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On the Fp-Normal Subgroups of Finite Groups

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***Abstract*—**Let G be a finite group, and let F be a formation of finite group. We say that a subgroup H of G is F_p-normal in G if there exists a normal subgroup T of G such that HT is a permutable Hall subgroup of G and $(H \cap T)H_G / H_G$ is contained in the F-hypercenter $Z_\infty^F(G/H_G)$ of G/H_G. In this note, we get some results about the F_p-normal subgroups and then use them to study the structure of finite groups.

***Keywords*—**Finite group, F_p-normal subgroup, Sylow subgroup, Maximal subgroup

I. Introduction

All groups considered in this paper are finite, and G denotes a finite group. The notation and terminology are standard, as in [1].Recall that, for a class F of groups, a chief factor H/K of a group G is called F-central (see [2]) if $[G/K](G/C_G(H/K)) \in F$. The symbol $Z_\infty^F(G)$ denotes the F-hypercenter of a group G, that is, the product of all such normal subgroups H of G whose G-chief factors are F- central. A subgroup H of G is said to be F-hypercenter [3] in G if $H \le Z_\infty^F(G)$ A class F of groups is called a formation if it is closed under a homomorphic image and a subdirect product. It is clear that every group G has a smallest normal subgroup (called F-residual of G and denoted by G^F) with quotient in F. A formation F is said to be saturated if it contains every group G with $G/\phi(G) \in F$. We use U and S to denote the formations of all supersoluble groups and soluble groups, respectively. Recall that a subgroup H of G is said to be complemented in G if G has a subgroup B such that $G = AB$ and $A \cap B = 1$ (see [4]). A subgroup H of G is said to be F_h-normal [5] (or c-normal [6], F_c-normal [7] or F_n-supplemented [8]) in G if there exists a normal subgroup T of G such that HT is a normal Hall subgroup of G, and $(H \cap T)H_G / H_G \le Z_\infty^F(G/H_G)$. By using the above subgroups, people have obtained some interesting results (see [5,7,8,9]}). As a development of this topic, we now introduce the following new concept.

Definition 1.1 Let F be a class of groups. A subgroup H of $\$G$ is said to be F_p-normal in G if there exists a normal subgroup T of G such that HT is a permutable Hall subgroup of G, and $(H \cap T)H_G / H_G \le Z_\infty^F(G/H_G)$ Obviously, all normal subgroups, c-normal subgroups, F_n-supplemented subgroups and F_h-normal subgroups are all F_p-normal in G for any nonempty saturated formation F. However, the following example shows that the converse is not true. For example, if a subgroup H is c-normal in G, then there exists a normal subgroup K such that $G = HK$ and

$$(H \cap T)H_G / H_G = 1 \le Z_\infty^F(G/H_G)$$

However, the following example shows that the converse is not true.

Example 1.2. Let S_4 be the symmetric group of degree 4 , P be the Sylow 3-subgroup of S_4 and Z be a group of order p with $p \ne 2,3$. Let $G = Z\imath S_4 = [K]S_4$ be a regular wreath product, where K is the base group of the regular wreath product G. Then PK is a permutable Hall subgroup and $P \cap K = 1$. Hence P is F_p-normal in G for any nonempty saturated formation F. However, it is easy to see that P is not normal, c-normal, S_h-normal and is not U_c-supplemented in G (in fact, for example, G is the only normal subgroup of G such that $PG = G$ and $P_G = 1$. However, since the unique minimal normal subgroup K of G is not cyclic, $P \cap G = P \not\le Z_\infty^S(G)$. Thus, P is not S_h-normal in G).

In this paper, we study the properties of F_p-normal subgroups and use them to give some new characterizations of some classes of groups. Some previously known results are generalized.

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II. Preliminaries

A formation F is said to be S-closed (resp. S_n-closed) if it contains all subgroups (resp. all normal subgroups) of all its groups.

We cite the following lemmas which will be useful in the sequel.

Lemma 2.1 [9, Lemma 2.1] Let G be a group and $A \leq G$. Let F be a non-empty saturated formation and $Z = Z_\infty^F(G)$. Then

(1) If A is normal in G, then $AZ/A \leq Z_\infty^F(G/A)$;

(2) If F is S-closed, then $Z \cap A \leq Z_\infty^F(G)$;

(3) If F is S_n-closed and A is normal in G, then $Z \cap A \leq Z_\infty^F(G)$;

(4) If $G \in F$, then $Z = G$.

Lemma 2.2 Let F be a saturated formation containing U and G a group with a normal subgroup E such that $G/E \in F$. If E is cyclic, then $G \in F$.

Proof. We assume that E is a minimal normal subgroup of G such that $E \not\subseteq \phi(G)$. Let M be a maximal subgroup such that $G = [E]M$ and let $C = C_G(E)$. Then $C \cap M = M_G \lhd G$, and so we have $G/M_G = [EM_G/M_G](M/M_G) \in F$. Hence $G \cong G/(E \cap M_G) \leq G/E \times G/M_G$ is supersoluble. Thus $G \in F$.
This completes the proof. □

Lemma 2.3 [14, Lemma 2.6] Let N be a soluble normal subgroup of G. If $N \cap \phi(G) = 1$, then $F(N)$ is the direct product of minimal normal subgroups of G contained in N.

Definition 2.4 Let H be a subgroup of G. A subgroup K of G is said to be a supersoluble supplement of H in G if K is a supersoluble subgroup of G such that $G = HK$

Lemma 2.5 [5, Lemma 2.6] Suppose that H has a supersoluble supplement in G.

(1) If $N \lhd G$, then HN/N has a supersoluble supplement in G/N.

(2) If $H \leq K \leq G$, then H has a supersoluble supplement in K.

Lemma 2.6 [5, Lemma 2.7] Suppose that G has a unique minimal normal subgroup N and $\phi(G) = 1$. If N is soluble, then there exist a maximal subgroup M of G such that $G = [N]M$, and $N = O_p(G) = F(G) = C_G(N)$ for some prime p.

Definition 2.7 Let F be a class of groups. A subgroup H of G is said to be F_p-normal in G if there exists a normal subgroup T of G such that HT is a permutable Hall subgroup of G, and $(H \cap T)H_G/H_G \leq Z_\infty^F(G/H_G)$.

Lemma 2.8 Let G be a group and $H \leq K \leq G$. Then

(1) H is F_p-normal in G if and only if G has a normal subgroup T such that HT is a permutable Hall subgroup of G, $H_G \leq T$ and $(H/H_G) \cap (T/H_G \leq Z_\infty^F(G/H_G)$.

(2) Suppose that H is normal in G. Then K/H is F_p-normal in G/H if and only if K is F_p-normal in G.

(3) Suppose that H is normal in G. Then for every F_p-normal subgroup E in G satisfying $(|H|,|E|) = 1$, the subgroup HE/H is F_p-normal in G/H.

(4) If H is F_p-normal in G and F is hereditary, then H is F_p-normal in K.

(5) If K is normal in G and F is normally hereditary, then H is F_p-normal in K.

(6) If $G \in F$, then every subgroup of G is F_p-normal in G.

Proof. (1) Assume that H is F_p-normal in G and let T be a normal subgroup of G such that HT is a permutable Hall subgroup of G and $(H \cap T)H_G/H_G \leq Z_\infty^F(G/H_G)$. Let $T_0 = TH_G$. Then T_0 is normal in G, $HT_0 = HTH_G = HT$ is a permutable Hall subgroup of G and

$$(T_0/H_G) \cap (H/H_G) = (T_0 \cap H)/H_G = (T \cap H)H_G/H_G \leq Z_\infty^F(G/H_G).$$

(2) First assume that K/H is F_p-normal in G/H. Then by (1), G/H has a normal subgroup T/H such that $(K/H)(T/H)$ is a permutable Hall subgroup of G/H, $(K/H)_{G/H} \leq T/H$ and

$$(T/H)/(K/H)_{G/H} \cap (K/H)/(K \cap H)_{G/H} \leq Z_\infty^F((G/H)/(K/H)_{G/H}).$$

Then T is normal in G and KT is a permutable Hall subgroup of G. Besides, since

$$(T/H)/(K/H)_{G/H} \cap (K/H)/(K/H)_{G/H}$$
$$= (T/H)/(K_G/H) \cap (K/H)/(K_G/H)$$
$$= ((T \cap K)/H)/(K_G/H)$$

and

$$Z_\infty^F((G/H)/(K/H)_{G/H}) = Z_\infty^F((G/H)/(K_G/H)),$$

we have

$$(T \cap K)/K_G = (T/K_G) \cap (K/K_G) \le Z_\infty^F(G/K_G).$$

Hence, K is F_p-normal in G. Analogously, one can show that if K is F_p-normal in G, then K/H is F_p-normal in G/H.

(3) Assume that E is F_p-normal in G. Then by (2), G has a normal subgroup T such that ET is a permutable Hall subgroup of G, $E_G \le T$ and $(E/E_G) \cap (T/E_G) \le Z_\infty^F(G/E_G)$. We will prove that HE/H is F_p-normal in G/H. By (1), we only need to show that HE is F_p-normal in G. Since $(|H|,|E|)=1$, we have $H \le T$ and so $T \cap HE = H(T \cap E) \le HZ$, where $Z/E_G = Z_\infty^F(G/E_G)$. Hence from the G-isomorphism

$$HZ/HE_G = HE_GZ/HE_G$$
$$\cong Z/(Z \cap HE_G) = Z/E_G(Z \cap H),$$

we have $HZ/HE_G \le X/HE_G = Z_\infty^F(G/HE_G)$ and so $(HE \cap T)/HE_G \le X/HE_G$. Let $D = HE_G$. By Lemma 2.3,

$$(X/HE_G)(D/HE_G)/(D/HE_G) \le Z_\infty^F(G/HE_G).$$

Therefore,

$$(TD/D) \cap (HE/D) = D(T \cap HE)/D \le Z_\infty^F(G/D)$$

and so HE is F_p-normal in G.

(4) Let T be a normal subgroup of G such that HT is a permutable Hall subgroup of G, $H_G \le T$ and $(H/H_G) \cap (T/H_G) \le Z_\infty^F(G/H_G)$. Let $T_1 = H_G(H \cap K)$. Since $K = K \cap HT = H(K \cap T)$, we have $K = HT_1$. By Lemma 2.2(3), T_1 is normal in K. Besides,

$$(T_1/H_G) \cap (H/H_G)$$
$$= H_G(H \cap T \cap K)/H_G$$
$$\le Z/H_G = Z_\infty^F(G/H_G) \cap K/H_G.$$

Assume that F is hereditary. Then by Lemma 2.3(2), $Z/H_G \le Z_\infty^F(G/H_G)$. By Lemma 2.3(1),

$$(Z/H_G)(HK/H_G)/(HK/H_G)$$
$$\le Z_\infty^F((K/H_G)/(H_K/H_G))$$

and so $(T_1/H_K) \cap (H/H_K) \le Z_\infty^F(K/H_K)$. Hence, H is F_p-normal in K

(5) See the proof of (4).

(6) Assume that $G \in F$ and let H be an arbitrary subgroup of G. By Lemma 2.3(6), $Z = Z_\infty^F(G) = G$, and so by Lemma 2.3(1), $Z_\infty^F(G/H_G) = G/H_G$. Hence, $H/H_G \le Z_\infty^F(G/H_G)$.

This completes the proof. □

Lemma 2.9 Let R be a soluble minimal normal subgroup of G. If there exists a maximal subgroup R_1 of R such that R_1 is U_p-normal in G, then R is a group of prime order.

Proof. Assume that $|R|$ is not a prime. Then $R_1 \ne 1$. Since R is a minimal normal subgroup R_1 of R, $(R_1)_G = 1$. Then, by hypothesis, there exists a normal subgroup K of G such that R_1K is a permutable Hall subgroup of G, $R_1 \cap K \le Z_\infty^F(G)$ and R_1K is subnormal in G. Hence $R \le R_1K$ since the minimality of R. Since $R \cap K \lhd G$, $R \cap K = 1$ or $R \cap K = R$. If $R \cap K = 1$, then $R = R \cap R_1K = R_1(R \cap K) = R_1$, a contradiction. If $R \cap K = R$, then $R \le K$. Hence, $R_1 \le K$ and therefore, $1 \ne R_1 = R_1 \cap K \le Z_\infty^U(G)$. Thus $1 \ne R_1 \le Z_\infty^U(G) \cap R \lhd G$. It follows, from the minimality of R, that $R \le Z_\infty^U(G)$. Consequently, $|R|$ is a prime.

The final contradiction completes the proof. □

III. Characterization of Supersoluble Groups

Theorem 3.1 Let F be a saturated formation containing U. Then $G \in F$ if and only if there exists a soluble normal subgroup H of G such that $G/H \in F$, and every maximal subgroup of every Sylow subgroup of $F(H)$, which has no supersoluble supplement in G, is U_p-normal in G.

Proof. The necessity part is obvious. We only need to prove the sufficienty part. Suppose that the assertion is false and let G be a contraexample with $|G||H|$ minimal. Let P be an arbitrary Sylow p-subgroup of $F(H)$. Since P char $F(H)$ char $H \lhd G$. We proceed the proof via the following steps.

Step 1. $\phi(G) = 1$.

If $\phi(G) \neq 1$, then $\phi(G) \leq F(H)$. Let $R = \phi(G)$. Clearly, $(G/R)/(H/R) \cong G/H \in F$. By [10, III, Theorem 3.5], we have that $F(H/R) = F(H)/R$. Let P/R be a Sylow p-subgroup of $F(H)/R$, and P_1/R be a maximal subgroup of P/R. Then P_1 is a maximal subgroup of P. By hypothesis, P_1 either has supersoluble supplement in G or is U_p-normal in G. It follows from Lemmas 2.5(1) and 2.8(2), P_1/R either has supersoluble supplement in G/R or is U_p-normal in G/R. Now Let Q/R be a maximal subgroup of some Sylow q-subgroup of $F(H)/R$, where $q \neq p$. Then $Q = Q_1R$, where Q_1 is a maximal subgroup of the Sylow q-subgroup of $F(H)$. By hypothesis, Q_1 either has supersoluble supplement in G or is U_p-normal in G. It follows from Lemmas 2.5(1) and 2.8(3), that Q_1R/R either has supersoluble supplement in G/R or is U_p-normal in G/R. Hence $(G/R, H/R)$ satisfies the hypothesis. The minimal choice of (G,H) inplies that $G/R \in F$. Since $G/\phi(G) = G/R$ and F is a saturated formation, we have $G \in F$, a contradiction.

Step 2. $P = Z_1 \times Z_2 \times \ldots \times Z_m$, where every Z_i is a normal subgroup of order p of G.

Since $P \lhd G$, by Step 1, $P \cap \phi(G) = 1$. Hence by Lemma 2.3, $P = Z_1 \times Z_2 \times \ldots \times Z_m$, where all Z_i are minimal normal subgroups of G. Next, we prove that all Z_i are of prime order p.

Assume that $|Z_i| > p$ for some i. Without loss of generality, let $|Z_1| > p$. Let Z_1^* be a maximal subgroup of Z_1. Then $Z_1^* \times Z_2 \times \ldots Z_m = P_1$ is a maximal subgroup of a Sylow p-subgroup P of $F(H)$. Set $T = Z_2 \times \ldots \times Z_m$, then $(P_1)_G = T$. If P_1 has a supersoluble supplement K in G, then $G = P_1K = Z_1^*Z_2\ldots Z_mK = Z_1^*TK$. Since $T \lhd G$, TK is a subgroup of G and so $|G:TK| \leq |Z_1^*| < |Z_1|$. Since

$$(Z_1 \cap TK)^G = (Z_1 \cap TK)^{Z_1TK}$$
$$= (Z_1 \cap TK)^{TK} = Z_1 \cap TK,$$

we have $Z_1 \cap TK \lhd G$. Hence, we have either $Z_1 \cap TK = 1$ or $Z_1 \cap TK = Z_1$. If $Z_1 \cap TK = 1$, then we have $|G:TK| = |Z_1| > |Z_1^*|$, a contradiction.
If $Z_1 \cap TK = Z_1$, then $Z_1 \leq TK$ and so $G = TK$. Since $T \lhd G$, then $K \cong G/T = TK/T = K/T \cap K$ is supersoluble as K is a supersoluble supplement and $Z_1 \cong Z_1T/T$ is chief factor of G/T. It follows that $|Z_1| = |Z_1T/T| = p$, which is a contradiction. Thus P_1 is U_p-normal in G, and so by Lemma 2.8(1), there exists a normal subgroup N of G such that $(P_1)_G \leq N$, P_1N is a permutable Hall subgroup of G, and $(P_1 \cap N)/(P_1)_G \leq Z_\infty^U(G/(P_1)_G)$. Hence $P_1N = Z_1^*Z_2\ldots Z_mN = Z_1^*(P_1)_GN = Z_1^*N$. We only think the following two cases.

Case 1. $Z_1^* \cap N = 1$.

In this case, $(Z_1^*)_G \leq Z_1^* \cap N$ and so $(Z_1)_G = 1 \leq N \lhd Z_\infty^F(G/(Z_1^*)_G)$, $Z_1^N = P_1N$ is a permutable subgroup of G, which implies that Z_1^* is U_p-normal in G. Hence By Lemma 2.9, Z_1 is a cyclic group of order p, a contradiction.

Case 2. $Z_1^* \cap N \neq 1$.

In this case, $1 < Z_1 \cap N \lhd G$ since Z_1 and N are both normal in G. By the minimality of Z_1, we have either $Z_1 \cap N = 1$ or $Z_1 \cap N = Z_1$. We assume that $Z_1 \cap N = 1$. Hence $Z_1^* \cap N = 1$, a contradiction. Hence $Z_1 \cap N = Z_1$ and so $Z_1 \leq N$. Thus $P_1N = Z_1^*N = N$. Consequently, $P_1 \leq N$. It follows that
$P_1/(P_1)_G = (P_1 \cap N)/(P_1)_G \leq Z_\infty^F(G/(P_1)_G) \cap P/(P_1)_G$. If $P_1 = (P_1)_G$, then $Z_1^* = 1$, which contradicts $Z_1^* \cap N \neq 1$. Hence $(P_1)_G < P_1$ and so

$$1 \neq P_1/(P_1)_G = (P_1 \cap N)/(P_1)_G \leq Z_\infty^F(G/(P_1)_G) \cap P_1/(P_1)_G.$$

Since $P = Z_1 \times \ldots Z_m = Z_1T = Z_1(P_1)_G$, then we have that $P/(P_1)_G = P/P_1 \cong Z_1$ and $P/(P_1)_G$ is a chief factor of

G because of the minimality of Z_1. This implies that $Z_\infty^F(G/(P_1)_G) \cap P/(P_1)_G$, and thereby, $P/(P_1)_G \le Z_\infty^F(G/(P_1)_G)$. It means that $|P/(P_1)_G| = p$. Therefore $P_1/(P_1)_G = 1$, which contradicts $P_1/(P_1)_G \ne 1$.

Step 3. $P = F(H)$.

Obviously, $P \le F(H)$. Assume that $P < F(H)$, then since $F(H)$ is nilpotent, there exists a Sylow q-subgroup Q of $F(H)$ and $Q \lhd G$. From Step 2, $P = Z_1 \times \ldots \times Z_m$ and $P = R_1 \times \ldots \times R_s$, where Z_i, R_j are a normal subgroup of G of order p, q respectively. Obviously, $F(H)/Z_i \le F(H/Z_i)$ and $F(H)/R_j \le F(H/R_j)$. From Step 1, we have $G/Z_i \in F$ and $G/R_j \in F$. It follows that $G/1 \cong G/Z_i \times G/R_j$, a contradiction.

Step 4. $G/F(H) \in F$.

From Steps 2 and 3, $P = F(H) = Z_1 \times Z_2 \times \ldots \times Z_m$, where every Z_i is a normal subgroup of G of order p. Since $G/C_G(Z_i)$ is isomorphic to a subgroup of $Aut(Z_i)$, $G/C_G(Z_i)$ is cyclic and so it lies in U for each i. It follows that $G/\cap_{i=1}^m C_G(Z_i) \in U$ Obviously, $C_G(F(H)) = \cap_{i=1}^m C_G(Z_i)$. Hence, $G/C_G(F(H)) \in U \subseteq F$. Consequently, $G/(H \cap C_G(F(H))) = G/C_H(F(H)) \in F$. Since $F(H)$ is abelian and H is soluble, we have that $F(H) \le C_G(F(H)) \le F(H)$. Thus, $F(H) = C_G(F(H)) = P$ and so $G/F(H) \in F$.

Step 5. If K is a miniml normal subgroup of G contained in H, then $G/K \in F$.

Since H is soluble, then $K \le F(H)$. By Lemmas 2.5(1) and 2.8(2), we have that every maximal subgroup of $F(H)/K$ either has a supersoluble supplement in G/K or is U_p-normal in G/K and $(G/K)/(H/K) \in F$. It follow that $(G/K, F(H)/K)$ satisfies the hypothesis. The minimal choice of (G, H) implies that $G/K \in F$.

Step 6. Final contradiction.

Since F is a saturated formation, then by Steps 2 and 4, we have that $F(H)$ is the uniue minimal ormal subgroup of G contained in H, and $F(H) = R_1$ is a cyclic group of order p. Hence by Step 4 and Lemma 2.2, $G \in F$, a contradiction. The final contradiction completes the proof. □

Corollary 3.2 [5] Let F be a saturated formation containing U. Then $G \in F$ if and only if there exists a soluble normal subgroup H of G such that $G/H \in F$, and every maximal subgroup of every Sylow subgroup of $F(H)$, which has no supersoluble supplement in G, is U_p-normal in G.

Corollary 3.3 [12] Let G be a soluble group having a normal subgroup H such that G/H is supersoluble. If every maximal subgroup of any Sylow subgroup of $F(H)$ $ either is normal in G or has a supersoluble supplement in G, then G is supersoluble.

Corollary 3.4 [14] Let G be a group. If H is a soluble normal subgroup of G with supersoluble quotient G/H and all maximal subgroups of all Sylow subgroups of $F(H)$ are c-normal in G, then G is supersoluble.

Corollary 3.5 [15] If G is a soluble group and all maximal subgroups of Sylow subgroups of $F(G)$ are normal in G, then G is supersoluble
.

Corollary 3.6 [16] Let F be a saturated formation containing U. Suppose that G is a group with a soluble normal subgroup H such that $G/H \in F$. If all maximal subgroups of any Sylow subgroup of $F(H)$ are c-normal in G, then $G \in F$.

IV. Characterization of Soluble Groups

Theorem 4.1 Let p be the smallest prime dividing |G|, and let P be some Sylow p-subgroup P. Then group G is soluble if and only if every maximal subgroup of P is U_p-normal in G.

Proof. Let P be an arbitrary Sylow subgroup of G. If G is soluble, then G/P_G is also soluble and so $Z_\infty^S(G/P_G) = G/P_G$. Hence, the necessity part obviously holds. We now prove the sufficiency part. Suppose that the assertion is false, and let G be a counterexample of minimal order. Then by the well-known Feit-Thompson's theorem, we have p=2. Now, we proceed the proof by the following steps.

Step 1. $O_2(G) = 1$.

Let $N = O_2(G) \neq 1$. Obviously, P/N is a Sylow 2-subgroup of G/N. Let M/N be a maximal subgroup of P/N. Then M is a maximal subgroup of P and so M is is Up-normal in G by hypothesis. By Lemma 2.8(2), M/N is Up-normal in G/N. The minimality of G implies that G/N is soluble. It follow that G is soluble, a contradiction.

Step 2. $O_{2'}(G) = 1$.

Let $D = O_{2'}(G) \neq 1$. Then PD/D is a Sylow 2-subgroup of G/D. Let M/D be a maximal subgroup of PD/D. Then there exists a maximal subgroup P_1 of P such that $M = P_1 D$ and so P_1 is Up-normal in G by hypothesis. By Lemma 2.8(3), M/D is Up-normal in G/D. Hence G/D is soluble by the minimal choice of G, and so G is soluble, a contradiction.

Step 3. Final contradiction.

Let P_1 be a maximal subgroup of P. By the hypothesis, there exists a normal subgroup K of G such that $P_1 K$ is a permutable Hall subgroup of G and
$(P_1 \cap K)(P_1)_G / (P_1)_G \leq Z_\infty^S (G/(P_1)_G)$.
By Steps 1 and 2, we have $(P_1)_G = 1$ and $Z_\infty^S(G) = 1$. This induces that $P_1 \cap K = 1$. Since $P_1 K$ is a permutable Hall subgroup of G, $|K_2| = 2$, where K_2 is some Sylow 2-subgroup of K. By \cite[Theorem 10.1.9]{robin}, we see that K is 2-nilpotent, and so K has a normal 2-complement $K_{2'}$. Since $K_{2'}$ char $K \lhd G$, $K_{2'} \lhd G$. Hence, by Step 2, $K_{2'} = 1$, and so $|K| = 2$, which contradicts Step 1.

This completes the proof. □

Corollary 4.2 Let M be a maximal subgroup of G with |G : M| = r, where r is a prime. Let p be the smallest prime dividing |M|. If there exists a Sylow p-subgroup P of M such that every maximal subgroup of P is Sp-normal in G, then G is soluble.

Proof. By Feit-Thompson's theorem, we may assume that 2||G|. If r =2, then M is normal in G. By Lemma 2.8(4), every maximal subgroup of P is Sp-normal in M. Hence, by Theorem 4.1, M is soluble. It follows that G is soluble. If r $\neq$ 2, then p =2$, and so P is a Sylow 2-subgroup of G. By Theorem 4.1, we have that G is soluble.

This completes the proof. □

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