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Extended Cubic B-spline Interpolation Method Applied to Linear Two-Point Boundary Value Problems

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Abstract—Linear two-point boundary value problem of order two is solved using extended cubic B-spline interpolation method. There is one free parameters, λ, that control the tension of the solution curve. For some λ, this method produced better results than cubic B-spline interpolation method.

Keywords—two-point boundary value problem, B-spline, extended cubic B-spline.

I. Introduction

CONSIDER the general form of linear two-point boundary value problem

$$u''(x) + p(x)u'(x) + q(x)u(x) = r(x),$$
$$x \in [a,b], \quad u(a) = \alpha, \quad u(b) = \beta. \tag{1}$$

This problem has a unique solution, $u(x)$, if $p, q, r \in C^1$ and $q(x) < 0$ [1]. Generally, this problem is difficult to solve analytically. Some of the most frequently used numerical methods are shooting, finite difference, finite element and finite volume methods [1], [2]. These methods, although requiring little computational time, evaluate the approximated solutions only at the collocation points, $u(x_i)$ for $i = 0, 1, ..., n$.

A different approach of solving linear two-point boundary value problem has first been suggested by Bickley in 1968 [3]. He used cubic spline interpolation to model the solution curve and applied the differential equation as well as the boundary conditions to solve for the unknown constants. As a result, a set of equations could be produced approximating the analytical solution. Further work on this approach can be found in [4], [5]. Thirty years later, Caglar et al. proposed the use of cubic B-spline interpolation to solve this problem. The basis function of B-spline is constructed using piecewise polynomial function that satisfies C^2 continuity. The definition and properties of the function as well as their approach can be found in [6] and the references therein. Continuing with this work, we applied the same procedure using extended cubic B-spline interpolation to solve the problem.

Extended B-spline is a generalization of B-spline. One free parameter, λ, is introduced within the basis function that can be used to change the shape of the produced curve. The value of λ is varied systematically and the results were analyzed. The value of λ producing the least error is identified. One example is provided at the end.

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II. Extended Cubic B-spline Basis Function

For a finite interval $[a, b]$, let $\{x_i\}_{i=0}^{n}$ be a partition of the interval with uniform step size, h. We can extend the partition using

$$h = \frac{b-a}{n}, \quad x_0 = a, \quad x_i = x_0 + ih, \quad i = \pm 1, \pm 2, \pm 3, ...$$

Extended cubic B-spline basis function is constructed by linear combination of the cubic B-spline basis function [7]. Here, blending function of degree 4, $EB_{3,i}(x)$, is considered and the resulting function is shown in (2).

$$\frac{1}{24h^4}\begin{cases} b_i(x), & x \in [x_i, x_{i+1}], \\ b_{i+1}(x), & x \in [x_{i+1}, x_{i+2}], \\ b_{i+2}(x), & x \in [x_{i+2}, x_{i+3}], \\ b_{i+3}(x), & x \in [x_{i+3}, x_{i+4}], \end{cases} \tag{2}$$

$$\begin{aligned} b_i(x) &= -4h(\lambda-1)(x-x_i)^3 + 3\lambda(x-x_i)^4, \\ b_{i+1}(x) &= (4-\lambda)h^4 + 12h^3(x-x_{i+1}) + \\ &\quad 6h^2(2+\lambda)(x-x_{i+1})^2 - 12h(x-x_{i+1})^3 - \\ &\quad 3\lambda(x-x_{i+1})^4, \\ b_{i+2}(x) &= (16+2\lambda)h^4 - 12h^2(2+\lambda)(x-x_{i+2})^2 + \\ &\quad 12h(1+\lambda)(x-x_{i+2})^3 - 3\lambda(x+x_{i+2})^4, \\ b_{i+3}(x) &= -(h+x_{i+3}-x)^3\left[h(\lambda-4) + 3\lambda(x-x_{i+3})\right]. \end{aligned}$$

Extended cubic B-spline basis will degenerate into cubic B-spline basis when $\lambda = 0$. For $\lambda \in [-8, 1]$, B-spline and extended B-spline share the same properties: local support, non-negativity, partition of unity and C^2 continuity.

III. Extended Cubic B-spline Interpolation

Given $\{x_i\}$, the extended cubic B-spline function, $S(x)$ is a linear combination of the extended cubic B-spline basis function,

$$S(x) = \sum_{i=-3}^{n-1} C_i EB_{3,i}(x), \quad x \in [x_0, x_n], \tag{3}$$

where C_i are unknown real coefficients. Since $EB_{3,i}(x_i)$ has a support on $[x_i, x_{i+4}]$, there are three nonzero basis function evaluated at each $x_i : EB_{3,i-3}(x_i)$, $EB_{3,i-2}(x_i)$ and $EB_{3,i-1}(x_i)$. Thus, from (3), for $i = 0, 1, ..., n$,

$$\begin{aligned} S(x_i) & \\ &= C_{i-3}E_{4,i-3}(x) + C_{i-2}E_{4,i-2}(x) + C_{i-1}E_{4,i-1}(x), \\ &= C_{i-3}\left(\tfrac{4-\lambda}{24}\right) + C_{i-2}\left(\tfrac{8+\lambda}{12}\right) + C_{i-1}\left(\tfrac{4-\lambda}{24}\right), \quad (4) \\ S'(x_i) & \\ &= C_{i-3}E'_{4,i-3}(x) + C_{i-2}E'_{4,i-2}(x) + C_{i-1}E'_{4,i-1}(x), \\ &= C_{i-3}\left(-\tfrac{1}{2h}\right) + C_{i-2}(0) + C_{i-1}\left(\tfrac{1}{2h}\right), \quad (5) \\ S''(x_i) & \\ &= C_{i-3}E''_{4,i-3}(x) + C_{i-2}E''_{4,i-2}(x) + C_{i-1}E''_{4,i-1}(x), \\ &= C_{i-3}\left(\tfrac{2+\lambda}{2h^2}\right) + C_{i-2}\left(-\tfrac{2+\lambda}{h^2}\right) + C_{i-1}\left(\tfrac{2+\lambda}{2h^2}\right). \quad (6) \end{aligned}$$

Returning to the two-point boundary value problem stated in (1), $S(x)$ is presumed to be the approximation of its solution, $u(x)$. Substituting $S(x)$ into (1), the equation becomes

$$\begin{aligned} u''(x) + p(x)u'(x) + q(x)u(x) = r(x), \\ x \in [a, b], \quad u(a) = \alpha, \quad u(b) = \beta. \end{aligned} \quad (7)$$

Substituting (4), (5) and (6) into (7) would result in a system of linear equations of order $(n + 3) \times (n + 3)$. The C_i's are solved from the system and are substituted in (3). The resulting equation becomes the approximated analytical solution for (1).

IV. Varying λ

The value of λ is varied systematically in the neighborhood of zero using brute force with suitable step size. At each trial, Max-norm and L^2-norm for the solution are calculated. The values of λ with the lowest norms are identified. Suppose that the true and approximated solution of (1) are $u(x)$ and $S(x)$, respectively. The norms are calculated using the following equations:

$$\text{Max-norm} = \max_{i=0}^{n} |S(x_i) - u(x_i)|,$$

$$L^2\text{-norm} = \sum_{i=0}^{n} [S(x_i) - u(x_i)]^2.$$

V. Numerical Example and Conclusion

Problem 5.1 **[6]**
$u''(x) - u'(x) = -e^{x-1} - 1, \quad x \in [0, 1], \quad u(0) = u(1) = 0.$
Exact solution: $u(x) = x\left(1 - e^{x-1}\right)$.

Problem 5.1 was solved using extended cubic B-spline interpolation method. The numerical results are shown in Table I. The first row is the norms when $\lambda = 0$, that is, for cubic B-spline interpolation method. Using $\lambda = 2.9762 \times 10^{-3}$, the approximated analytical solution is given in (8). The plots of $S(x)$ and $u(x)$ along with the error are presented in Figure 1.

TABLE I
The best values of λ for Example 5.1

λ	Max-Norm	L^2-Norm
0	2.8996×10^{-4}	6.6089×10^{-4}
2.9762×10^{-3}	3.1415×10^{-6}	7.2625×10^{-6}
2.9776×10^{-3}	3.2452×10^{-6}	7.2555×10^{-6}

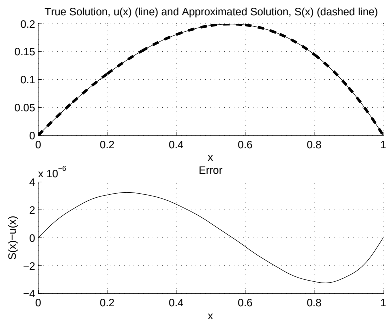

Fig. 1. Comparison between the exact and approximated solutions

$$S(x) =$$
$$\begin{cases} 3.683 \times 10^{-16} + 0.6321x - \\ \quad 0.3679x^2 - 0.1849x^3 - 0.05905x^4, & x \in [0.0, 0.1], \\ 1.380 \times 10^{-6} + 0.6321x - \\ \quad 0.3677x^2 - 0.1835x^3 - 0.06844x^4, & x \in [0.1, 0.2), \\ 2.621 \times 10^{-8} + 0.6322x - \\ \quad 0.3691x^2 - 0.1769x^3 - 0.07915x^4, & x \in [0.2, 0.3), \\ -5.759 \times 10^{-5} + 0.6331x - \\ \quad 0.3743x^2 - 0.1638x^3 - 0.09136x^4, & x \in [0.3, 0.4), \\ -3.515 \times 10^{-4} + 0.6362x - \\ \quad 0.3865x^2 - 0.1425x^3 - 0.1053x^4, & x \in [0.4, 0.5), \\ -0.001306 + 0.6439x - \\ \quad 0.4098x^2 - 0.1112x^3 - 0.1211x^4, & x \in [0.5, 0.6), \\ -0.003760 + 0.6601x - \\ \quad 0.4497x^2 - 0.06744x^3 - 0.1390x^4, & x \in [0.6, 0.7), \\ -0.009215 + 0.6905x - \\ \quad 0.5131x^2 - 0.008663x^3 - 0.1595x^4, & x \in [0.7, 0.8), \\ -0.02017 + 0.7434x - \\ \quad 0.6089x^2 + 0.06834x^3 - 0.1826x^4, & x \in [0.8, 0.9), \\ -0.04058 + 0.8306x - \\ \quad 0.7484x^2 + 0.1673x^3 - 0.2089x^4, & x \in [0.9, 1.0]. \end{cases} \quad (8)$$

These results show that extended cubic B-spline has potential to approximate the solution of two-point boundary value problems better than B-spline. Here, we used the exact solution of the problem as a reference to find good values of λ. Therefore, future work will focus on finding the values of λ that produce better approximation from the differential equation in (1) itself without using the exact solution. This study confirmed that for some problems, these values do exist.

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