\ &+ \tau b(x)\nabla\varepsilon_t^n + \tau d(x)\nabla\varepsilon^n)_{0,\Omega}. \end{aligned} \quad (16)$$

The right side of the above equation is

$$\begin{aligned} I_1 + I_2 \\ &= \tau(\varepsilon_t^n + \operatorname{div}\pi^n + q\varepsilon^n, \varepsilon_t^n + \operatorname{div}\pi^n + q\varepsilon^n)_{0,\Omega} \\ &+ (\pi^n - \pi^{n-1} + a(x)\nabla(\varepsilon_t^n - \varepsilon_t^{n-1}) + \tau b(x)\nabla\varepsilon_t^n \\ &+ \tau d(x)\nabla\varepsilon^n, \pi^n + a(x)\nabla\varepsilon_t^n + \tau b(x)\nabla\varepsilon_t^n + \tau d(x)\nabla\varepsilon^n)_{0,\Omega}. \end{aligned} \quad (17)$$

From the boundedness of $a(x)$, $b(x)$, $d(x)$ and $q(x)$, we obtain

$$\begin{aligned} I_2 = & c(\pi^n + \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n, \pi^n + \nabla\varepsilon_t^n \\ &+ \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega} - c(\pi^{n-1} + \nabla\varepsilon_t^{n-1}, \pi^n \\ &+ \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega} \\ &\geq \frac{c}{2}[(\pi^n + \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n, \pi^n \\ &+ \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega} \\ &- (\pi^{n-1}, \pi^{n-1})_{0,\Omega} - (\nabla\varepsilon_t^{n-1}, \nabla\varepsilon_t^{n-1})_{0,\Omega}] \\ &= \frac{c}{2}[(\pi^n, \pi^n)_{0,\Omega} - (\pi^{n-1}, \pi^{n-1})_{0,\Omega} + (\nabla\varepsilon_t^n, \nabla\varepsilon_t^n)_{0,\Omega} \\ &- (\nabla\varepsilon_t^{n-1}, \nabla\varepsilon_t^{n-1})_{0,\Omega} + 2(\pi^n, \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n \\ &+ \tau\nabla\varepsilon^n)_{0,\Omega} + 2(\nabla\varepsilon_t^n, \pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega} \\ &+ (\nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n, \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega} \\ &+ (\pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n, \pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega}]. \end{aligned}$$

Use $I_1 \geq \frac{c}{2}I_1$ ($c \leq 2$) to get

$$\begin{aligned} I_1 + I_2 \\ &\geq \frac{c}{2}[(\pi^n, \pi^n)_{0,\Omega} - (\pi^{n-1}, \pi^{n-1})_{0,\Omega} + (\nabla\varepsilon_t^n, \nabla\varepsilon_t^n)_{0,\Omega} \\ &- (\nabla\varepsilon_t^{n-1}, \nabla\varepsilon_t^{n-1})_{0,\Omega} + 2(\pi^n, \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega} \\ &+ 2(\nabla\varepsilon_t^n, \pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega} + (\nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n \\ &+ \tau\nabla\varepsilon^n, \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega} \\ &+ (\pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n, \pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega} \\ &+ \tau(\varepsilon_t^n + \operatorname{div}\pi^n + q\varepsilon^n, \varepsilon_t^n + \operatorname{div}\pi^n + q\varepsilon^n)_{0,\Omega}] \\ &= I_4. \end{aligned}$$

Summing from $n = 0$ to N , we obtain

$$\begin{aligned} \sum_{n=0}^N I_4 = & c(\|\pi^N\|_{0,\Omega}^2 - \|\pi^0\|_{0,\Omega}^2) + c(\|\nabla \varepsilon_t^N\|_{0,\Omega}^2 - \|\nabla \varepsilon_t^0\|_{0,\Omega}^2) \\ & + c \sum_{n=0}^N (\|\pi^n\|_{0,\Omega}^2 + \|\nabla \varepsilon_t^n\|_{0,\Omega}^2 + \tau^2 \|\nabla \varepsilon^n\|_{0,\Omega}^2 \\ & + \tau^2 \|\nabla \varepsilon_t^n\|_{0,\Omega}^2) \leq \sum_{n=0}^N I_3 \end{aligned}$$

Let $\|\pi^{N^*}\|_{0,\Omega}^2$ be a maximum of the $\|\pi^n\|_{0,\Omega}^2$ and $\|\nabla \varepsilon_t^{N^*}\|_{0,\Omega}^2$ be a maximum of the $\|\nabla \varepsilon_t^n\|_{0,\Omega}^2$, we obtain

$$\begin{aligned} \sum_{n=0}^N I_4 = & c(\|\pi^N\|_{0,\Omega}^2 + \|\nabla \varepsilon_t^N\|_{0,\Omega}^2) + c \sum_{n=0}^N (\|\pi^n\|_{0,\Omega}^2 \\ & + \|\nabla \varepsilon_t^n\|_{0,\Omega}^2 + \tau^2 \|\nabla \varepsilon^n\|_{0,\Omega}^2 + \tau^2 \|\nabla \varepsilon_t^n\|_{0,\Omega}^2). \end{aligned} \quad (18)$$

$$\begin{aligned} I_3 = & \tau(\rho_t^n + \operatorname{div} \theta^n + q\rho^n, \varepsilon_t^n + \operatorname{div} \pi^n + q\varepsilon^n)_{0,\Omega} \\ & + (\theta^n - \theta^{n-1} + a(x)\nabla(\rho_t^n - \rho_t^{n-1}) \\ & + \tau b(x)\nabla \rho_t^n + \tau d(x)\nabla \rho^n, \pi^n + a(x)\nabla \varepsilon_t^n \\ & + \tau b(x)\nabla \varepsilon_t^n + \tau d(x)\nabla \varepsilon^n)_{0,\Omega} \end{aligned}$$

$$\begin{aligned} = & \tau(\rho_t^n, \varepsilon_t^n)_{0,\Omega} + \tau(\rho_t^n, \operatorname{div} \pi^n)_{0,\Omega} + \tau(\rho_t^n, q\varepsilon^n)_{0,\Omega} \\ & + \tau(\operatorname{div} \theta^n, \varepsilon_t^n)_{0,\Omega} + \tau(\operatorname{div} \theta^n, \operatorname{div} \pi^n)_{0,\Omega} \\ & + \tau(\operatorname{div} \theta^n, q\varepsilon^n)_{0,\Omega} + \tau(q\rho^n, \varepsilon_t^n)_{0,\Omega} \\ & + \tau(q\rho^n, \operatorname{div} \pi^n)_{0,\Omega} + \tau(q\rho^n, q\varepsilon^n)_{0,\Omega} \\ & + (\theta^n - \theta^{n-1}, \pi^n)_{0,\Omega} + (\theta^n - \theta^{n-1}, a(x)\nabla \varepsilon_t^n)_{0,\Omega} \\ & + (\theta^n - \theta^{n-1}, \tau b(x)\nabla \varepsilon_t^n)_{0,\Omega} + (\theta^n - \theta^{n-1}, \tau d(x)\nabla \varepsilon^n)_{0,\Omega} \\ & + (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), \pi^n)_{0,\Omega} \\ & + (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), a(x)\nabla \varepsilon_t^n)_{0,\Omega} \\ & + (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), \tau b(x)\nabla \varepsilon_t^n)_{0,\Omega} \\ & + (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), \tau d(x)\nabla \varepsilon^n)_{0,\Omega} \\ & + (\tau b(x)\nabla \rho_t^n, \pi^n)_{0,\Omega} + (\tau b(x)\nabla \rho_t^n, a(x)\nabla \varepsilon_t^n)_{0,\Omega} \\ & + (\tau b(x)\nabla \rho_t^n, \tau b(x)\nabla \varepsilon_t^n)_{0,\Omega} \\ & + (\tau b(x)\nabla \rho_t^n, \tau d(x)\nabla \varepsilon^n)_{0,\Omega} + (\tau d(x)\nabla \rho^n, \pi^n)_{0,\Omega} \\ & + (\tau d(x)\nabla \rho^n, a(x)\nabla \varepsilon_t^n)_{0,\Omega} + (\tau d(x)\nabla \rho^n, \tau b(x)\nabla \varepsilon_t^n)_{0,\Omega} \\ & + (\tau d(x)\nabla \rho^n, \tau d(x)\nabla \varepsilon^n)_{0,\Omega}. \end{aligned}$$

From (15) and applying Green's formula, we know that

$$\begin{aligned} (\nabla \rho_t^n, \pi^n)_{0,\Omega} &= -(\rho_t^n, \operatorname{div} \pi^n)_{0,\Omega}, \\ (\tau d(x)\nabla \rho^n, a(x)\nabla \varepsilon_t^n)_{0,\Omega} &= 0, \end{aligned}$$

$$\begin{aligned} (\tau d(x)\nabla \rho^n, \tau b(x)\nabla \varepsilon_t^n)_{0,\Omega} &= 0, \\ (\tau d(x)\nabla \rho^n, \tau d(x)\nabla \varepsilon^n)_{0,\Omega} &= 0. \end{aligned}$$

So we obtain

$$\begin{aligned} I_3 = & \tau(\rho_t^n, \varepsilon_t^n)_{0,\Omega} + \tau(\rho_t^n, q\varepsilon^n)_{0,\Omega} + \tau(\operatorname{div} \theta^n, \varepsilon_t^n)_{0,\Omega} \\ & + \tau(\operatorname{div} \theta^n, \operatorname{div} \pi^n)_{0,\Omega} + \tau(\operatorname{div} \theta^n, q\varepsilon^n)_{0,\Omega} \\ & + \tau(q\rho^n, \varepsilon_t^n)_{0,\Omega} + \tau(q\rho^n, \operatorname{div} \pi^n)_{0,\Omega} + \tau(q\rho^n, q\varepsilon^n)_{0,\Omega} \\ & + (\theta^n - \theta^{n-1}, \pi^n)_{0,\Omega} + (\theta^n - \theta^{n-1}, a(x)\nabla \varepsilon_t^n)_{0,\Omega} \\ & + (\theta^n - \theta^{n-1}, \tau b(x)\nabla \varepsilon_t^n)_{0,\Omega} + (\theta^n - \theta^{n-1}, \tau d(x)\nabla \varepsilon^n)_{0,\Omega} \\ & + (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), \pi^n)_{0,\Omega} \\ & + (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), a(x)\nabla \varepsilon_t^n)_{0,\Omega} \\ & + (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), \tau b(x)\nabla \varepsilon_t^n)_{0,\Omega} \\ & + (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), \tau d(x)\nabla \varepsilon^n)_{0,\Omega} \\ & + (\tau b(x)\nabla \rho_t^n, a(x)\nabla \varepsilon_t^n)_{0,\Omega} \\ & + (\tau b(x)\nabla \rho_t^n, \tau b(x)\nabla \varepsilon_t^n)_{0,\Omega} \\ & + (\tau b(x)\nabla \rho_t^n, \tau d(x)\nabla \varepsilon^n)_{0,\Omega} + (\tau d(x)\nabla \rho^n, \pi^n)_{0,\Omega}. \end{aligned}$$

We obtain

$$\begin{aligned} \sum_{n=0}^N I_4 = & c(\|\pi^N\|_{0,\Omega}^2 + \|\nabla \varepsilon_t^N\|_{0,\Omega}^2) + c \sum_{n=0}^N (\|\pi^n\|_{0,\Omega}^2 \\ & + \|\nabla \varepsilon_t^n\|_{0,\Omega}^2 + \tau^2 \|\nabla \varepsilon^n\|_{0,\Omega}^2 + \tau^2 \|\nabla \varepsilon_t^n\|_{0,\Omega}^2) \\ & \leq \sum_{i=1}^{20} T_i. \end{aligned}$$

where

$$\begin{aligned} T_1 &= \sum_{n=0}^N \tau(\rho_t^n, \varepsilon_t^n)_{0,\Omega}, T_2 = \sum_{n=0}^N \tau(\rho_t^n, q\varepsilon^n)_{0,\Omega}, \\ T_3 &= \sum_{n=0}^N \tau(\operatorname{div} \theta^n, \varepsilon_t^n)_{0,\Omega}, T_4 = \sum_{n=0}^N \tau(\operatorname{div} \theta^n, \operatorname{div} \pi^n)_{0,\Omega}, \\ T_5 &= \sum_{n=0}^N \tau(\operatorname{div} \theta^n, q\varepsilon^n)_{0,\Omega}, T_6 = \sum_{n=0}^N \tau(q\rho^n, \varepsilon_t^n)_{0,\Omega}, \\ T_7 &= \sum_{n=0}^N \tau(q\rho^n, \operatorname{div} \pi^n)_{0,\Omega}, T_8 = \sum_{n=0}^N \tau(q\rho^n, q\varepsilon^n)_{0,\Omega}, \\ T_9 &= \sum_{n=0}^N (\theta^n - \theta^{n-1}, \pi^n)_{0,\Omega}, \\ T_{10} &= \sum_{n=0}^N (\theta^n - \theta^{n-1}, a(x)\nabla \varepsilon_t^n)_{0,\Omega}, \\ T_{11} &= \sum_{n=0}^N (\theta^n - \theta^{n-1}, \tau b(x)\nabla \varepsilon_t^n)_{0,\Omega}, \\ T_{12} &= \sum_{n=0}^N (\theta^n - \theta^{n-1}, \tau d(x)\nabla \varepsilon^n)_{0,\Omega}, \end{aligned}$$

$$T_{13} = \sum_{n=0}^N (a(x) \nabla(\rho_t^n - \rho_t^{n-1}), \pi^n)_{0,\Omega},$$

$$T_{14} = \sum_{n=0}^N (a(x) \nabla(\rho_t^n - \rho_t^{n-1}), a(x) \nabla \varepsilon_t^n)_{0,\Omega},$$

$$T_{15} = \sum_{n=0}^N (a(x) \nabla(\rho_t^n - \rho_t^{n-1}), \tau b(x) \nabla \varepsilon_t^n)_{0,\Omega},$$

$$T_{16} = \sum_{n=0}^N (a(x) \nabla(\rho_t^n - \rho_t^{n-1}), \tau d(x) \nabla \varepsilon_t^n)_{0,\Omega},$$

$$T_{17} = \sum_{n=0}^N (\tau b(x) \nabla \rho_t^n, a(x) \nabla \varepsilon_t^n)_{0,\Omega},$$

$$T_{18} = \sum_{n=0}^N (\tau b(x) \nabla \rho_t^n, \tau b(x) \nabla \varepsilon_t^n)_{0,\Omega},$$

$$T_{19} = \sum_{n=0}^N (\tau b(x) \nabla \rho_t^n, \tau d(x) \nabla \varepsilon_t^n)_{0,\Omega},$$

$$T_{20} = \sum_{n=0}^N (\tau d(x) \nabla \rho_t^n, \pi^n)_{0,\Omega},$$

$$|T_{17}| \leq \sum_{n=0}^N \tau \|\Delta \varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{18}| \leq \sum_{n=0}^N \tau^2 \|\Delta \varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{19}| \leq \sum_{n=0}^N \tau^2 \|\Delta \varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{20}| \leq \sum_{n=0}^N \tau \|\operatorname{div} \pi^n\|_{0,\Omega}^2 + ch^{2r},$$

So we have

$$\sum_{n=0}^N I_3 \leq c(\tau^2 + h^{2r}).$$

Note that

$$\begin{aligned} \|u^N\|_{0,\Omega}^2 &= \sum_{n=1}^N \frac{\|u^n\|_{0,\Omega}^2 - \|u^{n-1}\|_{0,\Omega}^2}{\tau} \tau \\ &\leq c \sum_{n=1}^N (\tau \|u^n\|_{0,\Omega}^2 + \|\omega^n\|_{0,\Omega}^2), \end{aligned}$$

From the properties cited in [18] and applying Green's formula, we obtain estimates of T_i :

$$|T_1| \leq \sum_{n=0}^N \tau \|\varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r}, |T_2| \leq \sum_{n=0}^N \tau \|\varepsilon^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_3| \leq \sum_{n=0}^N \tau \|\varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r}, |T_4| \leq \sum_{n=0}^N \tau \|\operatorname{div} \pi^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_5| \leq \sum_{n=0}^N \tau \|\varepsilon^n\|_{0,\Omega}^2 + ch^{2r}, |T_6| \leq \sum_{n=0}^N \tau \|\varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_7| \leq \sum_{n=0}^N \tau \|\operatorname{div} \pi^n\|_{0,\Omega}^2 + ch^{2r}, |T_8| \leq \sum_{n=0}^N \tau \|\varepsilon^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_9| \leq \sum_{n=0}^N \|\pi^n\|_{0,\Omega}^2 + ch^{2r}, |T_{10}| \leq \sum_{n=0}^N \|\nabla \varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{11}| \leq \sum_{n=0}^N \tau \|\nabla \varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{12}| \leq \sum_{n=0}^N \tau \|\nabla \varepsilon^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{13}| \leq \sum_{n=0}^N \|\operatorname{div} \pi^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{14}| \leq \sum_{n=0}^N \|\Delta \varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{15}| \leq \sum_{n=0}^N \tau \|\Delta \varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{16}| \leq \sum_{n=0}^N \tau \|\Delta \varepsilon^n\|_{0,\Omega}^2 + ch^{2r},$$

where $\omega^n = \frac{u^n - u^{n-1}}{\tau}$.

So we can obtain

$$\begin{aligned} \max_{0 \leq n \leq t/\tau} (\|\sigma^n - \sigma_h^n\|_{0,\Omega}^2 + \|u^n - u_h^n\|_{0,\Omega}^2 + \|\omega^n - \omega_h^n\|_{0,\Omega}^2) \\ \leq c(\tau^2 + h^{2r}) \end{aligned}$$

Thus, we obtain the following theorem

Theorem 5.2: For $\tau > 0$, we have

$$\begin{aligned} \max_{0 \leq n \leq t/\tau} (\|\sigma^n - \sigma_h^n\|_{0,\Omega}^2 + \|u^n - u_h^n\|_{0,\Omega}^2 + \|\omega^n - \omega_h^n\|_{0,\Omega}^2) \\ = O(\tau^2 + h^{2r}). \end{aligned} \quad (19)$$

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