

Some Collineations Preserving Cross-Ratio in some Moufang-Klingenberg Planes

Süleyman Ciftci, Atilla Akpınar and Basri Celik

Abstract—In this paper we are interested in Moufang-Klingenberg planes $\mathbf{M}(\mathcal{A})$ defined over a local alternative ring \mathcal{A} of dual numbers. We show that some collineations of $\mathbf{M}(\mathcal{A})$ preserve cross-ratio.

Keywords—Moufang-Klingenberg planes, local alternative ring, projective collineation, cross-ratio.

I. INTRODUCTION

The number of collineations of any projective plane is huge. For example; the Fano plane has 168 collineations, the non-Desarguesian projective Veblen-Wedderburn plane of order 9 (which is denoted by $\pi_N(9)$) has 311,040 collineations [14, p. 366]. It is easy to see that the composite of any two collineations is a collineation, as the inverses of any collineation. Function composition is always associative; thus the collineations of any projective or affine plane form a group. For more detailed information about these groups, the reader is referred to the books of [11], [14].

In the Euclidean plane, Desargues established the fundamental fact that cross-ratio (a concept originally introduced by Pappus of Alexandria c.300 B.C) is invariant under projection [3, p. 133]. For this reason, cross-ratio is one of the most important concepts of projective geometry.

In this paper we deal with the class (which we will denote by $\mathbf{M}(\mathcal{A})$) of Moufang-Klingenberg (MK) planes coordinatized by a local alternative ring

$$\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$$

(an alternative field \mathbf{A} , $\varepsilon \notin \mathbf{A}$ and $\varepsilon^2 = 0$) introduced by Blunck in [7]. We will show that some collineations of $\mathbf{M}(\mathcal{A})$ from [8] preserve cross-ratio. For more information about some well-known properties of cross-ratio in the case of Moufang planes or MK-planes $\mathbf{M}(\mathcal{A})$, respectively, it can be seen in the papers of [10], [4], [9] or [7], [1].

Section 2 includes some basic definitions and results from the literature.

In Section 3 we will give some collineations of $\mathbf{M}(\mathcal{A})$ from [8] and we show that the collineations preserve cross-ratio, the main result of the paper.

II. PRELIMINARIES

Let $\mathbf{M} = (\mathbf{P}, \mathbf{L}, \varepsilon, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \varepsilon)$ (points, lines, incidence) and an equivalence relation ' \sim ' (neighbour relation) on \mathbf{P} and on \mathbf{L} , respectively. Then

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\mathbf{M} is called a *projective Klingenberg plane* (PK-plane), if it satisfies the following axioms:

(PK1) If P, Q are non-neighbour points, then there is a unique line PQ through P and Q .

(PK2) If g, h are non-neighbour lines, then there is a unique point $g \cap h$ on both g and h .

(PK3) There is a projective plane $\mathbf{M}^* = (\mathbf{P}^*, \mathbf{L}^*, \varepsilon)$ and an incidence structure epimorphism $\Psi : \mathbf{M} \rightarrow \mathbf{M}^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim Q, \Psi(g) = \Psi(h) \Leftrightarrow g \sim h$$

hold for all $P, Q \in \mathbf{P}$, $g, h \in \mathbf{L}$.

A point $P \in \mathbf{P}$ is called *near* a line $g \in \mathbf{L}$ iff there exists a line $h \sim g$ such that $P \in h$.

Let $h, k \in \mathbf{L}$, $C \in \mathbf{P}$, C is not near to h, k . Then the well-defined bijection

$$\sigma := \sigma_C(k, h) : \begin{cases} h \rightarrow k \\ X \rightarrow XC \cap k \end{cases}$$

mapping h to k is called a *perspectivity* from h to k with center C . A product of a finite number of perspectivities is called a *projectivity*.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a *collineation* of \mathbf{M} .

A *Moufang-Klingenberg plane* (MK-plane) is a PK-plane \mathbf{M} that generalizes a Moufang plane, and for which \mathbf{M}^* is a Moufang plane (for the exact definition see [2]).

An *alternative ring (field)* \mathbf{R} is a not necessarily associative ring (field) that satisfies the alternative laws

$$a(ab) = a^2b, (ba)a = ba^2, \forall a, b \in \mathbf{R}.$$

An alternative ring \mathbf{R} with identity element 1 is called *local* if the set \mathbf{I} of its non-unit elements is an ideal.

We are now ready to give consecutively two important lemmas related to alternative rings.

Lemma 2.1: The subring generated by any two elements of an alternative ring is associative (cf. [13, Theorem 3.1]).

Lemma 2.2: The identities

$$\begin{aligned} x(y(xz)) &= (xyx)z \\ ((yx)z)x &= y(xzx) \\ (xy)(zx) &= x(yz)x \end{aligned}$$

which are known as Moufang identities are satisfied in every alternative ring (cf. [12, p. 160]).

We summarize some basic concepts about the coordinatization of MK-planes from [2].

Let \mathbf{R} be a local alternative ring. Then $\mathbf{MR} = (\mathbf{P}, \mathbf{L}, \in, \sim)$ is the incidence structure with neighbour relation defined as follows:

$$\begin{aligned} \mathbf{P} &= \{(x, y, 1) | x, y \in \mathbf{R}\} \cup \{(1, y, z) | y \in \mathbf{R}, z \in \mathbf{I}\} \\ &\quad \cup \{(w, 1, z) | w, z \in \mathbf{I}\}, \\ \mathbf{L} &= \{[m, 1, p] | m, p \in \mathbf{R}\} \cup \{[1, n, p] | p \in \mathbf{R}, n \in \mathbf{I}\} \\ &\quad \cup \{[q, n, 1] | q, n \in \mathbf{I}\}, \\ [m, 1, p] &= \{(x, xm + p, 1) | x \in \mathbf{R}\} \\ &\quad \cup \{(1, zp + m, z) | z \in \mathbf{I}\}, \\ [1, n, p] &= \{(yn + p, y, 1) | y \in \mathbf{R}\} \\ &\quad \cup \{(zp + n, 1, z) | z \in \mathbf{I}\}, \\ [q, n, 1] &= \{(1, y, yn + q) | y \in \mathbf{R}\} \\ &\quad \cup \{(w, 1, wq + n) | w \in \mathbf{I}\}, \\ P &= (x_1, x_2, x_3) \sim (y_1, y_2, y_3) = Q \\ &\Leftrightarrow x_i - y_i \in \mathbf{I} \quad (i = 1, 2, 3), \forall P, Q \in \mathbf{P}, \\ g &= [x_1, x_2, x_3] \sim [y_1, y_2, y_3] = h \\ &\Leftrightarrow x_i - y_i \in \mathbf{I} \quad (i = 1, 2, 3), \forall g, h \in \mathbf{L}. \end{aligned}$$

Now it is time to give the following theorem from [2].

Theorem 2.1: $\mathbf{M}(\mathbf{R})$ is an MK-plane, and each MK-plane is isomorphic to some $\mathbf{M}(\mathbf{R})$.

Let \mathbf{A} be an alternative field and $\varepsilon \notin \mathbf{A}$. Consider

$$\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$$

with componentwise addition and multiplication as follows:

$$(a_1 + a_2\varepsilon)(b_1 + b_2\varepsilon) = a_1b_1 + (a_1b_2 + a_2b_1)\varepsilon,$$

where $a_i, b_i \in \mathbf{A}$ for $i = 1, 2$. Then \mathcal{A} is a local alternative ring with ideal $\mathbf{I} = \mathbf{A}\varepsilon$ of non-units. The set of formal inverses of the non-units of \mathcal{A} is denoted as \mathbf{I}^{-1} . Calculations with the elements of \mathbf{I}^{-1} are defined as follows [6]:

$$\begin{aligned} (a\varepsilon)^{-1} + t &:= (a\varepsilon)^{-1} := t + (a\varepsilon)^{-1} \\ q(a\varepsilon)^{-1} &:= (aq^{-1}\varepsilon)^{-1} \\ (a\varepsilon)^{-1}q &:= (q^{-1}a\varepsilon)^{-1} \\ ((a\varepsilon)^{-1})^{-1} &:= a\varepsilon, \end{aligned}$$

where $(a\varepsilon)^{-1} \in \mathbf{I}^{-1}, t \in \mathcal{A}, q \in \mathcal{A} \setminus \mathbf{I}$. (Other terms are not defined.). For more information about \mathcal{A} and its relation to MK-planes, the reader is referred to the papers of Blunck [6], [7]. In [7], the centre $\mathbf{Z}(\mathcal{A})$ is defined to be the (commutative, associative) subring of \mathcal{A} which is commuting and associating with all elements of \mathcal{A} . It is $\mathbf{Z}(\mathcal{A}) := \mathbf{Z}(\varepsilon) = \mathbf{Z} + \mathbf{Z}\varepsilon$, where $\mathbf{Z} = \{z \in \mathbf{A} | za = az, \forall a \in \mathbf{A}\}$ is the centre of \mathbf{A} . If \mathbf{A} is not associative, then \mathbf{A} is a Cayley division algebra over its centre \mathbf{Z} .

Throughout this paper we assume $\text{char } \mathbf{A} \neq 2$ and we restrict ourselves to the MK-planes $\mathbf{M}(\mathcal{A})$.

Blunck [7] gives the following algebraic definition of the cross-ratio for the points on the line $g := [1, 0, 0]$ in $\mathbf{M}(\mathcal{A})$.

$$\begin{aligned} (A, B; C, D) &:= (a, b; c, d) \\ &= \langle (a-d)^{-1}(b-d) \rangle \langle (b-c)^{-1}(a-c) \rangle > \\ (Z, B; C, D) &:= (z^{-1}, b; c, d) \\ &= \langle (1-dz)^{-1}(b-d) \rangle \langle (b-c)^{-1}(1-cz) \rangle > \\ (A, Z; C, D) &:= (a, z^{-1}; c, d) \\ &= \langle (a-d)^{-1}(1-dz) \rangle \langle (1-cz)^{-1}(a-c) \rangle > \\ (A, B; Z, D) &:= (a, b; z^{-1}, d) \\ &= \langle (a-d)^{-1}(b-d) \rangle \langle (1-zb)^{-1}(1-za) \rangle > \\ (A, B; C, Z) &:= (a, b; c, z^{-1}) \\ &= \langle (1-za)^{-1}(1-zb) \rangle \langle (b-c)^{-1}(a-c) \rangle >, \end{aligned}$$

where $A = (0, a, 1), B = (0, b, 1), C = (0, c, 1), D = (0, d, 1), Z = (0, 1, z)$ are pairwise non-neighbour points of g and $\langle x \rangle = \{y^{-1}xy | y \in \mathcal{A}\}$.

In [6, Theorem 2], it is shown that the transformations

$$\begin{aligned} t_u(x) &= x + u; \quad u \in \mathcal{A} \\ r_u(x) &= xu; \quad u \in \mathcal{A} \setminus \mathbf{I} \\ i(x) &= x^{-1} \\ l_u(x) &= ux = (ir_u^{-1}i)(x); \quad u \in \mathcal{A} \setminus \mathbf{I} \end{aligned}$$

which are defined on the line g preserve cross-ratios. In [5, Corollary (iii)], it is also shown that the group generated by these transformations, which is denoted by Λ , equals to the group of projectivities of a line in $\mathbf{M}(\mathcal{A})$. The elements preserving cross-ratio of the group Λ defined on g will act a very important role in the proof of Theorem 3.1.

We give the following result from [1, Theorem 8]. This result states a simple way for calculation of the cross-ratio of the points on any line in $\mathbf{M}(\mathcal{A})$.

Theorem 2.2: Let $\{O, U, V, E\}$ be the basis of $\mathbf{M}(\mathcal{A})$ where $O = (0, 0, 1), U = (1, 0, 0), V = (0, 1, 0), E = (1, 1, 1)$ (see [2, Section 4]). Then, according to types of lines, the cross-ratio of the points on the line l can be calculated as follows:

If A, B, C, D and Z are the pairwise non-neighbour points

- of the line $l = [m, 1, k]$, where $A = (a, am + k, 1), B = (b, bm + k, 1), C = (c, cm + k, 1), D = (d, dm + k, 1)$ are not near to the line $UV = [0, 0, 1]$ and $Z = (1, m + zp, z)$ is near to UV ,
- of the line $l = [1, n, p]$, where $A = (an + p, a, 1), B = (bn + p, b, 1), C = (cn + p, c, 1), D = (dn + p, d, 1)$ are not neighbour to V and $Z = (n + zp, 1, z) \sim V$,
- of the line $l = [q, n, 1]$, where $A = (1, a, q + an), B = (1, b, q + bn), C = (1, c, q + cn), D = (1, d, q + dn)$ are not neighbour to V and $Z = (z, 1, zq + n) \sim V$,

then

$$\begin{aligned} (A, B; C, D) &= (a, b; c, d) \\ (Z, B; C, D) &= (z^{-1}, b; c, d) \\ (A, Z; C, D) &= (a, z^{-1}; c, d) \\ (A, B; Z, D) &= (a, b; z^{-1}, d) \\ (A, B; C, Z) &= (a, b; c, z^{-1}). \end{aligned}$$

We can give an important theorem, from [1, Theorem 9], about cross-ratio.

Theorem 2.3: In $\mathbf{M}(\mathcal{A})$, perspectivities preserve cross-ratios.

In the next section, we deal with some collineations preserving cross-ratio in $\mathbf{M}(\mathcal{A})$.

III. SOME COLLINEATIONS PRESERVING CROSS-RATIO.

In this section we would like to show that the following collineations we will introduce from [8] preserve cross-ratios. Now we start with giving the collineations, where $w, z, q, n \in \mathbf{A}$:

For any $u \notin \mathbf{I}$, the map L_u transforms points and lines as follows:

$$\begin{aligned} (x, y, 1) &\rightarrow (ux, uyu, 1) \\ (1, y, z\varepsilon) &\rightarrow (1, yu, (zu^{-1})\varepsilon) \\ (w\varepsilon, 1, z\varepsilon) &\rightarrow ((u^{-1}w)\varepsilon, 1, (u^{-1}zu^{-1})\varepsilon) \\ [m, 1, k] &\rightarrow [mu, 1, uk] \\ [1, n\varepsilon, p] &\rightarrow [1, (u^{-1}n)\varepsilon, up] \\ [q\varepsilon, n\varepsilon, 1] &\rightarrow [(qu^{-1})\varepsilon, (u^{-1}nu^{-1})\varepsilon, 1]. \end{aligned}$$

For any $u \notin \mathbf{I}$, the map F_u transforms points and lines as follows:

$$\begin{aligned} (x, y, 1) &\rightarrow (uxu, uy, 1) \\ (1, y, z\varepsilon) &\rightarrow (1, u^{-1}y, (u^{-1}zu^{-1})\varepsilon) \\ (w\varepsilon, 1, z\varepsilon) &\rightarrow ((wu)\varepsilon, 1, (zu^{-1})\varepsilon) \\ [m, 1, k] &\rightarrow [u^{-1}m, 1, uk] \\ [1, n\varepsilon, p] &\rightarrow [1, (nu)\varepsilon, up] \\ [q\varepsilon, n\varepsilon, 1] &\rightarrow [(u^{-1}qu^{-1})\varepsilon, (nu^{-1})\varepsilon, 1]. \end{aligned}$$

For any $\alpha, \beta \in \mathbf{Z}(\mathcal{A})$, $\alpha, \beta \notin \mathbf{I}$, the map $S_{\alpha, \beta}$ transforms points and lines as follows:

$$\begin{aligned} (x, y, 1) &\rightarrow (x\beta, y\alpha, 1) \\ (1, y, z\varepsilon) &\rightarrow (1, \beta^{-1}y\alpha, (\beta^{-1}z)\varepsilon) \\ (w\varepsilon, 1, z\varepsilon) &\rightarrow ((\alpha^{-1}w\beta)\varepsilon, 1, (\alpha^{-1}z)\varepsilon) \\ [m, 1, k] &\rightarrow [\beta^{-1}m\alpha, 1, k\alpha] \\ [1, n\varepsilon, p] &\rightarrow [1, (\alpha^{-1}n\beta)\varepsilon, p\beta] \\ [q\varepsilon, n\varepsilon, 1] &\rightarrow [(\beta^{-1}q)\varepsilon, (\alpha^{-1}n)\varepsilon, 1]. \end{aligned}$$

The map I_2 transforms points and lines as follows:

$$\begin{aligned} (x, y, 1) &\rightarrow (y^{-1}x, y^{-1}, 1) \text{ if } y \notin \mathbf{I} \\ (x, y, 1) &\rightarrow (1, x^{-1}, x^{-1}y) \text{ if } y \in \mathbf{I} \wedge x \notin \mathbf{I} \\ (x, y, 1) &\rightarrow (x, 1, y) \text{ if } y \in \mathbf{I} \wedge x \in \mathbf{I} \\ (1, y, z\varepsilon) &\rightarrow (y^{-1}, (y^{-1}z)\varepsilon, 1) \text{ if } y \notin \mathbf{I} \\ (1, y, z\varepsilon) &\rightarrow (1, z\varepsilon, y) \text{ if } y \in \mathbf{I} \\ (w\varepsilon, 1, z\varepsilon) &\rightarrow (w\varepsilon, z\varepsilon, 1) \\ [m, 1, k] &\rightarrow [-mk^{-1}, 1, k^{-1}] \text{ if } k \notin \mathbf{I} \\ [m, 1, k] &\rightarrow [1, -km^{-1}, m^{-1}] \text{ if } k \in \mathbf{I} \wedge m \notin \mathbf{I} \\ [m, 1, k] &\rightarrow [m, k, 1] \text{ if } k \in \mathbf{I} \wedge m \in \mathbf{I} \\ [1, n\varepsilon, p] &\rightarrow [p^{-1}, 1, -(np^{-1})\varepsilon] \text{ if } p \notin \mathbf{I} \\ [1, n\varepsilon, p] &\rightarrow [1, p, n\varepsilon] \text{ if } p \in \mathbf{I} \\ [q\varepsilon, n\varepsilon, 1] &\rightarrow [q\varepsilon, 1, n\varepsilon]. \end{aligned}$$

Now we are ready to give the main result of the paper.

Theorem 3.1: The collineations $L_u, F_u, S_{\alpha, \beta}$ and I_2 preserve cross-ratio.

Proof: Let A, B, C, D and Z be the points with the property given in the statement of Theorem 2.2. Then, it is obvious that

$$\begin{aligned} (A, B; C, D) &= (a, b; c, d) \\ (Z, B; C, D) &= (z^{-1}, b; c, d) \\ (A, Z; C, D) &= (a, z^{-1}; c, d) \\ (A, B; Z, D) &= (a, b; z^{-1}, d) \\ (A, B; C, Z) &= (a, b; c, z^{-1}), \end{aligned} \tag{1}$$

where $z \in \mathbf{I}$. In this case we must find the effect of φ to the points of any line where φ is any one of collineations $L_u, F_u, S_{\alpha, \beta}$, and I_2 .

i) Let $\varphi = L_u$. If $l = [m, 1, k]$, then

$$\begin{aligned} \varphi(X) &= \varphi(x, xm + k, 1) = (ux, u(xm + k), u, 1) \\ \varphi(Z) &= \varphi(1, m + zk, z) = (1, (m + zk)u, zu^{-1}) \end{aligned}$$

and $\varphi(l) = [mu, 1, uk]$. From (a) of Theorem 2.2, we obtain

$$\begin{aligned} (\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (ua, ub; uc, ud) \\ &=^\sigma (a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= (uz^{-1}, ub; uc, ud) \\ &=^\sigma (z^{-1}, b; c, d), \end{aligned}$$

where $\sigma = l_{u^{-1}} \in \Lambda$.

If $l = [1, n, p]$, then

$$\begin{aligned} \varphi(X) &= \varphi(xn + p, x, 1) = (u(xn + p), uxu, 1) \\ \varphi(Z) &= \varphi(n + zp, 1, z) = (u^{-1}(n + zp), 1, u^{-1}zu^{-1}) \end{aligned}$$

and $\varphi(l) = [1, u^{-1}n, up]$. From (b) of Theorem 2.2, we have

$$\begin{aligned} (\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (uau, ubu; ucu, udu) \\ &=^\sigma (a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= (uz^{-1}u, ubu; ucu, udu) \\ &=^\sigma (z^{-1}, b; c, d), \end{aligned}$$

where $\sigma = l_{u^{-1}} \circ r_{u^{-1}} \in \Lambda$.

If $l = [q, n, 1]$, then

$$\begin{aligned} \varphi(X) &= \varphi(1, x, q + xn) = (1, xu, (q + xn)u^{-1}) \\ \varphi(Z) &= \varphi(z, 1, zq + n) = (u^{-1}z, 1, u^{-1}(zq + n)u^{-1}) \end{aligned}$$

and $\varphi(l) = [qu^{-1}, u^{-1}nu^{-1}, 1]$. From (c) of Theorem 2.2, we obtain

$$\begin{aligned} (\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (au, bu; cu, du) \\ &=^\sigma (a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= (z^{-1}u, bu; cu, du) \\ &=^\sigma (z^{-1}, b; c, d), \end{aligned}$$

where $\sigma = r_{u^{-1}} \in \Lambda$.

ii) Let $\varphi = F_u$. If $l = [m, 1, k]$, then

$$\begin{aligned} \varphi(X) &= \varphi(x, xm + k, 1) = (uxu, u(xm + k), 1) \\ \varphi(Z) &= \varphi(1, m + zk, z) = (1, u^{-1}(m + zk), u^{-1}zu^{-1}) \end{aligned}$$

and $\varphi(l) = [u^{-1}m, 1, uk]$. From (a) of Theorem 2.2, we have

$$\begin{aligned} (\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (uau, ubu; ucu, udu) \\ &=^\sigma (a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= (uz^{-1}u, ubu; ucu, udu) \\ &=^\sigma (z^{-1}, b; c, d), \end{aligned}$$

where $\sigma = l_{u^{-1}} \circ r_{u^{-1}} \in \Lambda$.

If $l = [1, n, p]$, then

$$\begin{aligned} \varphi(X) &= \varphi(xn + p, x, 1) = (u(xn + p)u, ux, 1) \\ \varphi(Z) &= \varphi(n + zp, 1, z) = ((n + zp)u, 1, zu^{-1}) \end{aligned}$$

and $\varphi(l) = [1, nu, upu]$. From (b) of Theorem 2.2, we obtain

$$\begin{aligned} (\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (ua, ub; uc, ud) \\ &=^\sigma (a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= (uz^{-1}, ub; uc, ud) \\ &=^\sigma (z^{-1}, b; c, d), \end{aligned}$$

where $\sigma = l_{u^{-1}} \in \Lambda$.

If $l = [q, n, 1]$, then

$$\begin{aligned} \varphi(X) &= \varphi(1, x, q + xn) = (1, u^{-1}x, u^{-1}(q + xn)u^{-1}) \\ \varphi(Z) &= \varphi(z, 1, zq + n) = (zu, 1, (zq + n)u^{-1}) \end{aligned}$$

and $\varphi(l) = [u^{-1}qu^{-1}, nu^{-1}, 1]$. From (c) of Theorem 2.2, we have

$$\begin{aligned} (\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (u^{-1}a, u^{-1}b; u^{-1}c, u^{-1}d) =^\sigma (a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= (u^{-1}z^{-1}, u^{-1}b; u^{-1}c, u^{-1}d) =^\sigma (z^{-1}, b; c, d), \end{aligned}$$

where $\sigma = l_u \in \Lambda$.

iii) Let $\varphi = S_{\alpha, \beta}$. If $l = [m, 1, k]$, then

$$\begin{aligned} \varphi(X) &= \varphi(x, xm + k, 1) = (x\beta, (xm + k)\alpha, 1) \\ \varphi(Z) &= \varphi(1, m + zk, z) = (1, \beta^{-1}(m + zk)\alpha, \beta^{-1}z) \end{aligned}$$

and $\varphi(l) = [\beta^{-1}m\alpha, 1, k\alpha]$. From (a) of Theorem 2.2, we obtain

$$\begin{aligned} (\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (a\beta, b\beta; c\beta, d\beta) \\ &=^\sigma (a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= (z^{-1}\beta, b\beta; c\beta, d\beta) \\ &=^\sigma (z^{-1}, b; c, d), \end{aligned}$$

where $\sigma = r_{\beta^{-1}} \in \Lambda$.

If $l = [1, n, p]$, then

$$\begin{aligned} \varphi(X) &= \varphi(xn + p, x, 1) = ((xn + p)\beta, x\alpha, 1) \\ \varphi(Z) &= \varphi(n + zp, 1, z) = (\alpha^{-1}(n + zp)\beta, 1, \alpha^{-1}z) \end{aligned}$$

and $\varphi(l) = [1, \alpha^{-1}n\beta, p\beta]$. From (b) of Theorem 2.2, we have

$$\begin{aligned} (\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (a\alpha, b\alpha; c\alpha, d\alpha) \\ &=^\sigma (a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= (z^{-1}\alpha, b\alpha; c\alpha, d\alpha) \\ &=^\sigma (z^{-1}, b; c, d), \end{aligned}$$

where $\sigma = r_{\alpha^{-1}} \in \Lambda$.

If $l = [q, n, 1]$, then

$$\begin{aligned} \varphi(X) &= \varphi(1, x, q + xn) = (1, \beta^{-1}x\alpha, \beta^{-1}(q + xn)) \\ \varphi(Z) &= \varphi(z, 1, zq + n) = (\alpha^{-1}z\beta, 1, \alpha^{-1}(zq + n)) \end{aligned}$$

and $\varphi(l) = [\beta^{-1}q, \alpha^{-1}n, 1]$. From (c) of Theorem 2.2, we obtain

$$\begin{aligned} (\varphi(A), \varphi(B); \varphi(C), \varphi(D)) &= (\beta^{-1}a\alpha, \beta^{-1}b\alpha; \beta^{-1}c\alpha, \beta^{-1}d\alpha) =^\sigma (a, b; c, d) \\ (\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) &= (\beta^{-1}z^{-1}\alpha, \beta^{-1}b\alpha; \beta^{-1}c\alpha, \beta^{-1}d\alpha) =^\sigma (z^{-1}, b; c, d), \end{aligned}$$

where $\sigma = l_{\beta} \circ r_{\alpha^{-1}} \in \Lambda$.

iv) Let $\varphi = I_2$. If $l = [m, 1, k]$, then

$$\begin{aligned} \varphi(X) &= \varphi(x, xm + k, 1) \\ &= \left((xm + k)^{-1}x, (xm + k)^{-1}, 1 \right), \end{aligned}$$

where $xm + k \notin \mathbf{I}$

$$\begin{aligned} \varphi(X) &= \varphi(x, xm + k, 1) \\ &= (1, x^{-1}, x^{-1}(xm + k)), \end{aligned}$$

where $xm + k \in \mathbf{I}$ and $x \notin \mathbf{I}$

$$\begin{aligned} \varphi(X) &= \varphi(x, xm + k, 1) \\ &= (x, 1, xm + k), \text{ where } xm + k \in \mathbf{I} \text{ and } x \in \mathbf{I} \end{aligned}$$

$$\begin{aligned} \varphi(Z) &= \varphi(1, m + zk, z) \\ &= \left((m + zk)^{-1}, (m + zk)^{-1}z, 1 \right), \end{aligned}$$

where $m + zk \notin \mathbf{I}$

$$\begin{aligned} \varphi(Z) &= \varphi(1, m + zk, z) \\ &= (1, z, m + zk), \text{ where } m + zk \in \mathbf{I} \end{aligned}$$

and

$$\begin{aligned} \varphi(l) &= [-mk^{-1}, 1, k^{-1}], \text{ where } k \notin \mathbf{I} \\ \varphi(l) &= [1, -km^{-1}, m^{-1}], \text{ where } k \in \mathbf{I} \text{ and } m \notin \mathbf{I} \\ \varphi(l) &= [m, k, 1], \text{ where } k \in \mathbf{I} \text{ and } m \in \mathbf{I}. \end{aligned}$$

In this case, from (a) of Theorem 2.2, the cross-ratio of the points of $[-mk^{-1}, 1, k^{-1}]$ is as follows:

$$\begin{aligned} &(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) \\ &= ((am+k)^{-1}a, (bm+k)^{-1}b; \\ &(cm+k)^{-1}c, (dm+k)^{-1}d) \\ &=^\sigma (a, b; c, d) \end{aligned}$$

$$\begin{aligned} &(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) \\ &= ((m+zk)^{-1}, (bm+k)^{-1}b; \\ &(cm+k)^{-1}c, (dm+k)^{-1}d) \\ &=^\sigma (z^{-1}, b; c, d), \end{aligned}$$

where $\sigma = i \circ r_{k^{-1}} \circ t_{-m} \circ i \in \Lambda$. From (b) of Theorem 2.2, the cross-ratio of the points of $[1, -km^{-1}, m^{-1}]$ is as follows:

$$\begin{aligned} &(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) \\ &= ((am+k)^{-1}, (bm+k)^{-1}; \\ &(cm+k)^{-1}, (dm+k)^{-1}) \\ &=^\sigma (a, b; c, d) \end{aligned}$$

$$\begin{aligned} &(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) \\ &= ((m+zk)^{-1}z, (bm+k)^{-1}; \\ &(cm+k)^{-1}, (dm+k)^{-1}) \\ &=^\sigma (z^{-1}, b; c, d), \end{aligned}$$

where $\sigma = r_{m^{-1}} \circ t_{-k} \circ i \in \Lambda$. From (c) of Theorem 2.2, the cross-ratio of the points of $[m, k, 1]$ is as follows:

$$\begin{aligned} &(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = (a^{-1}, b^{-1}; c^{-1}, d^{-1}) \\ &=^\sigma (a, b; c, d) \\ &(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) = (z, b^{-1}; c^{-1}, d^{-1}) \\ &=^\sigma (z^{-1}, b; c, d), \end{aligned}$$

where $\sigma = i \in \Lambda$.

If $l = [1, n, p]$, then

$$\begin{aligned} \varphi(X) &= \varphi(xn+p, x, 1) \\ &= (x^{-1}(xn+p), x^{-1}, 1), \text{ where } x \notin \mathbf{I} \end{aligned}$$

$$\begin{aligned} \varphi(X) &= \varphi(xn+p, x, 1) \\ &= (1, (xn+p)^{-1}, (xn+p)^{-1}x), \\ &\text{ where } x \in \mathbf{I} \text{ and } xn+p \notin \mathbf{I} \end{aligned}$$

$$\begin{aligned} \varphi(X) &= \varphi(xn+p, x, 1) \\ &= (xn+p, 1, x), \text{ where } x \in \mathbf{I} \text{ and } xn+p \in \mathbf{I} \end{aligned}$$

$$\varphi(Z) = \varphi(n+zp, 1, z) = (n+zp, z, 1)$$

and

$$\begin{aligned} \varphi(l) &= [p^{-1}, 1, -np^{-1}], \text{ where } p \notin \mathbf{I} \\ \varphi(l) &= [1, p, n], \text{ where } p \in \mathbf{I}. \end{aligned}$$

In this case, from (a) of Theorem 2.2, the cross-ratio of the points of $[p^{-1}, 1, -np^{-1}]$ is as follows:

$$\begin{aligned} &(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = (a^{-1}(an+p), b^{-1}(bn+p); \\ &c^{-1}(cn+p), d^{-1}(dn+p)) \\ &=^\sigma (a, b; c, d) \end{aligned}$$

$$\begin{aligned} &(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) = (n+zp, b^{-1}(bn+p); \\ &c^{-1}(cn+p), d^{-1}(dn+p)) \\ &=^\sigma (z^{-1}, b; c, d), \end{aligned}$$

where $\sigma = i \circ r_{p^{-1}} \circ t_{-n} \in \Lambda$. From (b) of Theorem 2.2, the cross-ratio of the points of $[1, p, n]$ is as follows:

$$\begin{aligned} &(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = (a^{-1}, b^{-1}; c^{-1}, d^{-1}) \\ &=^\sigma (a, b; c, d) \end{aligned}$$

$$\begin{aligned} &(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) = (z, b^{-1}; c^{-1}, d^{-1}) \\ &=^\sigma (z^{-1}, b; c, d), \end{aligned}$$

where $\sigma = i \in \Lambda$.

If $l = [q, n, 1]$, then

$$\begin{aligned} \varphi(X) &= \varphi(1, x, q+xn) \\ &= (x^{-1}, x^{-1}(q+xn), 1), \text{ where } x \notin \mathbf{I} \end{aligned}$$

$$\begin{aligned} \varphi(X) &= \varphi(1, x, q+xn) \\ &= (1, q+xn, x), \text{ where } x \in \mathbf{I} \end{aligned}$$

$$\varphi(Z) = \varphi(z, 1, zq+n) = (z, zq+n, 1)$$

and $\varphi(l) = [q, 1, n]$. In this case, from (a) of Theorem 2.2, the cross-ratio of the points of $[q, 1, n]$ is as follows:

$$\begin{aligned} &(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = (a^{-1}, b^{-1}; c^{-1}, d^{-1}) \\ &=^\sigma (a, b; c, d) \end{aligned}$$

$$\begin{aligned} &(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) = (z, b^{-1}; c^{-1}, d^{-1}) \\ &=^\sigma (z^{-1}, b; c, d), \end{aligned}$$

where $\sigma = i \in \Lambda$.

Consequently, by considering other all cases we get

$$\begin{aligned} &(\varphi(A), \varphi(B); \varphi(C), \varphi(D)) = (a, b; c, d) \\ &(\varphi(Z), \varphi(B); \varphi(C), \varphi(D)) = (z^{-1}, b; c, d) \\ &(\varphi(A), \varphi(Z); \varphi(C), \varphi(D)) = (a, z^{-1}; c, d) \\ &(\varphi(A), \varphi(B); \varphi(Z), \varphi(D)) = (a, b; z^{-1}, d) \\ &(\varphi(A), \varphi(B); \varphi(C), \varphi(Z)) = (a, b; c, z^{-1}) \end{aligned}$$

for every collineation φ . Combining the last result and the result of (1), the proof is completed. ■

Remark 3.2: In the present paper we show that the collineations $L_u, F_u, S_{\alpha, \beta}$, and I_2 , given in [8], preserve cross-ratio. A paper related to the result that the other collineations of [8] ($T_{u, v}, I_1, F$ and G_u) preserve cross-ratio, is under review.

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