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An Estimation of the Performance of HRLS Algorithm

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Abstract—The householder RLS (HRLS) algorithm is an $O(N^2)$ algorithm which recursively updates an arbitrary square-root of the input data correlation matrix and naturally provides the LS weight vector. A data dependent householder matrix is applied for such an update. In this paper a recursive estimate of the eigenvalue spread and misalignment of the algorithm is presented at a very low computational cost. Misalignment is found to be highly sensitive to the eigenvalue spread of input signals, output noise of the system and exponential window. Simulation results show noticeable degradation in the misalignment by increase in eigenvalue spread as well as system's output noise, while exponential window was kept constant.

Keywords—HRLS algorithm, eigenvalue spread, misalignment.

I. Introduction

Adaptive filtering is a modeling procedure in signal processing which allows for the adaptation of model parameters with respect to incoming input signals. For adaptive transversal filters, it is a standard procedure to formulate the problem using the method of least squares, extended for recursive solution update. Recursive least squares (RLS) based methods offer good numerical stability and are closely related to algorithms for Kalman filtering problems [1]. The HRLS algorithm is related to Potter's square-root covariance filter, which was the first square-root Kalman filter implementation, developed in the early 1960s [2]. The performance of the algorithm depends on the orthogonalization capabilities of the householder transformation used to process the input for the next update. Fast convergence rate of HRLS algorithm make it numerically robust[3], but its misalignment is greatly affected by the eigenvalue spread of the input signals and system output noise.

The rate of convergence , misalignment and numerical stability of the algorithms depend on the condition number of the input signal covariance matrix [4]. The eigenvalue spread of the covariance matrix is a measure of the condition number [5], [6], and this spread controls the convergence rate of the LMS based algorithms[7].

In this paper we examine the performance of the HRLS algorithm by recursively estimating the input eigenvalue spread and misalignment at a low computational cost. Although rate of convergence of HRLS algorithm is invariant to the eigenvalue spread of the correlation matrix, its misalignment is highly sensitive to this spread. Affect of the output signal noise on misalignment as well as convergence rate is also observed. Simulations are presented to show the affect of spectral condition number and output noise (SNR) on the misalignment as well as mean square error of the algorithm.

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II. The HRLS Algorithm

In this section we briefly derive the HRLS algorithm for adaptive least squares problem of the form:

$$\min_{w\in\mathbf{R}^N} J_n(w_n) = \sum_{i=0}^{n} \lambda^{n-i}(w_n^T a_i - s(i))^2 \qquad (1)$$

where $s(i) \in \mathbf{R}$ is the reference signal, and $y(i) = w_n^T a_i$ is the prediction of $s(i)$ for $1 \leq i \leq n$, while n is the current time value.

For a transversal finite impulse response (FIR) adaptive filter, vectors $a_i \in \mathbf{R}^N$ are formed by the input $u(i)$, such that

$$a_i = [\ u(i) \quad u(i-1) \quad \ldots \quad u(i-N+1)\ ]^T$$

and vector $w_n \in \mathbf{R}^N$ is an estimate of the filter tap vector, which is updated by minimizing the sum of squared error cost function $J_n(w_n)$. The constant $\lambda \in [0, 1]$ is known as the forgetting factor.

Define the $n \times N$ data matrix A_n by

$$\begin{pmatrix} u(1) & 0 & \ldots & 0 \\ u(2) & u(1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ u(n-1) & u(n-2) & \ldots & u(n-N) \\ u(n) & u(n-1) & \ldots & u(n-N+1) \end{pmatrix}$$

and the diagonal matrix Λ_n by

$$\Lambda_n = diag[\sqrt{\lambda^{n-1}}, \sqrt{\lambda^{n-2}}, \ldots, \sqrt{\lambda}, 1]$$

The above definitions allow us to write the minimization problem in (1) as

$$J_n(w_n) = \|\Lambda_n d_n - \Lambda_n A_n w_n\|_2^2 \qquad (2)$$

where

$$d_n = [\ s(1) \quad s(2) \quad \ldots \quad s(n)\ ]^T + \eta$$

η is the white Gaussian noise of the system output. If R_n denotes the correlation matrix of input data A_n, then following relation holds for the square-root factor S_n of R_n ,

$$R_n = S_n^T S_n$$

Let us define $N \times 1$ vector x_n by considering new data vector a_n and the square-root factor of the previous instant as:

$$x_n = \frac{S_{n-1}^T a_n}{\sqrt{\lambda}}$$

We are now in a position to consider the $(N+1) \times (N+1)$ orthogonal matrix P_n, presented in [2], to formulate the equation for updating S_{n-1}^{-T} to S_n^{-T}. i.e.,

$$P_n \begin{pmatrix} x_n & \lambda^{-\frac{1}{2}} S_{n-1}^{-T} \\ 1 & 0^T \end{pmatrix} = \begin{pmatrix} 0 & S_n^{-T} \\ \delta_n & u_n^T \end{pmatrix} \quad (3)$$

where $\delta_n = \sqrt{1 + \| x_n \|^2} = \frac{\sqrt{\lambda + a_n^T R_{n-1}^{-1} a_n}}{\sqrt{\lambda}}$.

The orthogonal matrix P_n , used to annihilate the vector x_n in (3), is a Householder matrix [2]. Here,

$$u_n = \frac{R_{n-1}^{-1} a_n}{\lambda \delta_n}$$

is a scaled version of the Kalman gain vector [8], which can be used to update the filter tap-weight vector.

Householder RLS(HRLS) algorithm can be deduced by computing the a priori error $e(n)$, and then updating the filter tap-weight w_n by [2],

$$e(n) = (s(n) + \eta(n)) - w_{n-1}^T a_n \quad (4)$$

$$w_n = w_{n-1} - \frac{e(n)}{\delta_n} u_n$$

III. Recursive Estimation of the Performance

A. Eigenvalue Spread of Input data

The eigenvalue spread of the data matrix R_n is the ratio of the maximum eigenvalue of R_n to its minimum eigenvalue [6].i.e.,

$$\xi_n = \frac{\lambda_{max}(R_n)}{\lambda_{min}(R_n)}$$

The eigenvalue spread is a measure of the condition number of the covariance matrix R_n. When this spread is increased, R_n becomes contaminated and its condition number is increased. The eigenvalue spread of the correlation matrix of the input vector plays a fundamental role in limiting the convergence performance of the LMS based algorithms. In case of HRLS algorithm, although its rate of convergence is invariant to the eigenvalue spread, its misalignment is found to suffer terribly from it.

B. Recursive Computation of Misalignment

We define the misalignment vector at time n as :

$$m_n = \tilde{w} - w_n$$

where $\tilde{w}$ is the true impulse response of the system. Rewriting HRLS algorithm in terms of misalignment gives:

$$m_n = m_{n-1} - \frac{e(n)}{\delta_n} u_n \quad (5)$$

Taking the l_2 norm and mathematical expectation on both sides of (4), we obtain

$$E\{\|m_n\|_2^2\} = E\{\|m_{n-1}\|_2^2\} - E\{\|\frac{e(n)}{\delta_n} u_n\|_2^2\}$$

Here we have assumed that $E\{\|\frac{e(n)}{\delta_n} u_n m_{n-1}\|_2^2\} = 0$. According to [3] , the second term on the right hand side of the last equation is positive, so we have:

$$E\{\|m_n\|_2^2\} - E\{\|m_{n-1}\|_2^2\} \leq 0$$

This shows that the length of the misalignment vector is nonincreasing.

Now rewriting equation (4) in terms of misalignment , we have

$$e(n) = m_{n-1}^T a_n + \eta(n)$$

If a_n are independent, then $\lim_{n \to \infty} E\{e(n)^2\} = 0$, which implies that

$$\lim_{n \to \infty} E\{\|m_n\|_2^2\} = 0$$

C. Eigenvalue Spread and Misalignment

Misalignment of the RLS algorithm is found to depend on three terms: the exponential window, the level of system output noise and the condition number in [4]. The $\|.\|_E$ norm was used for the computation of the condition number. We notice the same behavior of the misalignment of the HRLS algorithm. A high level of system output noise, high condition number and exponential window far from one, degrade the misalignment of the HRLS algorithm.

In this paper, we consider the behavior of misalignment subject to the changes in the eigenvalue spread, output noise of the system and exponential window. To ease the computations we are considering the eigenvalue spread in place of the condition number with $\|.\|_E$ norm. Our simulation results show that with a fixed exponential window misalignment is highly sensitive to the system output noise as well as the eigenvalue spread of the input correlation matrix R_n.

IV. Simulation Results

Computer simulation consists of estimating an unknown finite impulse response $\tilde{w}$ of length N. The adaptive filter is assumed to have the same number of taps. The input signal $u(n)$ is obtained by filtering a white, zero mean, Gaussian random sequence through the model [7],

$$H(z) = \frac{\sqrt{1 - \alpha^2}}{1 - \alpha z^{-1}}$$

where $|\alpha| < 1$ is the input signal to noise ratio.

The parameter α controls the eigenvalue spread of the input autocorrelation matrix. $\alpha = 0$ gives uncorrelated sequence (white) with eigenvalue spread ≈ 1. The eigenvalue spread increases as α moves away from 0. Figure1 shows the results of the eigenvalue spread of input correlation matrix for three different values: $\alpha = 0$, 0.5, and 0.9. Significant increase in the eigenvalue spread is observed for α close to 1. During

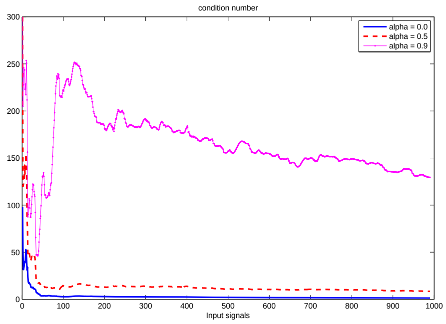

Fig. 1. Spectral Condition Number of Input correlation matrix for $\alpha = 0.0, 0.5, 0.9$

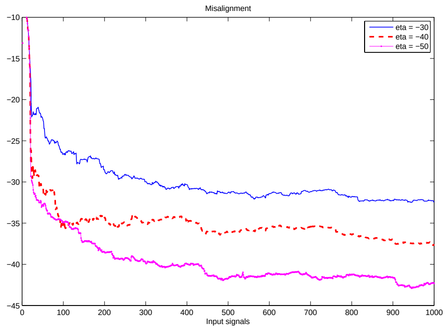

Fig. 2. Misalignment of HRLS algorithm for $\alpha = 0.0$.

this process we have fixed output noise η at -30 dB, and exponential window at $1 - \frac{1}{5N}$.

Misalignment of the HRLS algorithm is computed recursively to estimate the behavior of the algorithm. Keeping exponential window fixed at $1 - \frac{1}{5N}$, we compute the misalignment in dB, and notice that when eigenvalue spread close to 1, the misalignment is approximately equal to the output signal noise η in dB.

- Figure 2 shows the degradation of misalignment with the increase in the eigenvalue spread, with fixed output noise. For α close to 0, misalignment is close to -30 dB, and for $\alpha = 0.9$, it is degraded to -27.
- Figure 3 shows the degradation of misalignment with the increase in the output signal noise of the system for eigenvalue spread close to 1, i.e. for $\alpha = 0$. Here misalignment is close to the corresponding signal to noise ratio η of output.

So far the convergence rate of the HRLS algorithm is concerned, it is invariant to the eigenvalue spread of the input data, but is affected by the output noise η. Figure 4 shows the delay in convergence with the increase in the out put noise. A steady state solution is achieved faster with smaller value of η.

V. Conclusion

Performance of the numerically robust HRLS algorithm is estimated recursively at a low computational cost, with respect

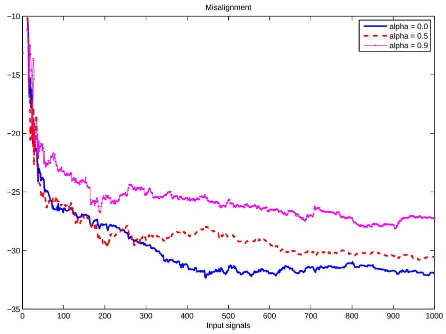

Fig. 3. Misalignment of HRLS algorithm for $\eta = -30dB$.

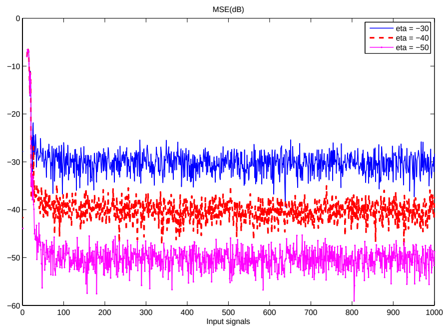

Fig. 4. Mean square error of HRLS algorithm for $\eta = -30dB, -40dB, -50dB$.

to the eigenvalue spread of the input covariance matrix. Inspite of fast convergence, misalignment of the algorithm is found to be affected by the systems output as well as input noise. Recursive estimate of misalignment is computed and is shown to be affected by the changes in the eigenvalue spread.

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