

Some Discrete Propositions in IVSs

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Abstract—The aim of this paper is to exhibit some properties of local topologies of an IVS. Also, we introduce ISG structure as an interesting structure of semigroups in IVSs.

Keywords—IVS, ISG, Local topology, Lebesgue number, Lindelof theorem

I. INTRODUCTION

THE concept of IVS (*Indexed Variable System*) has been introduced in [2, 4] and some properties of such systems has studied. We use the following definition:

A. Definition

An IVS is a triple (X, Ξ, R) which satisfies the following conditions,

- 1) X is a nonempty set;
 Ξ is a collection of membership-congruent relations $\{=_r\}_{r \in R}$ where R is a subset of interval $[0,1]$ such that $1 \in R$ and,
- 2) for each $x, y \in X$ there exists $r \in R$ such that $x =_r y$;
- 3) If $x =_r y$ then $y =_r x$;
- 4) $x =_1 y$ iff x and y are not different objects. Viz, the set $\{x, y\}$ has one unique element.

In [2, 4] we have seen that every IVS is a metric space and conversely. Also; the relation indexed identity is not an equivalence relation. Moreover, we have seen in [3] that every one-to one fuzzy set [5] on a nonempty set X can introduce an IVS on X . Also; we have the following interesting result

B. Theorem[2].

Let X be an IVS and for nonempty set Y ; $f : X \rightarrow Y$ is an arbitrary function. Then $f(X)$ be an IVS.

II. LOCAL TOPOLOGY

In this section we review the main results in [3] and state some consequences.

A. Theorem.

Let X be an IVS. For each $x \in X$, there exists a local topology on X (called Local Topology with respect to x or generated by x).

Proof. Let (X, Ξ, R) be an IVS, for each $x \in X$ and $r \in R$ define

$$N_r(x) = \{y \in X ; \exists s \in R : x =_s y, s \geq r\}$$

It's obvious that $N_r(x)$ is a nonempty set for each $x \in X$ and $N_0(x) = X$. Also for each $r, r' \in R$; such that $r \leq r'$ we have $N_r(x) \cap N_{r'}(x) = N_{r'}(x)$. Moreover, if $(r_i)_{i \in I}$ is a sequence of elements of R then $\bigcup_{i \in I} N_{r_i}(x) = N_s(x)$; where $s = \inf\{r_i ; i \in I\}$ so, the set $\tau(x) = \{N_r(x) ; r \in R\} \cup \{\Phi\}$ is a topology on X .

B. Example.

Assume that $X = \{x_1, x_2, \dots, x_5\}$ be an IVS and the set of properties $\Xi = \{=_r\}_{r \in R}$ is defined by table below:

TABLE I
INDEXED IDENTITY RELATIONS BETWEEN X ELEMENTS

	x_1	x_2	x_3	x_4	x_5
x_1	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
x_2	$\frac{2}{3}$	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
x_3	$\frac{2}{3}$	$\frac{2}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$
x_4	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	1	$\frac{2}{3}$
x_5	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	1

For the element x_1 of X we have:

$$N_1(x_1) = N_{\frac{3}{3}}(x_1) = \{x_1\},$$

$$N_{\frac{2}{3}}(x_1) = \{x_1, x_2, x_3\},$$

$$N_{\frac{1}{3}}(x_1) = \{x_1, x_2, x_3, x_4, x_5\} = X, \quad N_0(x) = X$$

And hence;

$$\tau(x_1) = \left\{ \Phi, N_1(x_1), N_{\frac{2}{3}}(x_1), N_{\frac{1}{3}}(x_1) = N_0(x_1) = X \right\}$$

Also,

$$\begin{aligned}\tau(x_2) &= \left\{ \Phi, N_1(x_2), N_{\frac{2}{3}}(x_2) = X \right\}, \\ \tau(x_3) &= \left\{ \Phi, N_1(x_3), N_{\frac{2}{3}}(x_3), N_{\frac{1}{3}}(x_3) = X \right\}, \\ \tau(x_4) &= \left\{ \Phi, N_1(x_4), N_{\frac{2}{3}}(x_4), N_{\frac{1}{3}}(x_4) = X \right\}, \\ \tau(x_5) &= \left\{ \Phi, N_1(x_5), N_{\frac{2}{3}}(x_5), N_{\frac{1}{3}}(x_5) = X \right\}\end{aligned}$$

In addition we state some discrete properties of local topologies.

C.Theorem.

Every local topology is a topological base.

Proof. It's clear.

By the set theory, we can state the next result.

D.Theorem.

Every local topology is a chain. Moreover, that is a lattice.

Proof. Between each two arbitrary elements of a local topology there exists relation \subseteq . For each $x \in X$ and $N_{r_1}(x), N_{r_2}(x) \in \tau(x)$ either $N_{r_1}(x) \subseteq N_{r_2}(x)$ or $N_{r_2}(x) \subseteq N_{r_1}(x)$. Also, if $r_1 \leq r_2$; we have:

$$N_{r_1}(x) \cup N_{r_2}(x) = N_{r_1}(x) \text{ and}$$

$$N_{r_1}(x) \cap N_{r_2}(x) = N_{r_2}(x).$$

Thus; local topology has the supremum and infimum properties. Hence it's a lattice.

E. Theorem

Assume that A and B are two closed subsets of a local topology $\tau(x)$. Then either $A \subseteq B$ or $B \subseteq A$.

Proof. If $A \subseteq B$ the proof is complete; if not, A^c and B^c are open. Consequently; there exists $N_r(x), N_{r'}(x)$ such that

$A^c = N_r(x), B^c = N_{r'}(x)$. If $A \not\subseteq B$ then $B^c \not\subseteq A^c$ and so

$N_{r'}(x) \not\subseteq N_r(x)$, From other wise for every

$N_r(x), N_{r'}(x)$ either $N_{r'}(x) \subseteq N_r(x)$ or

$N_r(x) \subseteq N_{r'}(x)$; thus we obtain that $N_r(x) \subseteq N_{r'}(x)$.

Hence, $A^c \subseteq B^c$. It shows that $B \subseteq A$.

Our next result is about Lebesgue number [1] of a local topology:

F. Theorem.

The Lebesgue number (denote by ε) of each open cover in a local topology is not grater than one.

Proof. By the definition of $N_r(x)$ (the elements of a local topology), one can see that when r decreasing, $N_r(x)$ will be grater. In fact, 1 is the greatest radiuses of neighborhoods and other radiuses are less than 1. So, its clear that $\varepsilon \leq 1$.

In addition, we explain and exhibits some concepts and results of topologies.

By the Lindelof theorem [1], every open cover of a subset of \mathfrak{R}^n , can be reduced to an at most countable subcover. This theorem extend to each IVS as below:

G.Theorem. (Lindelof)

Every subcover of a subset A of an IVS X by a local topology $\tau(x)$ can be reduce to an open subset.

Proof. Let $\{O_i\}_{i \in I}$ (where $I \subseteq \mathfrak{R}$ is the set of indexes) is an open cover of A .i.e. $O_i \in \tau(x)$ for each $i \in I$. By theorem E $\{O_i\}_{i \in I}$ has a supremum member O^* such that $\forall i \in I; O_i \subseteq O^*, O^* \in \{O_i\}$. Hence, O^* is open and the proof is complete.

H.Corollary.

Let X be an IVS and $\tau(x)$ be a local topology for X .

Every subset E of X is compact iff there exists

$N_r(x) \in \tau(x)$ such that $E \subseteq N_r(x)$.

Proof. By theorem G its clear.

An equivalence proposition to above corollary is as follow:

Every subset E of an IVS X is compact iff for each open cover $G = \{N_r(x)\}_{r \in R}$ of E ; infimum of set

$$\eta = \{r \in R; N_r(x) \in G\}$$

is again in η .

III. INDEXED SEMIGROUP STRUCTURE (ISG)

Our results in this section will limit to some interesting examples. First attend to following definition:

A. Definition.

Let X be an IVS and $*: X \times X \rightarrow X$ is a function satisfies the next properties:

- 1) $x * y \in X$; $\forall x, y \in X$;
- 2) $x * (y * z) = (x * y) * z$; $\forall x, y, z \in X$ (Associativity);
- 3) there exists $e \in X$ such that for each $x \in X, r \in R$ if $x * e =_r x$ then $e * x =_r x$ (Identity element);
- 4) for each $x \in X$ there exists $x^{-1} \in X$ such that $x * x^{-1} =_r x$ implies that $x^{-1} * x =_r x$. (Inverse element).

$(X, *)$ is called *Indexed Semigroup*. (ISG for short). Moreover, $(X, *)$ is called *Abelian ISG* if it has the additional property:

$$5) \quad x * y = y * x ; \quad \forall x, y \in X.$$

B. Corollary.

Every group $(X, *)$ is an ISG where X is an IVS.

Proof. Its clear.

The converse of the above Corollary is not true. Counter example is in below:

C. Example.

Let $X = \{0, 1, 2\}$ by the following properties.

$$\Xi = \{(a) \text{ is even}, (b) \text{ is odd}, (c) \text{ is prime}, (d) \text{ is component}\}$$

By the assumption 0 and 1 both not prime and component. The table of indexed relations between members of X is:

	0	1	2
0	1	$\frac{2}{4}$	$\frac{3}{4}$
1	$\frac{2}{4}$	1	$\frac{1}{4}$
2	$\frac{3}{4}$	$\frac{1}{4}$	1

Thus, $R = \left\{1, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}\right\}$ and (X, Ξ, R) is an IVS.

Define action on X by the following Table III. $(X, *)$ is an ISG. It is easy to see that X is closed and associative. 0 is an identity element of X and we can get

$$(0)^{-1} = 0, (1)^{-1} = 1, (2)^{-1} = 2;$$

Because

$$(1) * (1)^{-1} = (1)^{-1} * (1) = 2 = \frac{3}{4} 0 ;$$

$$(2) * (2)^{-1} = (2)^{-1} * (2) = 2 = \frac{3}{4} 0$$

and

$$(0) * (0)^{-1} = (0)^{-1} * (0) = 0 = 1 0.$$

From the table its obvious that $*$ is abelian. In the above ISG, inverse of members is not unique. If we get $(1)^{-1} = 2$ and $(2)^{-1} = 1$ then

$$(1) * (1)^{-1} = (1)^{-1} * (1) = 2 = \frac{3}{4} 0 ;$$

$$(2) * (2)^{-1} = (2)^{-1} * (2) = 2 = \frac{3}{4} 0$$

$*$	0	1	2
0	0	1	2
1	1	2	2
2	2	2	2

This examples show that an ISG may be not a group.

There are many examples of ISG's that persuade us to study the ISG structures. One of the importance is the set of all propositions by actions \vee, \wedge , where \vee, \wedge are the conjunction and disjunction of propositions and this is the propose of next studies.

REFERENCES

- [1] C. D. Aliprantis and O. Burkinshaw, *Principal of Real Analysis*, (Academic Press 1990).
- [2] M. H. Moslemy Koupaie and A. Pouhassani, *Indexed Identity and Fuzzy Set Theory*, Proceeding of the 6th Iranian Conference on Fuzzy Systems and 1st Islamic World Conference on Fuzzy Systems (Farsi Version), 2006.
- [3] A. Pouhassani and M. H. Moslemy Koupaie, *Topological Entropy of an IVS*, Iranian Mathematical Conference, (2007).
- [4] A. S. Sant' Anna, *Indexed Identity and Fuzzy Set Theory*, Available: (arXiv: math. Lo/9910182 V1, 1 Nov 1999).
- [5] H. J. Zimmermann, *Fuzzy Set Theory and its Applications*, Third Editions, (Kluwer Acad. Pub. Boston, Dordrecht, London, 1998).



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