International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:5, 2013

Prime Cordial Labeling on Graphs

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Abstract—A prime cordial labeling of a graph G with vertex set V is a bijection f from V to $\{1, 2, ..., |V|\}$ such that each edge uv is assigned the label 1 if $gcd(f(u), f(v)) = 1$ and 0 if $gcd(f(u), f(v)) > 1$, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. In this paper we exhibit some characterization results and new constructions on prime cordial graphs.

Keywords—Prime cordial, tree, Euler, bijective, function.

I. Introduction

Graph labeling is a strong relation between numbers and structure of graphs. A useful survey to know about the numerous graph labeling methods is given by J.A. Gallian [7]. By combining the relatively prime concept in number theory and cordial labeling [6] concept in graph labeling, Sundaram, Ponraj and Somasundaram [9] have introduced the concept called prime cordial labeling. A prime cordial labeling of a graph G with vertex set V is a bijection f from V to $\{1, 2, ..., |V|\}$ such that each edge $uv \in E$

$$f(uv) = \begin{cases} 1 & \text{if } gcd(f(u), f(v)) = 1 \\ 0 & \text{if } gcd(f(u), f(v)) > 1 \end{cases}$$

then $|e(0) - e(1)| \leq 1$ where $e(0)$ is the number of edges labeled with 0 and $e(1)$ is the number of edges labeled with 1. In [4], [5], [9], the following graphs are proved to have prime cordial labeling: C_n if and only if $n \geq 6$; P_n if and only if $n \neq 3$ or 5; $K_{1,n}$ (n odd); the graph obtained by subdividing each edge of $K_{1,n}$ if and only if $n \geq 3$; bistars; dragons; crowns; triangular snakes if and only if the snake has at least three triangles; ladders, sun graph, kite graph $(n > t)$ and coconut tree, $S_n^{(1)} : S_n^{(2)}$, full binary trees from second level, $K_2 \Theta C_n$ (C_n) and $K_{1,n,n}$ for $n \geq 3$.

S.K. Vaidya in [11], [12] proved that the square graph of path P_n is a prime cordial graph for $n = 6$ and $n \geq 8$ while the square graph of cycle C_n is a prime cordial graph for $n \geq 10$. Also they show that the shadow graph of $K_{1,n}$ for $n \geq 4$ and the shadow graph of $B_{n,n}$ are prime cordial graphs, certain cycle related graphs, the graphs obtained by mutual duplication of a pair of edges as well as mutual duplication of a pair of vertices from each of two copies of cycle C_n admit prime cordial labeling. In [8] Haque proved that prime cordial labeling of generalized Petersen graph. Also in [3], prime cordial labeling for some class of cactus graphs and discuss with duality of prime cordial labeling.

Sundaram and Somasundaram [10] and Youssef observed that for $n > 3$, K_n is not prime cordial provided that the inequality $\varphi(2) + \varphi(3) + \cdots + \varphi(n) > n(n1)/4 + 1$ is valid for $n > 3$. The definitions and resuts we used in our paper is given as follows

Theorem 1: [13] In any binary edge labeling the following are equivalent. (i) $|e(0) - e(1)| \leq 1$
(ii) $e(1) = \begin{cases} \frac{q}{2} & \text{if } q \text{ is even} \\ \lceil \frac{q}{2} \rceil or \lceil \frac{q}{2} \rceil - 1 & \text{if } q \text{ is odd} \end{cases}$
where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. (iii) $\sum_{j=1}^{q} e_j + (q \mod 2)e_d = \lceil \frac{q}{2} \rceil$ where e_d is the binary label of a dummy edge which is introduced only when q is odd.

Definition 1: [2] If $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ are two connected graphs, $G_1 \hat{o} G_2$ is obtained by superimposing any selected vertex of G_2 on any selected vertex of G_1. The resultant graph $G = G_1 \hat{o} G_2$ consists of $p_1 + p_2 - 1$ vertices and $q_1 + q_2$ edges.

II. Characterization Results on Prime Cordial Graphs

In this section we give some characterization results for prime cordial graphs.

Theorem 2: A (p, q)-graph G is prime cordial. Let $f : V(G) to \{1, 2, \ldots, p\}$ be a prime cordial labeling then for each edge $uv \in E(G)$,

$$\sum_{uv \in E(G)} f(u) + f(uv) + f(v)$$
$$= \begin{cases} \sum_{i=1}^{p} f(v_i)deg(v_i) + \dfrac{q}{2} & \text{if } q \text{ is even} \\ \sum_{i=1}^{p} f(v_i)deg(v_i) + \left(\lceil \dfrac{q}{2} \rceil \text{ or } \lceil \dfrac{q}{2} \rceil - 1\right) & \text{if } q \text{ is odd} \end{cases}$$

Proof: Let f be a prime cordial labeling of $G(p, q)$. Consider

$$\sum_{uv \in E(G)} f(u) + f(uv) + f(v)$$
$$= \sum_{uv \in E(G)} f(u) + \sum_{uv \in E(G)} f(uv) + \sum_{uv \in E(G)} f(v)$$

Since f is a bijective function, each vertex label $f(u)$, $u \in V(G)$ occurs exactly $deg(u)$ times once in the summation and each label $uv \in E(G)$ admits binary labeling $\{0, 1\}$ by the concept of prime cordial which implies $\sum_{uv \in E(G)} f(uv) = e(1)$. Since by (ii) of 1.1,

$$= \sum_{i=1}^{p} f(v_i)deg(v_i) + e(1)$$

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$$\sum_{uv\in E(G)} f(u)+f(uv)+f(v)$$
$$=\begin{cases}\sum_{i=1}^{p} f(v_i)deg(v_i)+\frac{q}{2} & \text{if } q \text{ is even}\\ \sum_{i=1}^{p} f(v_i)deg(v_i)+\left(\lceil\frac{q}{2}\rceil \text{ or } \lceil\frac{q}{2}\rceil-1\right) & \text{if } q \text{ is odd}\end{cases}$$

■

Corollary 1: Let G be an r-regular prime cordial graph on p vertices and $S=\sum_{uv\in E(G)} f(u)+f(uv)+f(v)$ then

$$S=\begin{cases} r\frac{p(p+1)}{2}+\frac{q}{2} & \text{if } q \text{ is even}\\ r\frac{p(p+1)}{2}+\left(\lceil\frac{q}{2}\rceil \text{ or } \lceil\frac{q}{2}\rceil-1\right) & \text{if } q \text{ is odd}\end{cases}$$

Corollary 2: If a (p,q)-graph G is Prime cordial and $S=\sum_{uv\in E(G)} f(u)+f(uv)+f(v)$ then

$$S-\frac{p(p+1)}{2}$$
$$=\begin{cases}\sum_{i=1}^{p} f(v_i)deg(v_i-1)+\frac{q}{2} & \text{if } q \text{ is even}\\ \sum_{i=1}^{p} f(v_i)deg(v_i-1)+\left(\lceil\frac{q}{2}\rceil \text{ or } \lceil\frac{q}{2}\rceil-1\right) & \text{if } q \text{ is odd}\end{cases}$$

Theorem 3: If G is a prime cordial graph, then $G-e$ is also prime cordial
(i) for all $e\in E(G)$ when q is even
(ii) for some $e\in E(G)$ when q is odd

Proof: Consider the prime cordial graph G with p vertices and q edges.

Case (i): when q is even
Let q be the even size of the prime cordial graph G. Then it follows that $e(0)=e(1)=q/2$. Let e be any edge in G which is labeled either 0 or 1. Then in $G-e$, we have either $e(0)=e(1)+1$ or $e(1)=e(0)+1$ and hence $|e(0)-e(1)|\le 1$. Thus $G-e$ is prime cordial for all $e\in E(G)$.

Case (ii): when q is odd
Let q be the odd size of the prime cordial graph G. Then it follows that either $e(0)=e(1)+1$ or $e(1)=e(0)+1$. If $e(0)=e(1)+1$ then remove the edge e which is labeled as 0 and if $e(1)=e(0)+1$ then remove the edge e which is labeled 1 from G. Then it follows that $e(0)=e(1)$. Thus $G-e$ is prime cordial for some $e\in E(G)$. ■

Remark 1: G is a prime cordial graph. Similarly we can prove prime cordiality for the graph which is obtained by adding one edge to G.

It becomes an interesting problem to investigate the maximum number of possible edges that can be constructed in a graph with n vertices having prime cordial labeling property. We interpret this problem in number theory to find total number of relatively prime pair of integers in the set $\{1,2,3,\dots,n\}$ and find the maximum number of edges in prime cordial graph using Euler's phi function $\varphi(n)$. In [1], the total number of relatively prime pairs for the set $S_n=\{1,2,3,\dots,n\}$ is given by $|S_n|=\sum_{k=2}^{n}\varphi(k)$. For n vertices, the number of possible edges labeled with 1 is $e(1)=\sum_{k=2}^{n}\varphi(k)$ then the number of possible edges labeled with 0 is $e(0)=nC_2-\sum_{k=2}^{n}\varphi(k)$. By the definition of prime cordiality, if $e(0)$ is $nC_2\sum_{k=2}^{n}\varphi(k)$ then $e(1)$ is at most $nC_2-\sum_{k=2}^{n}\varphi(k)+1$.

Theorem 4: A maximum number of edges in a simple prime cordial graph with n vertices is $n^2-n+1+2\sum_{k=2}^{n}\varphi(k)$.

Proof: Consider a graph $G(V,E)$ with n vertices is a simple graph with no loops and parallel edges. The vertices $\{v_1,v_2,\dots,v_n\}$ are labeled with the integers $\{1,2,\dots,n\}$ such that each edge $v_iv_j\in E$ is assigned the label 1 if $gcd(f(u),f(v))=1$ and 0 if $gcd(f(u),f(v))>1$, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1.

Number of relatively prime pairs for the set $\{1,2,\dots,n\}$ is $\sum_{k=2}^{n}\varphi(k)$. For n vertices, the number of possible non relatively prime pairs is $nC_2-\sum_{k=2}^{n}\varphi(k)$. By the definition of prime cordial, if $e(0)$ is $nC_2-\sum_{k=2}^{n}\varphi(k)$ then $e(1)$ is at most $nC_2-\sum_{k=2}^{n}\varphi(k)+1$. For any prime cordial graph, the maximal number of edges is

$$\begin{aligned} e(0)+e(1) &= nC_2-\sum_{k=2}^{n}\varphi(k)+nC_2-\sum_{k=2}^{n}\varphi(k)+1\\ &= 2nC2+1-2\sum_{k=2}^{n}\varphi(k)\\ &= n(n-1)+1-2\sum_{k=2}^{n}\varphi(k)\\ &= n^2-n+1-2\sum_{k=2}^{n}\varphi(k). \end{aligned}$$

Hence maximum number of edges in a simple prime cordial graph with n vertices is $n^2-n+1-2\sum_{k=2}^{n}\varphi(k)$. ■

III. Prime Cordial Labeling for $G_1\hat{o}G_2$

In the following theorems we consider the prime cordial graph G and glue a vertex of some class of graphs to one of the selected vertex of G and check whether the new graph retain the property of prime cordial. For that we need the theorem given by [13] wieslet.

Theorem 5: If G has prime cordial labeling then, $G\hat{o}K_{1,m}$ admits prime cordial labeling for
(i) m is even and G is of any size q.
(ii) m is odd and
(a) G is of even size q.
(b) G is of odd size q with $e(1)=\lceil\frac{q}{2}\rceil$ and order is odd.
(c) G is of odd size q with $e(1)=\lceil\frac{q}{2}\rceil-1$ and order is even.

Proof: Let $G(p,q)$ be a prime cordial graph. Let $w\in V$ be the vertex whose label is $f(w)=2$. Consider the star $K_{1,m}$ with vertex set $\{x_0,x_i : 1\le i\le m\}$ and edge set $\{x_0x_i : 1\le i\le m\}$. We superimpose the vertex x_0 of the star $K_{1,m}$ graph on the vertex $w\in V$ of G. Now we define the new graph called $G_1=G\hat{o}K_{1,m}$ with vertex set $V_1(G_1)=V(G)\cup\{x_i : 1\le i\le m\}$ and $E_1(G_1)=E(G)\cup\{vx_i : 1\le i\le m\}$. Consider the bijective function $g:V_1\to\{1,2,\dots,p,p+1,p+2,\dots,p+m\}$ defined

by
$g(v_i) = f(v_i)$; $1 \leq i \leq p$
By our construction, $f(w) = g(x_0) = 2$.
$g(x_i) = p + i$, for $1 \leq i \leq m$.
We have to show that the graph $G_1 = G\hat{o}K_{1,m}$ has prime cordial labeling in various cases.
Case (i): m is even and G is of any size q
The edge set is defined as
$g(uv) = f(uv)$ for all $uv \in E(G)$.
If G is of even order then $g(wx_{2i-1}) = 1$ for $1 \leq i \leq m/2$; $g(wx_{2i}) = 0$ for $1 \leq i \leq m/2$.
If G is of odd order then $g(wx_{2i-1}) = 0$ for $1 \leq i \leq m/2$; $g(wx_{2i}) = 1$ for $1 \leq i \leq m/2$.
Hence the total number of edges labeled with 1's are given by $e(1) = m/2$ and the total number of edges labeled with 0's are given by $e(0) = m/2$ in the edge set $\{wx_i : 1 \leq i \leq m\}$. Also given that G is already prime cordial. Hence by 1.1, one can easily verify that the edge set $E_1(G_1)$ satisfy the condition $|e(0) - e(1)| \leq 1$ for the graph $G_1 = G\hat{o}K_{1,m}$.
Case (ii): m is odd
Subcase **(*i*):** G is of even size q
The bijective function $g : V_1 \rightarrow \{1, 2, \ldots, p, p+1, p+2, \ldots, p+m\}$ defined as given in above.
The edge set is defined as $g(uv) = f(uv)$ for all $uv \in E(G)$.
If G is of even order then $g(wx_{2i-1}) = 1$ for $1 \leq i \leq \lceil \frac{m}{2} \rceil$; $g(wx_{2i}) = 0$ for $1 \leq i \leq \lceil \frac{m}{2} \rceil - 1$.

The total number of edges labeled with 1's are given by $e(1) = \lceil \frac{m}{2} \rceil$ and the total number of edges labeled with 0's are given by $e(0) = \lceil \frac{m}{2} \rceil - 1$. Since G is prime cordial labeling of even size, $e(1) = \frac{q}{2}$ and $e(0) = \frac{q}{2}$. Hence for the graph $G_1 = G\hat{o}K_{1,m}$, the total number of edges labeled with 1's are given by $e(1) = \frac{q}{2} + \lceil \frac{m}{2} \rceil$ and the total number of edges labeled with 0's are given by $e(0) = \frac{q}{2} + \lceil \frac{m}{2} \rceil - 1$. Therefore the total difference between 1's and 0's is given by $|e(1) - e(0)| = \left| \frac{q}{2} + \lceil \frac{m}{2} \rceil - \left(\frac{q}{2} + \lceil \frac{m}{2} \rceil \right) \right| = 1$.

If G is of odd order then $g(vx_{2i-1}) = 0$ for $1 \leq i \leq \lceil \frac{m}{2} \rceil$; $g(vx_{2i}) = 1$ for $1 \leq i \leq \lceil \frac{m}{2} \rceil - 1$. The total number of edges labeled with 1's are given by $e(1) = \lceil \frac{m}{2} \rceil - 1$ and the total number of edges labeled with 0's are given by $e(0) = \lceil \frac{m}{2} \rceil$. Since G is prime cordial labeling of even size, $e(1) = \frac{q}{2}$ and $e(0) = \frac{q}{2}$. Hence for the graph $G_1 = G\hat{o}K_{1,m}$, the total number of edges labeled with 1's are given by $e(1) = \frac{q}{2} + \lceil \frac{m}{2} \rceil - 1$ and the total number of edges labeled with 0's are given by $e(0) = \frac{q}{2} + \lceil \frac{m}{2} \rceil$. Therefore the total difference between 1's and 0's is given by $|e(1) - e(0)| = \left| \frac{q}{2} + \lceil \frac{m}{2} \rceil - 1 - \left(\frac{q}{2} + \lceil \frac{m}{2} \rceil \right) \right| = 1$.
Subcase **(*ii*):** G is of odd size q with $e(1) = \lceil \frac{q}{2} \rceil$ and order is odd.
The bijective function $g : V_1 \rightarrow \{1, 2, \ldots, p, p+1, p+2, \ldots, p+m\}$ defined as given in above. The edge set is defined as $g(uv) = f(uv)$ for all $uv \in E(G)$.
$g(wx_{2i-1}) = 0$ for $1 \leq i \leq \lceil \frac{m}{2} \rceil$; $g(wx_{2i}) = 1$ for $1 \leq i \leq -1$.
The total number of edges labeled with 1's are given by $e(1) = \lceil \frac{m}{2} \rceil - 1$ and the total number of edges labeled with 0's are given by $e(0) = \lceil \frac{m}{2} \rceil$. Since G is prime cordial labeling of odd size q with $e(1) = \lceil \frac{q}{2} \rceil$, then $e(0) = \lceil \frac{q}{2} \rceil - 1$. Hence for the graph $G_1 = G\hat{o}K_{1,m}$, the total number of edges labeled with 1's are given by $e(1) = \lceil \frac{q}{2} \rceil + \lceil \frac{m}{2} \rceil - 1$ and the total number of edges labeled with 0's are given by $e(0) = \lceil \frac{q}{2} \rceil - 1 + \lceil \frac{m}{2} \rceil$. Therefore the total difference between 1's and 0's is given by $|e(1) - e(0)| = \left| \frac{q}{2} + \lceil \frac{m}{2} \rceil - 1 - \left(\frac{q}{2} + \lceil \frac{m}{2} \rceil - 1 \right) \right| = 0$.
Subcase **(*iii*):** G is of odd size q with $e(1) = \lceil \frac{q}{2} \rceil - 1$ and order is even.
The bijective function $g : V_1 \rightarrow \{1, 2, \ldots, p, p+1, p+2, \ldots, p+m\}$ defined as given in above. The edge set is defined as $g(uv) = f(uv)$ for all $uv \in E(G)$.
$g(wx_{2i-1}) = 1$ for $1 \leq i \leq \lceil \frac{m}{2} \rceil$; $g(wx_{2i}) = 0$ for $1 \leq i \leq \lceil \frac{m}{2} \rceil - 1$.
The total number of edges labeled with 1's are given by $e(1) = \lceil \frac{m}{2} \rceil$ and the total number of edges labeled with 0's are given by $e(0) = \lceil \frac{m}{2} \rceil - 1$. Since G is prime cordial labeling of odd size q with $e(1) = \lceil \frac{m}{2} \rceil - 1$, then $e(0) = \lceil \frac{q}{2} \rceil$. Hence for the graph $G_1 = G\hat{0}K_{1,m}$, the total number of edges labeled with 1's are given by $e(1) = \lceil \frac{q}{2} \rceil - 1 + \lceil \frac{m}{2} \rceil$ and the total number of edges labeled with 0's are given by $e(0) = \lceil \frac{q}{2} \rceil + \lceil \frac{m}{2} \rceil - 1$. Therefore the total difference between 1's and 0's is given by $|e(1) - e(0)| = \left| \frac{q}{2} - 1 + \lceil \frac{m}{2} \rceil - \left(\frac{q}{2} + \lceil \frac{m}{2} \rceil - 1 \right) \right| = 0$.

Hence using the above cases $G\hat{0}K_{1,m}$ has prime cordial labeling with the above conditions. ■

Theorem 6: If G has prime cordial labeling then, $G\hat{0}P_m$ admits prime cordial labelinag for
(i) m is even and G is of size q.
(ii) m is odd and
(a) G is of even size q.
(b) G is of odd size q with $e(1) = \lceil \frac{q}{2} \rceil$ and order is odd.
(c) G is of odd size q with $e(1) = \lceil \frac{q}{2} \rceil - 1$ and order is even.

Proof: Let $G(p, q)$ be a prime cordial graph. Let $w \in V$ be the vertex whose label is $f(w) = 2$. Consider the path P_m with vertex set $\{x_i : 1 \leq i \leq m\}$ and edge set $\{x_i x_{i+1} : 1 \leq i \leq m-1\}$. We superimpose one of the pendent vertex of the path P_m say x_1 on the vertex $w \in V$ of G. Now we define the new graph called $G_1 = G\hat{0}P_m$ with vertex set $V_1(G_1) = V(G) \cup \{x_i : 1 \leq i \leq m\}$ and $E_1(G_1) = E(G) \cup \{wx_2\} \cup \{x_i x_{i+1} : 2 \leq i \leq m-1\}$. Consider the bijective function $g : V_1 \rightarrow \{1, 2, \ldots, p, p+1, p+2, \ldots, p+m-1\}$ defined by
$g(v_i) = f(v_i)$; $1 \leq i \leq p$
By our construction, $f(w) = g(x_1) = 2$.
When p is odd and m is even,
$g(x_i) = p + 2i - 3$, for $2 \leq i \leq \frac{m+2}{2}$; $g(x_{\frac{m}{2}+i}) = p + m - 2(i-1)$, for $2 \leq i \leq \frac{m}{2}$.
When p is odd and m is odd,
$g(x_i) = p + 2i - 3$, for $2 \leq i \leq \lceil \frac{m}{2} \rceil$; $g(x_{\lfloor \frac{m}{2} \rfloor + i}) = p + m - 2i + 3$, for $2 \leq i \leq \lceil \frac{m}{2} \rceil$
When p is even and m is even,
$g(x_i) = p + 2(i-1)$, for $2 \leq i \leq \frac{m}{2}$; $g(x_{(\frac{m}{2}-1)+i}) = p + m - 2i + 3$, for $2 \leq i \leq \frac{m}{2} + 1$.
When p is even and m is odd,
$g(x_i) = p + 2(i-1)$, for $2 \leq i \leq \frac{m}{2}$; $g(x_{\lfloor \frac{m}{2} \rfloor + i}) = p + m - 2(i-1)$, for $2 \leq i \leq \lceil \frac{m}{2} \rceil$.

Using the above labeling and similar method of Theorem ??, one can easily verify that the graph $G_1 = G\hat{o}P_m$ has prime cordial labeling, all the cases given in hypothesis. ■

Theorem 7: If G has prime cordial labeling then, $G\hat{o}F_{1,m}$ admits prime cordial labeling for

(i) m is even and

(a) G is of even size q.

(b) G is of odd size with $e(1) = \lceil \frac{q}{2} \rceil - 1$.

(ii) m is odd and

(c) G is of even size q and odd order.

(d) G is of odd size q with $e(1) = \lceil \frac{q}{2} \rceil$ and order is odd.

Proof: Let $G(p, q)$ be a prime cordial graph. Let $w \in V$

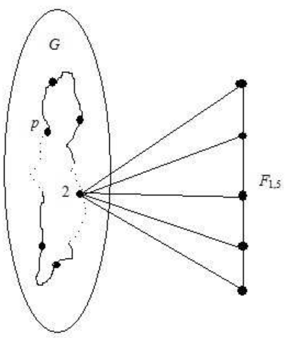

Fig. 1. $G\hat{o}F_{1,5}$

be the vertex whose label is $f(w) = 2$. Consider the Fan $F_{1,m}$ with vertex set $\{x_0, x_i : 1 \leq i \leq m\}$ and edge set $\{x_0x_i : 1 \leq i \leq m\} \cup \{x_ix_{i+1} : 1 \leq i \leq m-1\}$. We superimpose the vertex x_0 of the Fan $F_{1,m}$ graph on the vertex $w \in V$ of G. Now we define the new graph called $G_1 = G\hat{o}F_{1,m}$ with vertex set $V_1(G_1) = V(G) \cup \{x_0, x_i : 1 \leq i \leq m\}$ and $E_1(G_1) = E(G) \cup \{wx_i : 1 \leq i \leq m\} \cup \{x_ix_{i+1} : 1 \leq i \leq m-1\}$. Consider the bijective function $g : V_1 \to \{1, 2, \ldots, p, p+1, p+2, \ldots, p+m\}$ defined by

$g(v_i) = f(v_i)$; $1 \leq i \leq p$

By our construction, $f(w) = g(x_0) = 2$.

When p is odd and m is even,

$g(x_i) = p+2i-1$, for $1 \leq i \leq \frac{m}{2}$; $g(x_{\frac{m}{2}+i}) = p+m-2(i-1)$, for $1 \leq i \leq \frac{m}{2}$.

When p is odd and m is odd,

$g(x_i) = p + 2i - 1$, for $1 \leq i \leq \lceil \frac{m}{2} \rceil$; $g(x_{\lceil \frac{m}{2} \rceil + i}) = p + m - 2i + 1$, for $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$.

When p is even and m is even,

$g(x_i) = p + 2i$, for $1 \leq i \leq \frac{m}{2}$; $g(x_{\frac{m}{2}+i}) = p + m - 2i + 1$, for $1 \leq i \leq \frac{m}{2}$.

When p is even and m is odd,

$g(x_i) = p + 2i$, for $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$; $g(x_{\lfloor \frac{m}{2} \rfloor + i}) = p + m - 2(i-1)$, for $2 \leq i \leq \lceil \frac{m}{2} \rceil$.

We have to show that the graph $G_1 = G\hat{o}F_{1,m}$ has prime cordial labeling in various cases.

Case (i): m is even

$Subcase$ **(**i**)**: G is of even size q.

The edge set is defined as

$g(uv) = f(uv)$ for all $uv \in E(G)$.

For any p, $g(vx_i) = 0$ for $1 \leq i \leq m/2$; $g(vx_i) = 1$ for $\frac{m}{2} + 1 \leq i \leq m$ and

$g(x_ix_{i+1}) = 0$ for $1 \leq i \leq \frac{m}{2} - 1$; $g(x_ix_{i+1}) = 1$ for $\frac{m}{2} \leq i \leq m - 1$.

Hence the total number of edges labeled with 1's are given by $e(1) = m$ and the total number of edges labeled with 0's are given by $e(0) = m - 1$. Since G is prime cordial labeling of even size, $e(1) = \frac{q}{2}$ and $e(0) = \frac{q}{2}$. Hence for the graph $G_1 = G\hat{o}F_{1,m}$, the total number of edges labeled with 1's are given by $e(1) = \frac{q}{2} + m$ and the total number of edges and 0's is given by $e(0) = \frac{q}{2} + m - 1$. Therefore the total difference between 1's and 0's is given by $|e(1) - e(0)| = \left|\frac{q}{2} + m - \left(\frac{1}{2} + m - 1\right)\right| = 1$.

$Subcase$ **(**ii**):** G is of odd size with $e(1) = \lceil \frac{q}{2} \rceil - 1$.

The edge set is defined as in subcase (i).

Since G is prime cordial labeling of odd size with $e(1) = \lceil \frac{q}{2} \rceil - 1$ then $e(0) = \lceil \frac{q}{2} \rceil$. Hence for the graph $G_1 = G\hat{o}F_{1,m}$, the total number of edges labeled with 1's are given by $e(1) = \lceil \frac{q}{2} \rceil - 1 + m$ and the total number of edges labeled with 0's are given by $e(0) = \lceil \frac{q}{2} \rceil + m - 1$. Therefore the total difference between 1's and 0's is given by $|e(1) - e(0)| = \left|\lceil \frac{q}{2} \rceil - 1 + m - \left(\lceil \frac{q}{2} \rceil + m - 1\right)\right| = 0$.

Case (ii): m is odd.

$Subcase$ **(**i**):** G is of even size q and order is odd.

The bijective function $g : V_1 \to \{1, 2, \ldots, p, p+1, p+2, \ldots, p+m\}$ defined as given in above.

The edge set is defined as $g(uv) = f(uv)$ for all $uv \in E(G)$.

$g(vx_i) = 0$ for $1 \leq i \leq \lceil \frac{m}{2} \rceil$; $g(vx_i) = 1$ for $\lceil \frac{m}{2} \rceil + 1 \leq i \leq m$ and

$g(x_ix_{i+1}) = 0$ for $1 \leq i \leq \lceil \frac{m}{2} \rceil - 1$; $g(x_ix_{i+1}) = 1$ for $\lceil \frac{m}{2} \rceil \leq i \leq m - 1$.

The total number of edges labeled with 1's are given by $e(1) = m - 1$ and the total number of edges labeled with 0's are given by $e(0) = m$. Since G is prime cordial labeling of even size, $e(1) = \frac{q}{2}$ and $e(0) = \frac{q}{2}$. Hence for the graph $G_1 = G\hat{o}F_{1,m}$, the total number of edges labeled with 1's are given by $e(1) = \frac{q}{2} + m - 1$ and the total number of edges labeled with 0's are given by $e(0) = \frac{q}{2} + m$. Therefore the total difference between 1's and 0's is given by $|e(1) - e(0)| = \left|\frac{q}{2} + m - 1 - \left(\frac{q}{2} + m\right)\right| = 1$.

$Subcase$ **(**ii**):** G is of odd size q with $e(1) = \lceil \frac{q}{2} \rceil$ and order is odd.

The edge set is defined as same as the subcase (i).

Since G is prime cordial labeling of odd size q with $e(1) = \lceil \frac{q}{2} \rceil$, then $e(0) = \lceil \frac{q}{2} \rceil - 1$. Hence for the graph $G_1 = G\hat{o}F_{1,m}$, the total number of edges labeled with 1's are given by $e(1) = \lceil \frac{q}{2} \rceil + m - 1$ and the total number of edges labeled with 0's are given by $e(0) = \lceil \frac{q}{2} \rceil - 1 + m$. Therefore the total difference between 1's and 0's is given by $|e(1) - e(0)| = \left|\lceil \frac{q}{2} \rceil + m - 1 - \left(\lceil \frac{q}{2} \rceil - 1 + m\right)\right| = 0$. Hence the graph $G\hat{o}F_{1,m}$ has prime cordial labeling with above conditions. ■

IV. Prime Cordial Labeling for Tree Related Graphs

Theorem 8: The tree $K_{1,m} \odot P_n$ $(n, m > 2)$ obtained by replacing each edge of $K_{1,m}$ by equal length of path has prime cordial labeling.

Proof: Consider the tree $K_{1,m} \odot P_n$ with vertex set $V = \{v_i : i = 1, 2, 3, \ldots, nm + 1\}$ and the edge set $E = \{E_1 \cup E_2 \cup E_3 \cup E_4 \cup \cdots \cup E_m \cup E_{m+1}\}$ where $E_1 = \{v_iv_{i+1} : 2 \le i \le n\}$, $E_2 = \{v_iv_{i+1} : n+2 \le i \le 2n\}$, $\ldots$, $E_m = \{v_iv_{i+1} : n(m-1)+2 \le i \le nm\}$ and $E_{m+1} = \{v_1v_{(i-1)n+2} : 1 \le i \le m\}$. We prove this theorem in three cases.

Case (i): For even m and any value of n.
The bijective function $f : V \to \{1, 2, \ldots, nm+1\}$ defined as
$f(v_i) = 2i$ for $1 \le i \le \lfloor \frac{nm+1}{2} \rfloor$;
$f(v_{\lfloor \frac{nm+1}{2} \rfloor + 1}) = nm + 1$; $f(v_{\lfloor \frac{nm+1}{2} \rfloor + 2}) = 1$; $f(v_{\lfloor \frac{nm+1}{2} \rfloor + 3}) = 3$; $f(v_{\lfloor \frac{nm+1}{2} \rfloor + 4}) = 9$; $f(v_{\lfloor \frac{nm+1}{2} \rfloor + 5}) = 5$; $f(v_{\lfloor \frac{nm+1}{2} \rfloor + 6}) = 7$; $f(v_{\lfloor \frac{nm+1}{2} \rfloor + i}) = 2i - 3$ for $7 \le i \le \lceil \frac{nm+1}{2} \rceil$;
By the above labeling, the $\frac{nm}{2}$ edges have label 0 and $\frac{nm}{2}$ edges have label 1. Hence the difference between the total number of edges labeled with 1's and 0's is given by $\left| \frac{nm}{2} - \frac{nm}{2} \right| = 0$.

Case (ii): For odd m and even n.
The bijective function $f : V \to \{1, 2, \ldots, nm+1\}$ defined as
$f(v_i) = 2i$ for $1 \le i \le \lfloor \frac{nm+1}{2} \rfloor$;
$f(v_{\lfloor \frac{nm+1}{2} \rfloor + 1}) = 1$; $f(v_{\lfloor \frac{nm+1}{2} \rfloor + 2}) = 3$; $f(v_{\lfloor \frac{nm+1}{2} \rfloor + 3}) = 9$; $f(v_{\lfloor \frac{nm+1}{2} \rfloor + 4}) = 5$; $f(v_{\lfloor \frac{nm+1}{2} \rfloor + 5}) = 7$; $f(v_{\lfloor \frac{nm+1}{2} \rfloor + i}) = 2i - 1$ for $6 \le i \le \lceil \frac{nm+1}{2} \rceil$;
By the above labeling, the $\frac{nm}{2}$ edges have label 0 and $\frac{nm}{2}$ edges have label 1. Hence the difference between the total number of edges labeled with 1's and 0's is given by $\left| \frac{nm}{2} - \frac{nm}{2} \right| = 0$.

Case (iii): For odd m and odd n.
The bijective function $f : V \to \{1, 2, \ldots, nm+1\}$ defined as
$f(v_i) = 2i$ for $1 \le i \le \frac{nm+1}{2}$;
$f(v_{\lfloor \frac{nm+1}{2} \rfloor + i}) = 2i - 1$ for $6 \le i \le \left(\frac{nm+1}{2}\right)$;
By the above labeling, the $\frac{nm}{2} - 1$ edges have label 0 and $\frac{nm}{2} - 1$ edges have label 1. Hence the difference between the total number of edges labeled with 1's and 0's is given by $\left| \left(\lfloor \frac{nm}{2} \rfloor \right) - \left(\lfloor \frac{nm}{2} \rfloor + 1 \right) \right| = 1$.
Hence the tree $K_{1,m} \odot P_n$ $(n, m > 2)$ has prime cordial labeling. ∎

Theorem 9: The tree $P_n \odot P_m$ $(n \ge 4)$ obtained by replacing each vertex of P_n by equal length of path has prime cordial labeling.

Proof: Consider the tree $P_n \odot P_m$ with the vertex set $V = \{v_1, v_2, \ldots, v_m, v_{m+1}, \ldots, v_{2m}, v_{2m+1}, \ldots, v_{(n-1)m}, v_{(n-1)m+1}, \ldots, v_{nm}\}$ and the edge set $E = E_1 \cup E_2 \cup \cdots \cup E_n \cup E_{n+1}$ where $E_1 = \{v_iv_{i+1} : 1 \le i \le m-1\}$; $E_2 = \{v_iv_{i+1} : m+1 \le i \le 2m-1\}$; $\ldots$; $E_{n-1} = \{v_iv_{i+1} : (n-2)m+1 \le i \le (n-1)m-1\}$; $E_n = \{v_iv_{i+1} : (n-1)m+1 \le i \le nm-1\}$ and $E_{n+1} = \{v_{im+1}v_{(i+1)m+1} : 0 \le i \le n-2\}$. We prove this theorem in three cases.

Case (i): For any value of n and $m = 2$
The bijective function $f : V \to \{1, 2, \ldots, nm\}$ defined as
$f(v_{2i-1}) = 2i$ for $1 \le i \le n$;
$f(v_{2i}) = 2i - 1$ for $1 \le i \le n$.
In view of the labeling pattern defined above, one can easily verify that $|e(0) - e(1)| = 1$.

Case (ii): For even n and any value of $m > 2$
The bijective function $f : V \to \{1, 2, \ldots, nm\}$ defined as
$f(v_i) = 2i$ for $1 \le i \le \frac{nm}{2}$;
$f(v_{\frac{nm}{2}+i}) = 2i - 1$ for $1 \le i \le \frac{nm}{2}$.
By the definition of prime cordial and the pattern of labeling given above, we have the edges in the sets $E_1 \cup E_2 \cup \cdots \cup E_{n/2}$ and the edges in the set $\{v_{im+1}v_{(i+1)m+1} : 0 \le i \le \frac{n-4}{2}\}$ have edge label 0. The edge $v_{\left(\frac{n}{2}-1\right)}v_{\left(\frac{n}{2}\right)}$ have induced label 1. Let $E_{n+1} = \{v_{im+1}v_{(i+1)m+1} : 0 \le i \le n-2\}$. The vertices from v_1 to $v_{\left(\frac{n}{2}-1\right)}$ in E_{n+1} have even numbers and the vertices from $v_{\left(\frac{n}{2}\right)}$ to $v_{\left(\frac{n}{2}\right)}v_{\left(\frac{n}{2}-1\right)}$ have odd numbers.
Claim: Prove that the labeling f for any edge $uv \in \{v_{im+1}v_{(i+1)m+1} : \frac{n}{2} + 1 \le i \le n-2\}$ is 1 (i.e, $f(u)$ and $f(v)$ is relatively prime).

For that consider the labeling of $\{v_{\left(\frac{n}{2}\right)m+1}, v_{\left(\frac{n}{2}\right)m+m+1}, v_{\left(\frac{n}{2}\right)m+2m+1}, \ldots, v_{(n-1)m+1}\}$ be $\{1, 2m+1, 4m+1, \ldots, 2\left(\left(\left(\frac{n}{2}\right) - 1\right)m\right) + 1\}$ which is increasing order. Suppose the labeling f for any edge $uv \in \{v_{im+1}v_{(i+1)m+1} : \frac{n}{2} + 1 \le i \le n-2\}$ is not relatively prime i.e., $gcd(f(u), f(v)) > 1$.
Then for any integer p,

$$f(u) \equiv 0 \mod p \text{ and } f(v) \equiv 0 \mod p \Rightarrow |f(u) - f(v)| \equiv 0 \mod p \quad (1)$$

Since both $f(u)$ and $f(v)$ are odd, p is also odd and according to our labeling,

$$|f(u) - f(v)| \equiv 0 \mod 2 \quad (2)$$

By (1) and (2), $p | f(u) - f(v)|$ and $2/|f(u) - f(v)| \Rightarrow p$ is a multiple of $2 \Rightarrow p$ is even. Which is contradiction to the fact that p is odd. So the labeling f for any edge $uv \in \{v_{im+1}v_{(i+1)m+1} : \frac{n}{2} + 1 \le i \le n-2\}$ is relatively prime.

Therefore the total number of edges having edge label 0 is $\frac{nm}{2} - 1$. The remaining $\frac{nm}{2}$ edges in the edge set E have edge label 1. Hence the difference between the total number of edges labeled with 1's and 0's is given by $\left| \left(\frac{nm}{2} - 1 \right) - \frac{nm}{2} \right| = 1$.

Case (iii): For odd n and any value of $m > 2$.
Subcase **(*i*):** For m is even
The bijective function $f : V \to \{1, 2, \ldots, nm\}$ defined as
$f(v_i) = 2i$ for $1 \le i \le \frac{nm}{2}$;
$f(v_{\frac{nm}{2}+i}) = nm - 2i$ for $1 \le i \le \frac{m}{2}$;
$f(v_{\lceil \frac{n}{2} \rceil m+i}) = 2i - 1$ for $1 \le i \le \frac{nm}{2} - \frac{m}{2}$;
By the above labeling, the total number of edges having edge label 0 is $e(0) = \frac{nm}{2} - 1$ and the total number of edges having edge label 1 is $e(1) = \frac{nm}{2}$. Hence the difference between the total number of edges labeled with 1's and 0's is given by $|e(1) - e(0)| = \left| \left(\frac{nm}{2} - 1 \right) - \frac{nm}{2} \right| = 1$.
Subcase **(*ii*):** For m is odd.
The bijective function $f : V \to \{1, 2, \ldots, |V|\}$ defined as
$f(v_i) = 2i$ for $1 \le i \le \lfloor \frac{nm}{2} \rfloor$;

$f(v_{\lfloor \frac{nm}{2} \rfloor + i}) = nm - 2(i-1)$, for $1 \leq i \leq \lceil \frac{m}{2} \rceil$.
$f(v_{\lfloor \frac{n}{2} \rfloor m+1}) = 1$; $f(v_{\lfloor \frac{n}{2} \rfloor m+2}) = 3$; $f(v_{\lfloor \frac{n}{2} \rfloor m+3}) = 9$; $f(v_{\lfloor \frac{n}{2} \rfloor m+4}) = 5$; $f(v_{\lfloor \frac{n}{2} \rfloor m+5}) = 7$;
$f(v_{\lfloor \frac{n}{2} \rfloor m+i}) = 2i - 1$; for $6 \leq i \leq \lfloor \frac{nm}{2} \rfloor - \lfloor \frac{m}{2} \rfloor$;

By the above labeling, the total number of edges having edge label 0 is $e(0) = \lfloor \frac{nm}{2} \rfloor$ and the total number of edges having edge label 1 is $e(1) = \lfloor \frac{nm}{2} \rfloor$. Hence the difference between the total number of edges labeled with 1's and 0's is given by $|e(1) - e(0)| = \left| \lfloor \frac{nm}{2} \rfloor - \lfloor \frac{nm}{2} \rfloor \right| = 0$. The edge $v_{(\lceil \frac{n}{2} \rceil - 1)m+1} v_{(\lceil \frac{n}{2} \rceil)m+1}$ have induced label 1. Let $E_{n+1} = \{v_{im+1} v_{(i+1)m+1} : 0 \leq i \leq \lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor - 2\}$. The edges from v_1 to $v_{(\lceil \frac{n}{2} \rceil - 1)m+1}$ in E_{n+1} have even numbers and the edges from $v_{(\lceil \frac{n}{2} \rceil)m+1}$ to $v_{\lceil \frac{n}{2} \rceil m + (\lfloor \frac{n}{2} \rfloor - 1)m+1}$ have odd numbers. We can prove that the labeling f for any edge $uv \in \{v_{im+1} v_{(i+1)m+1} : \lceil \frac{n}{2} \rceil + 1 \leq i \leq \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor - 2\}$ is relatively prime similar to the claim given in case (ii). Hence $P_n \odot P_m$ has prime cordial labeling. ■

Corollary 3: The comb graph $P_n \odot K_2$ has prime cordial labeling.

V. Conclusion

According to literature survey, more work has been done in prime cordial labeling for cycle related graphs. In our work we determine the prime cordial labeling for certain classes of trees and also exhibit some characterization results and new constructions of prime cordial graph.

Acknowledgment

The first author Ms. S. Babitha is grateful to the University Grants Commission (New Delhi, India) for their financial support, under Research Fellowship in Science for Meritorious Students Research Scholars (Sanction No: F.4-1/2006(BSR)/11-2008(BSR)).

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