

Some Rotational Flows of an Incompressible Fluid of Variable Viscosity

Rana Khalid Naeem, Waseem Ahmed Khan, Muhammad Akhtar and Asif Mansoor

Abstract—The Navier Stokes Equations (NSE) for an incompressible fluid of variable viscosity in the presence of an unknown external force in Von-Mises system (x, ψ) are transformed, and some new exact solutions for a class of flows characterized by equation $y = f(x) + a\psi + b$ for an arbitrary state equation are determined, where $f(x)$ is a function, ψ the stream function, $a (\neq 0)$ and b are the arbitrary constants. In three, out of four cases, the function $f(x)$ is arbitrary, and the solutions are the solutions of the flow equations for all the flows characterized by the equation $y = f(x) + a\psi + b$. Streamline patterns for some forms of $f(x)$ in unbounded and bounded regions are given.

Keywords—Bounded and unbounded region, Exact solution, Navier Stokes equations, Streamline pattern, Variable viscosity, Von-Mises system

I. INTRODUCTION

A vast amount of work has been done on the Navier-Stokes equations (NSE) in the absence of external forces, and very small amount of work has appeared in literature on NSE with known external forces. This can be found in references [1-22], and references there in. Recently in [23] some work has been done on the NSE for viscous fluid of variable viscosity in the presence of external force in the Martin's coordinates (ϕ, ψ) , where $\psi = \text{constant}$ represents the stream lines and $\phi = \text{constant}$ are the arbitrary curves. The exact solutions of NSE are determined for radial, circular and parallel flows by making the coordinate system (ϕ, ψ) orthogonal. The expressions for the unknown external force are also determined for these flows, and these contain arbitrary function(s) enabling to construct a large number of external forces, and hence a large number of solutions to the flow equations. The objective of this paper is to present some exact solutions of the equations governing the motion of an incompressible fluid of variable viscosity in the presence of an unknown external force for a class of flows characterized by

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the equation $y = f(x) + a\psi + b$ for an arbitrary state equation, where $f(x)$ is a function, ψ the streamfunction, $a (\neq 0)$ and b are the arbitrary constants. The plan of the paper is as follows:

In section II, we consider the non-dimensional equations describing the motion of an incompressible fluid of variable viscosity in the presence of an unknown external force for an arbitrary state equation. We first transform the basic flow equation into the Martin's system (ϕ, ψ) using transformation defined by (15) and then into von-Mises system (x, ψ) . In section III, we determine the solutions of the flow equations in Von-Mises system (x, ψ) for the class of flows under considerations. We also give streamline patterns for some flows in the unbounded and bounded regions. In section IV, we give conclusion on the present work.

II. FLOW EQUATIONS

The non-dimensional equations describing the steady plane flow of an incompressible fluid of variable viscosity in the presence of an unknown external force with no heat addition are:

$$u_x + v_y = 0 \quad (1)$$

$$uu_x + vv_y = -p_x + \frac{1}{\text{Re}}(2\mu u_x)_x + \frac{1}{\text{Re}}\{\mu(u_y + v_x)\}_y + \lambda f_1 \quad (2)$$

$$uv_x + vv_y = -p_y + \frac{1}{\text{Re}}(2\mu v_y)_y + \frac{1}{\text{Re}}\{\mu(u_y + v_x)\}_x + \lambda f_2 \quad (3)$$

$$uT_x + vT_y = \frac{1}{\text{RePr}}(T_{xx} + T_{yy}) + \frac{Ec}{\text{Re}} \left\{ \frac{2\mu(u_x^2 + v_y^2)}{\mu(u_y + v_x)^2} + \right\} \quad (4)$$

$$\mu = \mu(T) \quad (5)$$

where u, v are the velocity components, T the temperature, μ the viscosity, P the pressure, Re the Reynolds number, Pr the Prandtl number, E_c the Eckert number, ρ the density of the fluid and f_1 and f_2 are the components of the external force. In

(2) and (3), λ is a non dimensional number, and in case of motion under the gravitational force, λ is called Froude number (F_r). On substituting

$$\omega = v_x - u_y \tag{6}$$

$$H = p + \frac{1}{2}(u^2 + v^2) - 2\frac{\mu u_x}{Re} \tag{7}$$

The system (1-5) is replaced by the following system of equations

$$u_x + v_y = 0 \tag{8}$$

$$H_x = v\omega + \frac{1}{Re}\{\mu(u_y + v_x)\}_y + F_1 \tag{9}$$

$$H_y = -u\omega - \frac{4(\mu u_x)_y}{Re} + \frac{1}{Re}\{\mu(u_y + v_x)\}_x + F_2 \tag{10}$$

$$\omega = v_x - u_y \tag{11}$$

$$uT_x + vT_y = \frac{1}{RePr}(T_{xx} + T_{yy}) + \frac{Ec}{Re}\{2\mu(u_x^2 + v_y^2) + \mu(u_y + v_x)^2\} \tag{12}$$

$$\mu = \mu(T) \tag{13}$$

Equations (8-13) constitute a system of six equations in eight unknowns $u, v, \mu, H, T, \omega, F_1, F_2$ as functions of x, y . In (9) and (10) for convenience we have put $F_1 = \lambda f_1, F_2 = \lambda f_2$. Equation (8) implies the existence of the stream function $\psi(x, y)$ such that

$$u = \psi_y, v = -\psi_x \tag{14}$$

$\psi = \text{constant}$ defines the family of streamlines and if we assume $\phi(x, y) = \text{constant}$ defines some family of curves such that it generates a curvilinear net (ϕ, ψ) with $\psi = \text{constant}$, then the transformation

$$x = x(\phi, \psi), y = y(\phi, \psi) \tag{15}$$

defines the curvilinear net in the physical plane. The squared element of arc length along any curve is defined by

$$ds^2 = E(\phi, \psi)d\phi^2 + 2F(\phi, \psi)d\phi d\psi + G(\phi, \psi)d\psi^2 \tag{16}$$

where

$$E = x_\phi^2 + y_\phi^2, G = x_\psi^2 + y_\psi^2, F = x_\phi x_\psi + y_\phi y_\psi \tag{17}$$

Equation (15) can be solved to get

$$\phi = \phi(x, y), \psi = \psi(x, y) \tag{18}$$

such that

$$x_\phi = J\psi_y, x_\psi = -J\phi_y, y_\phi = -J\psi_x, y_\psi = J\phi_x \tag{19}$$

provided J , the transformation Jacobin, is finite and non-zero. The transformation Jacobian J is given by

$$J = x_\phi y_\psi - x_\psi y_\phi \tag{20}$$

Assuming α to be the local angle of inclination of the tangent to the coordinate line $\psi = \text{constant}$, directed in the sense of increasing ϕ , then from differential geometry, we have

$$J = \pm \omega, x_\phi = \sqrt{E} \cos \alpha, y_\phi = \sqrt{E} \sin \alpha, x_\psi = \frac{F \cos \alpha - J \sin \alpha}{\sqrt{E}}, y_\psi = \frac{F \sin \alpha + J \cos \alpha}{\sqrt{E}} \tag{21}$$

$$\alpha_\phi = \frac{J}{E} \Gamma_{11}^2, \alpha_\psi = \frac{J}{E} \Gamma_{12}^2$$

where

$$\Gamma_{11}^2 = \frac{1}{2W^2}(-FE_\phi + 2EF_\phi - EE_\psi), \Gamma_{12}^2 = \frac{1}{2W^2}(EG_\phi - FE_\psi) \tag{22}$$

$$W = (EG - F^2)^{\frac{1}{2}}$$

The functions $E(\phi, \psi), G(\phi, \psi)$ and $F(\phi, \psi)$ satisfy the Gauss equation

$$\left(\frac{W}{E} \Gamma_{11}^2\right)_\psi - \left(\frac{W}{E} \Gamma_{12}^2\right)_\phi = 0 \tag{23}$$

Equations (8-13), utilizing (14-22) are replaced by the following equations

$$q = \frac{\sqrt{E}}{W} \tag{24}$$

$$J_\omega = -JL_\psi + A_\phi \left\{ (F^2 - J^2) \frac{\sin 2\alpha}{2E} + \frac{FJ \cos 2\alpha}{E} \right\} - A_\psi \{ F \sin \alpha \cos \alpha + J \cos^2 \alpha \} + B_\phi \left\{ -\frac{F^2 \cos 2\alpha}{E} + \frac{2FJ \sin 2\alpha}{E} + \frac{J^2 \cos 2\alpha}{E} \right\} + B_\psi \{ F \cos 2\alpha - J \sin 2\alpha \} + \frac{J}{\sqrt{E}} \{ F(F_1 \cos \alpha + F_2 \sin \alpha) + J(F_2 \cos \alpha + F_1 \sin \alpha) \} \tag{25}$$

$$0 = -JL_\phi + A_\psi \{ F \sin \alpha \cos \alpha - J \sin^2 \alpha \} - A_\phi \{ E \sin \alpha \cos \alpha \} + B_\phi \{ J \sin 2\alpha - F \cos 2\alpha \} + B_\psi \{ E \cos 2\alpha \} + J\sqrt{E}(F_1 \cos \alpha + F_2 \sin \alpha) \tag{26}$$

$$\frac{\left(\frac{GT_\phi}{J} - \frac{FT_\psi}{J}\right)_\phi + \left(\frac{ET_\psi}{J} - \frac{FT_\phi}{J}\right)_\psi}{JRePr} = -\frac{EcRe(A^2 + 4B^2)}{4\mu} + \frac{qT_\phi}{\sqrt{E}} \tag{27}$$

$$\omega = \frac{\left(\frac{F}{W}\right)_\phi - \left(\frac{E}{W}\right)_\psi}{W} \tag{28}$$

$$K = \frac{\left(\frac{W\Gamma_{11}^2}{E}\right)_\psi - \left(\frac{W\Gamma_{12}^2}{E}\right)_\phi}{W} = 0 \tag{29}$$

$$\mu = \mu(T) \tag{30}$$

in which ϕ and Ψ are considered as independent variables. This is a system of seven equations in ten unknown functions E, F, G, W, L, T, q, μ , F_1 , F_2 . In (25-27), the functions A and B are given by

$$A = \frac{4\mu}{JRe} \left[\left(\frac{F \sin \alpha}{\sqrt{E}} + \frac{J \cos \alpha}{\sqrt{E}} \right) (q_\phi \cos \alpha - q \alpha_\psi \sin \alpha) - \sqrt{E} \sin \alpha (q_\psi \cos \alpha - q \alpha_\psi \sin \alpha) \right] \quad (31)$$

$$B = \frac{\mu}{JRe} \left[q_\phi \left(-\frac{F \cos 2\alpha}{\sqrt{E}} + \frac{J \sin 2\alpha}{\sqrt{E}} \right) + q_\psi \sqrt{E} \cos 2\alpha + q \left(\frac{F \sin 2\alpha}{\sqrt{E}} + \frac{J \cos 2\alpha}{\sqrt{E}} \right) \alpha_\phi - q \sqrt{E} \alpha_\psi \sin 2\alpha \right] \quad (32)$$

Equations (24-30) constitute a system of seven equations in ten unknown functions E, F, G, W, H, T, q, μ , F_1 and F_2 . Since our objective is to determine exact solutions of the equations for a class of flows characterized by the equation

$$y = f(x) + a\psi + b, \quad a \neq 0 \quad (33)$$

For arbitrary state equation $\mu = \mu(T)$, in (24-30), utilizing (33), and taking $\phi = x$ are replaced by the following system of equations

$$q = \frac{\sqrt{1+y_x^2}}{a} \quad (34)$$

$$\frac{f''}{a^2} = -H_v + B_x - f' B_v + F_2 \quad (35)$$

$$H_x = f' B_x + (1-f'^2) B_v + (F_1 + f' F_2) \quad (36)$$

$$T_{xx} - f'' T_v - f' T_{vx} + (1+f'^2) T_{vv} - f' T_{xv} = -\frac{Ec Re \mu}{a^2} f''^2 + \frac{Re Pr}{a} T_x \quad (37)$$

In above equations $v = a\psi + b$ and the function B is given by

$$B = \frac{\mu f''}{a Re} \quad (38)$$

Equations (34-37) are the required flow equations in Von-Mises system (x, ψ) for the flow under considerations.

III. SOLUTION

In this section, we determine the solutions of (34-37) as follows: The compatibility condition $H_{vx} = H_{xv}$, yields

$$\frac{f'''}{a^2} = B_{xx} - (1-f'^2) B_{vv} - 2f' B_{vx} - f'' B_v + F_{2x} - (F_{1v} + f' F_{2v}) \quad (39)$$

This is the equation in which f, μ , F_1 and F_2 must satisfy for the flow under consideration. Once a solution of this equation is determined the function H and T are determined from (35-37). Equation (39) possesses solution for the following cases:

Case I. $B = B(x)$.

Case II. $B = B(v)$.

Case III. $B = n v + Z(x), n \neq 0, Z''(x) \neq 0$.

Case IV. $B = n v + m x, n \neq 0, m \neq 0$.

We consider these four cases separately

Case I.

In this case (39) becomes

$$\frac{f'''}{a^2} = M'' + F_{2x} - (F_{1v} + f' F_{2v}) \quad (40)$$

where

$$B(x, y) = M(x)$$

This holds for all X and U provided

$$F_1 = n_1 v + Q(x) \quad (41)$$

$$F_2 = n_2 v + Q(x) \quad (42)$$

$$\frac{f'''}{a^2} = M'' + R' - (n_1 + n_2 f') \quad (43)$$

In (41-43), n_1 and n_2 are both non-zero arbitrary constants, and $R(x)$, $Q(x)$ are unknown functions to be determined. Equation (43) yields

$$M = \frac{f'}{a^2} - \int R(x) dx + \frac{n_1 x^2}{2} + n_2 \int f(x) dx + c_1 x + c_2 \quad (44)$$

where C_1 and C_2 are arbitrary constants. Now (35), using the fact that $B(x, y) = M(x)$, and (42) yields

$$H = -\frac{v f''}{a^2} + v M' + \frac{n_2 v^2}{2} + v R(x) + I(x) \quad (45)$$

where $I(x)$ is an unknown function to be determined using (45) and (36). Differentiating (45) with respect to x and using (36), we get

$$I'(x) = f' M' + Q(x) + f' R(x) \quad (46)$$

whose solution is given by:

$$I(x) = \int (f' M' + Q(x) + f' R(x)) dx + m_1 \quad (47)$$

where $m_1(x)$ is an arbitrary constant.

Since $B(x, y) = M(x)$, (38) implies that μ is a function of x-alone, and therefore the R.H.S of (37) suggests to seek solution of it of the form

$$T = t_1 v + W_1(x) \quad (48)$$

This on substituting in (37), we obtain

$$W_1'' - \frac{Re Pr}{a} W_1' = t_1 f'' - \frac{Ec Pr \mu}{a^2} f''^2 \quad (49)$$

whose solution is

$$W_1 = W_2 \left[\int W_2^{-1} \left\{ t_1 f'' - \frac{Ec Pr}{a^2} \int \mu f''^2 dx + t_2 \right\} dx + t_3 \right] \quad (50)$$

where

$$W_2 = \exp\left(\frac{Re Pr}{a} x\right)$$

and t_1, t_2, t_3 are all non-zero arbitrary constants. We note that the solution of (35-37) involve arbitrary functions $Q(x)$ and $R(x)$ for given $f(x)$. This means that for a given flow pattern there exist large number of expressions for the functions H, T, μ and the external force (F_1, F_2) .

Case II.

When $B(x, v)$ is a function of v alone, (35) and (36), become

$$H_v = -\frac{f''}{a^2} - f' V'(v) + F_2 \quad (51)$$

$$H_x = (1-f'^2) V'(v) + (F_1 + f' F_2) \quad (52)$$

where

$$B(x, v) = V(v)$$

On eliminating H from (51) and (52), we get

$$\frac{f'''}{a^2} = -f'' V'(v) + F_{2x} - (1-f'^2) V''(v) - (F_{1v} + f' F_{2v}) \quad (53)$$

This holds for all x and v provided.

$$V = c_3 v \quad (54)$$

$$F_2 = n_3 v + R_2(x) \quad (55)$$

$$F_1 = \left(-\frac{f'''}{a^2} - c_3 f'' + R_2' - n_3 f' \right) v + c_4 \quad (56)$$

Equations (35) and (36), utilizing (54-56), provide

$$H = -\frac{f'' v}{a^2} - c_3 v f' + \frac{n_3 v^2}{2} + v R_2(x) + c_3 \int (1-f'^2) dx + \int f' R_2(x) dx + c_4 x + c_5 \quad (57)$$

As B is a function of v alone, therefore (38) implies that f'' must be constant and therefore

$$f = K_1 \frac{x^2}{2} + K_2 x + K_3 \quad (58)$$

$$\mu = \frac{c_3 v \text{Re}}{a K_1} \quad (59)$$

In above equations c₃, c₄, c₅, n₃, K₁, K₂, K₃ are all non-zero arbitrary constants. In this case the function T is given by

$$T = t_4 v + t_5 v^2 + \exp\left(\frac{\text{Re Pr}}{a} x\right) \left[\int \left[\int \left(t_4 f' - 2t_5 \int (1+f'^2) \right) dx + t_6 \right] \exp\left(-\frac{\text{Re Pr}}{a} x\right) dx + t_7 \right] \quad (60)$$

where

$$t_5 = \frac{c_3 Ec \text{Pr Re}}{2a}$$

Case III.

In this case (39) yields

$$\frac{f'''}{a^2} = Z'' - n f'' + F_{2x} - (F_{1v} + f' F_{2v}) \quad (61)$$

which gives

$$Z(x) = \frac{f'}{a^2} + n f - \int F_2 dx - \iint \left(\begin{matrix} Fv \\ + f' F_{2v} \end{matrix} \right) dx dx + K_1 x + K_2 \quad (62)$$

where K₁ and K₂ are both non-zero arbitrary constants. Also (35) and (36) yield

$$H_v = -\frac{f''}{a^2} + Z' - n f' + F_2 \quad (63)$$

$$H_x = f' Z' + n(1-f'^2) + (F_1 + f' F_2) \quad (64)$$

Equations (62-64), proceeding in the same manner as in previous cases, provide

$$F_2 = l_1 \quad (65)$$

$$F_1 = G_1(x) - l_1 f' \quad (66)$$

$$H = -\frac{v f''}{a^2} + Z' v - n v f' + l_1 v + \int \left[\frac{f' f''}{a^2} + n f'^2 - l_1 f' + K_1 f' + n(1-f'^2) + F_1 + l_1 f' \right] dx + K_3 \quad (67)$$

$$Z = \frac{f'}{a^2} + n f - l_1 x + K_1 x + K_2 \quad (68)$$

where l₁, K₁, K₂, K₃ are non-zero arbitrary constants and G₁(x) is an arbitrary function.

The viscosity μ in this case is given by

$$\mu = \frac{a \text{Re}(n v + Z(x))}{f''}, f'' \neq 0. \quad (69)$$

The temperature distribution T is given by

$$T = l_2 v^2 + \exp\left(\frac{\text{Re Pr}}{a} x\right) \left\{ \int \exp\left(-\frac{\text{Re Pr}}{a} x\right) \left[-\frac{Ec \text{Pr Re}}{a} \int \left(z f'' + f'^2 + 1 \right) dx + l_3 \right] dx + l_4 \right\} \quad (70)$$

In (70), l₃ and l₄ are both non-zero arbitrary constants. We note that the solution of the flow equations involves two arbitrary functions G₁(x) and f(x), and therefore we can construct a large number of solutions to the flow equations.

Case IV.

In this case we, proceeding in manner similar to previous cases, we find

$$F_1 = n_1 x + n_2 v$$

$$F_2 = n_3 v + s_1(x)$$

$$\mu = \frac{a \text{Re}(m x + n v)}{f''} \quad (71)$$

$$s_1(x) = \frac{f''}{a^2} + n f' + n_2 x + n_3 f + n_4$$

$$T = \frac{Ec \text{Pr Re } n}{2a} v^2 + n_5 v + s_2(x)$$

$$s_2 = \exp\left(\frac{\text{Re Pr } x}{a}\right) s_4(x) - \frac{a n_6}{\text{Re Pr}} + n_7 \exp\left(\frac{\text{Re Pr } x}{a}\right)$$

where

$$s_4 = \int \exp\left(-\frac{\text{Re Pr } x}{a}\right) \left[\int s_3(x) dx \right] dx$$

$$s_3 = -\frac{Ec \text{Re Pr } m x f''}{a} + n_5 f'' - \frac{Ec \text{Re Pr} (1+f'^2)}{a}$$

We note that n_5 , n_6 and n_7 are arbitrary constants and $f(x)$ is an arbitrary function. The streamline patterns for some forms of $f(x)$ in unbounded and bounded regions are given in Figs. (1-6). Fig. 2, represents the flow pattern for the fluid impinging on a porous wall $x=0$ satisfying boundary conditions $u(0, y) = \text{constant}$, $v(0, y) = 0$. Fig. 4, depicts the streamline pattern for the fluid flow on the left of an infinite plate $x=0$, satisfying boundary conditions $u(0, y) = \frac{1}{a}$, $v(0, y) = 0$. For $a < 0$, we have injection at $x=0$ and for $a > 0$, we have suction at $x=0$.

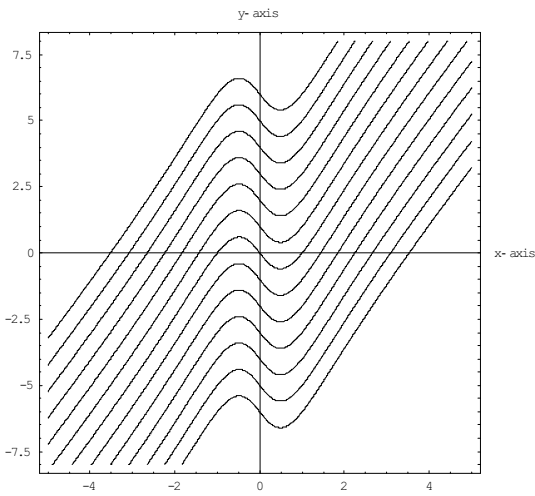


Fig. 1 Streamline pattern for $y - \frac{2x^3 - 2x}{1 + x^2} = \text{constant}$ in unbounded domain

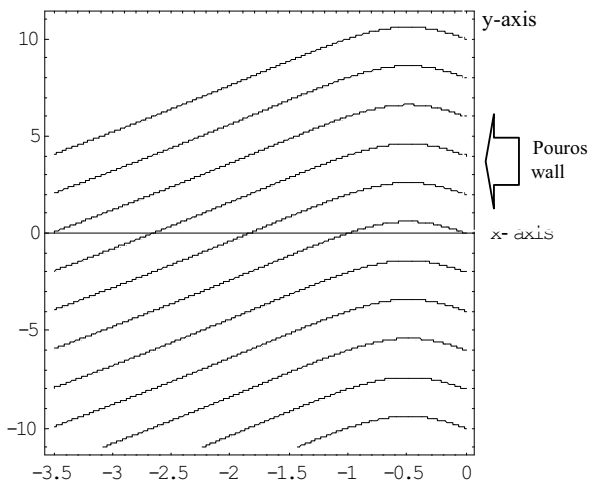


Fig. 2 Streamline pattern for $y - \frac{2x^3 - 2x}{1 + x^2} = \text{constant}$ for boundary value problem

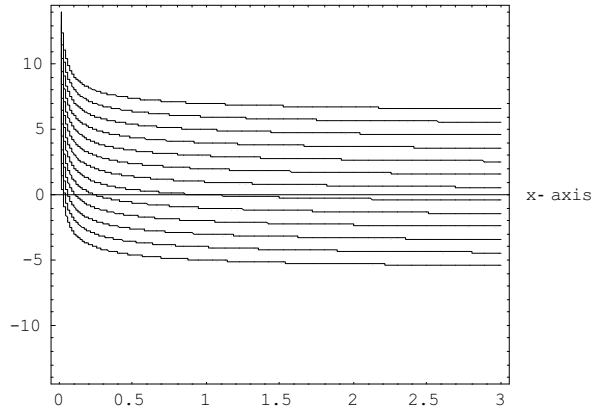


Fig. 3 Streamline pattern for $y - \frac{1}{\sqrt{x}} = \text{constant}$

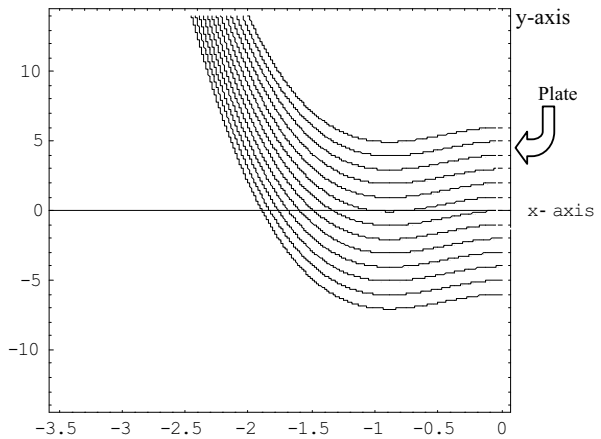


Fig. 4 Streamline pattern for $y + 3x^3 + 4x^2 = \text{constant}$ for boundary value problem

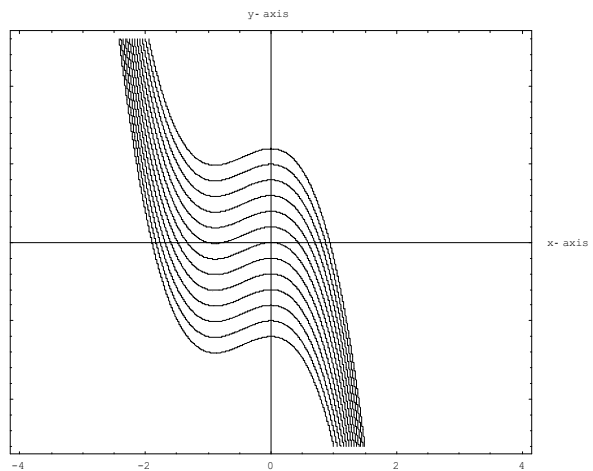


Fig. 5 Streamline pattern for $y + 3x^3 + 4x^2 = c$ in unbounded domain

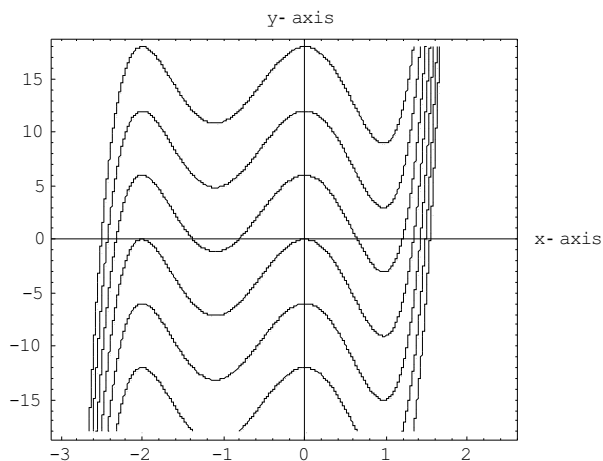


Fig. 6. Streamline pattern for $y - x^2(3x-4)(x+2)^2 = \text{constant}$ in unbounded domain

IV. CONCLUSION

In this paper the equations describing the steady plane flow of an incompressible fluid of variable viscosity in the presence of unknown external force are considered. The flow equations are transformed in (ϕ, ψ) system in which the curves $\psi = \text{constant}$ represents family of streamlines and $\phi = \text{constant}$, a family of arbitrary curves. To determine the exact solutions of these equations for a class of flows characterized by the equation $y = f(x) + a\psi + b$ for an arbitrary state equation $\mu = \mu(T)$, ϕ is taken equal to x , and employing the integrability condition $H_{x\psi} = H_{\psi x}$ the equation which the functions f , μ , F_1 , F_2 must satisfy is obtained. This equation possesses solutions for four possible cases. The solution for each case is determined. All solutions involve arbitrary function or functions, and this arbitrariness of the function(s) enables to construct a large number of solutions to the flow equations. It is worth noting that the function $f(x)$ in cases I, II and IV is arbitrary, and therefore the solutions determined are the solutions of the flow equations for all flows characterized by

$y = f(x) + a\psi + b$. Streamline patterns for some flows in unbounded and bounded region are also given.

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