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# Sign pattern matrices that admit $P_0$ matrices

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*Abstract*—A $P_0$-matrix is a real square matrix all of whose principle minors are nonnegative. In this paper, we consider the class of $P_0$-matrix. Our main aim is to determine which sign pattern matrices are admissible for this class of real matrices.

*Keywords*—Sign pattern matrices, $P_0$ matrices, graph, digraph

## I. Introduction

An $n \times n$ sign pattern $A$ is a matrix whose entries are from $\{+, -, 0\}$. The sign of a given number $a$, denoted by sign $(a)$ is $+$, $-$, or 0 depending on $a > 0$, $a < 0$ or $a = 0$. Denote by $Q_n$ the set of all $n \times n$ sign pattern matrices. For a real matrix $B$, sign $(B)$ is the sign pattern matrix obtained by replacing each entry with its sign. If $A \in Q_n$, then the qualitative class of $A$ is defined by

$$Q(A) = \{B : \text{sign}(B) = A\}.$$

If $P$ is a property referring to a realmatrix, then a sign pattern $A$ requires $P$ if every realmatrix in $Q(A)$ has property $P$, or allows $P$ if some real matrix in $Q(A)$ has property $P$. In the literature, one can find, recently, an increasing interest in problems that arise from the basic question of whether a certain sign pattern matrix requires (or allows) a certain property (see, for instance, [2, 6-9]).

For an $n \times n$ matrix $A$, $A[\alpha|\beta]$ denotes the submatrix of $A$ containing rows numbered by $\alpha$ and columns numbered by $\beta$ where $\alpha, \beta \subseteq \{1, 2, \cdots, n\}$. When $\alpha = \beta$, $A[\alpha|\alpha]$ is abbreviated as $A[\alpha]$. Therefore, a real $n \times n$ matrix $A$ is a $P_0$-matrix if $\det A[\alpha] > 0$, for all $\alpha \subseteq \{1, 2, \cdots, n\}$.

In [8], the authors characterized the sign pattern matrices that admit $N$-matrices, $P$-matrices and $M$-matrices. In [9], the authors characterized the sign pattern matrices that admit $P_0$-matrix. Recall that an $n \times n$ real matrix $A$ is called an $N$-matrix if all of its principal minors are negative while $A$ is said to be a $P$-matrix if all of its principal minors are positive. If $Z_n$ is the set of all square real matrices of order $n$ whose off-diagonal entries are non-positive, then a matrix $A \in Z_n$ is an $M$-matrix if and only if $A$ is a $P$-matrix. A nonsingular matrix $A$ is said to be an inverse $M$-matrix if $A^{-1}$ is an $M$-matrix. See [1, 5] for more information on these classes of matrices. A $P_0$-matrix is a real square matrix all of whose principle minors are nonnegative.

In this paper, we will consider the class of $P_0$-matrix. Our main aim is to determine which sign pattern matrices are admissible for this class of real matrices.

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## II. Notation and preliminaries

A sign pattern matrix $A = (a_{ij})$ is said to be combinatorially symmetric is $a_{ij} \neq 0$ if and only if $a_{ji} \neq 0$ for all choices of $i, j$, $i \neq j$, and not combinatorially symmetric otherwise.

A permutation pattern is a square sign pattern matrix with entries 0 and $+$, where the entry $+$ occurs precisely once in each row and in each column. A permutational similarity of the (square) sign pattern $A$ is a product of the form $P^T AP$, where $P$ is permutation pattern.

Let $A = (a_{ij})$ be an $n \times n$ sign pattern matrix. The digraph of $A$, denoted $D(A)$, is the digraph with vertex set $\{1, \cdots, n\}$, such that there is an arc $(i, j)$ from $i$ to $j$ if and only if $a_{i,j} \neq 0$. Let $E(D(A))$ and $V(D(A))$ denote the arc set and vertex set of the digraph $D(A)$, respectively. By a path $W = x \rightarrow y$ (a path from $x$ to $y$) in digraph $D$, we mean a sequence of vertices $x, v_1, \cdots, v_l, y \in V(D)$ and a sequence of arcs $(x, v_1), (v_1, v_2), \cdots, (v_l, y) \in E(D)$ where the vertices and arcs are distinct.

In this study, we will use directed graphs, but in the case of combinatorially symmetric sign pattern matrices we will treat the graphs as undirected for convenient.

A simple cycle of length $k$ (or a $k$-cycle) in the digraph $D(A)$ is a sequence of arcs of the form $C = (i_1, i_2)$, $(i_2, i_3)$, $\cdots$, $(i_k, i_1)$, where the set $\{i_1, \cdots, i_k\}$ contains no repeated vertices. The simple cycle $C = (i_1, i_2)$, $(i_2, i_3)$, $\cdots$, $(i_k, i_1)$ in $D(A)$ can also be defined as a formal product of the form $C = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}$. The sign (positive or negative) of a simple cycle in a sign pattern $A$ is the actual product of the entries in the cycle, following the obvious rules that multiplication is commutative and associative, and $(+)(+) = +, (+)(-) = -$.

A composite cycle $C$ in $A$ is a product of simple cycles, say $C = C_1 C_2 \cdots C_m$ where the index sets of the $C_i$'s are mutually disjoint. If the length of $C_i$ is $l_i$, then the length of $C$ is $\sum_{i=1}^{m} l_i$ and the sign of $C$ is $(-1)^{\sum_{i=1}^{m} (l_i - 1)}$. From now on, the term cycle always refers to a composite cycle (which as a special case could be a simple cycle). The weight of a cycle $C$ of length $k$ in $D(A)$, is defined as the product of the entries in $C$. The signed weight of a cycle $C$ in $D(A)$, denoted by $w(C)$, is defined as the product of its sign and its weight.

## III. Main results

**Definition 3.1.** *We say that a sign pattern matrix $A = (a_{ij})$ has the 2-cycle property if $a_{ij} a_{ji} < 0$, whenever $(i, j, (j, i) \in E(G)$, where $G$ is the graph describing $P$.*

In this section we first give another proof of the theorem 3.2 of [9], which we think is simpler.

**Theorem 3.2.** *Let $A = (a_{ij})$ be an $n \times n$ sign pattern matrix, with $a_{ii} = 0$ for all $i$, whose associated graph $D(A)$*

*is an undirected cycle. There exists a $P_0$-matrix in $Q(A)$ if and only if $A$ has the 2-cycle property and the entry according to the loop is positive.*

**Proof.** The necessary condition is obvious. Conversely, assume that $A$ has the 2-cycle property. We can assume, by permutation similarity, that $A$ is of the following form

$$A = \begin{pmatrix} 0 & a_{12} & 0 & \cdots & 0 & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & 0 & 0 \\ 0 & a_{23} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1,n} \\ a_{n,1} & 0 & 0 & \cdots & a_{n,n-1} & 0 \end{pmatrix} \quad (1)$$

Consider $B \in \mathcal{Q}(A)$ of the form

$$B = \begin{pmatrix} 0 & b_{12} & 0 & \cdots & 0 & b_{1n} \\ b_{21} & 0 & b_{23} & \cdots & 0 & 0 \\ 0 & b_{23} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b_{n-1,n} \\ b_{n,1} & 0 & 0 & \cdots & b_{n,n-1} & 0 \end{pmatrix} \quad (2)$$

As a cycle of length $k$ in the digraph $D(B)$ corresponds to a term in the determinant of the $k \times k$ principal submatrix associated with the vertices of the cycle, we only have to consider the cycles in the digraph $D(B)$.

In the digraph $D(B)$, there are $n$ 2-cycles, $C_i = b_{ii+1}b_{i+1i}$ for $i = 1, 2, \cdots, n-1$ and $C_n = b_{n1}b_{1n}$ and 2 $n$-cycles $C^{'} = b_{12}b_{23}\cdots b_{n-1n}b_{n1}$ and $C^{''} = b_{1n}b_{nn-1}\cdots b_{21}$.

Let $\alpha \subset \{1, 2, \cdots, n\}$. Assume that $|b_{ii+1}| = |b_{i+1i}| = |b_{n1}| = |b_{1n}| = 1$ where $i = 1, 2, \cdots, n-1$. Then, we show that $\det B \geq 0$ and $\det B[\alpha] \geq 0$. Since $A$ has the 2-cycle property, we have that $w(C_i) = 1$ for $i = 1, 2, \cdots, n$. First, we show that $\det B \geq 0$. Now we consider the the following two cases.

**Case 1.** When $n$ is even:

If $n = 2$, it is clear. If $n \geq 4$, then there are 2 $n$-cycles $C^{'}$ and $C^{''}$ and 2 composite cycles $C^* = C_1C_3\cdots C_{n-1}$ and $C^{**} = C_2C_4\cdots C_n$ of lengths $n$. Since $w(C_i) = 1$, we have that $w(C^{'})w(C^{''}) = 1$ and $w(C^*) = w(C^{**}) = 1$. It follows that $w(C^{'}) + w(C^{''}) \geq -2$. Therefore, $\det B = w(C^*) + w(C^{**}) + w(C^{'}) + w(C^{''}) \geq 0$.

**Case 2.** When $n$ is odd:

If $n$ is odd, then there are only two $n$-cycles $C^{'}$ and $C^{''}$. Since the $w(C^{''})w(C^{''}) = -1$ and $|w(C^{''})| = |w(C^{''})| = 1$, we have $\det B = 0$. If $|\alpha|$ is even and there exists no cycle of length whose vertex set equals to $\alpha$, then $\det B[\alpha] = 0$.

Next we show that $\det B[\alpha] \geq 0$. Since there exists no odd cycle in the digraph $D(B)$, if $|\alpha|$ is odd, then $\det B[\alpha] = 0$. If $|\alpha|$ is even and there exists no cycle of length whose vertex set equals to $\alpha$, then $\det B[\alpha] = 0$. If $|\alpha|$ is even and there exists $m_1 \geq 1$ composite cycles of length $|\alpha|$ whose vertex set equals to $\alpha$. Each cycle of these $m_1$ composite cycles is a product 2-cycles. Since $w(C_i) = 1$, $\det B[\alpha] \geq 1$. $\square$

**Theorem 3.3.** *Let $A = (a_{ij})$ be an $n \times n$ sign pattern matrix. The associated graph $D(A)$ is an undirected cycle with only one loop. There exists a $P_0$-matrix in $Q(A)$ if and only if $A$ has the 2-cycle property and the sign of the loop is positive.*

**Proof.** The necessary condition is obvious. Conversely, assume that $A$ has the 2-cycle property. We can assume, by permutation similarity, that $A$ is of the following form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & 0 & 0 \\ 0 & a_{23} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1,n} \\ a_{n,1} & 0 & 0 & \cdots & a_{n,n-1} & 0 \end{pmatrix} \quad (3)$$

Consider $B \in \mathcal{Q}(A)$ of the form

$$B = \begin{pmatrix} b_{11} & b_{12} & 0 & \cdots & 0 & b_{1n} \\ b_{21} & 0 & b_{23} & \cdots & 0 & 0 \\ 0 & b_{23} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b_{n-1,n} \\ b_{n,1} & 0 & 0 & \cdots & b_{n,n-1} & 0 \end{pmatrix} \quad (4)$$

In the digraph $D(B)$, there are $n$ 2-cycles, $C_i = b_{ii+1}b_{i+1i}$ for $i = 1, 2, \cdots, n-1$ and $C_n = b_{n1}b_{1n}$ and 2 $n$-cycles $C^{'} = b_{12}b_{23}\cdots b_{n-1n}b_{n1}$ and $C^{''} = b_{1n}b_{nn-1}\cdots b_{21}$ and a loop $b_{11}$.

Assume that $|b_{ii+1}| = |b_{i+1i}| = |b_{n1}| = |b_{1n}| = b_{11} = 1$ where $i = 1, 2, \cdots, n-1$. Then, we show that $\det B \geq 0$ and $\det B[\alpha] \geq 0$. Since $A$ has the 2-cycle property, we have that $w(C_i) = 1$ where $i = 1, 2, \cdots, n$.

First, we show that $\det B \geq 0$. Now we consider the following two cases.

**Case 1.** When $n$ is even:

If $n = 2$, it is clear. If $n \geq 4$, then there are 2 $n$-cycles $C^{'}$, $C^{'}$ and 2 composite $C^* = C_1C_3\cdots C_{n-1}$ and $C^{**} = C_2C_4\cdots C_n$ of lengths $n$. Since $w(C_i) = 1$, we have that $w(C^*) = w(C^{**}) = 1$ and $w(C^{'})w(C^{''}) = 1$. It follows that $w(C^{'}) + w(C^{''} \geq -2$. Therefore, $\det B = w(C^{'}) + w(C^{''} + w(C^*) + w(C^{**}) \geq 0$.

**Case 2.** When $n$ is odd:

In this case, there are 2 $n$-cycles $C^{'}$ and $C^{''}$ and one composite $C^{'''} = b_{11}C_2C_4\cdots C_{n-1}$. Since $w(C_i) = 1$, we have $w(C^{'''}) = 1$ and $w(C^{'}) + w(C^{''}) = 0$. Hence, $\det B[\alpha] = w(C^{'''}) + w(C^{'}) + w(C^{''}) = 1$.

Next we show that $\det B[\alpha] \geq 0$. If $|\alpha|$ is even and there exists no cycle of length $|\alpha|$ whose vertex set equals to $\alpha$, then $\det B[\alpha] = 0$. If $|\alpha|$ is even and there exists at least one cycle of length $|\alpha|$ whose vertex set equals to $|\alpha|$, since each even composite cycle of length $\alpha$ is a product of some 2-cycles, we have that $\det B[\alpha] \geq 1$.

If $|\alpha|$ is odd, every cycle of length $|\alpha|$ is a product of the loop $b_{11}$ and 2-cycles. If there exits no cycle of length $|\alpha|$ whose vertex set equals to $\alpha$, then $\det B[\alpha] = 0$. If there exists at least one cycle of length $|\alpha|$ whose vertex set equals to $\alpha$, since each odd composite cycle of length $|\alpha|$ is a product of the loop $b_{11}$ and 2-cycles, we have that $\det B[\alpha] \geq 1$.

**Theorem 3.4.** *Let $A = (a_{ij})$ be an $n\times n$ sign pattern matrix. The associated graph $D(A)$ is an undirected cycle with only two loops whose signs are positive. Denote the vertices of the two loops by $i_0$ and $j_0$. For convenience, assume that $i_0 > j_0$.*

*(1) If $i_0$ and $j_0$ are consecutive (For convenience, we say $n$ and 1 are consecutive), then there exists a $P_0$-matrix in $Q(A)$ if and only if the weights of the 2-cycles whose vertices in $\{1, 2, \cdots, n\} \setminus \{i_0, j_0\}$ are negative and $a_{i_0i_0} = a_{j_0j_0} = +$.*

*(2) If $i_0$ and $j_0$ are not consecutive, then there exists a $P_0$-matrix in $Q(A)$ if and only if $A$ has the 2-cycle property and $a_{i_0i_0} = a_{j_0j_0} = +$.*

**Proof.** Let $C_i = b_{ii+1}b_{i+1i}$ for $i = 1, 2, \cdots, n-1$ and $C_n = b_{n1}b_{1n}$. Let $C^{''} = b_{12}b_{23}\cdots b_{n-1n}b_{n1}$ and $C^{'} = b_{1n}b_{nn-1}\cdots b_{32}b_{21}$. Let $\alpha \subset \{1, 2, \cdots, n\}$.

(1) If $i_0$ and $j_0$ are consecutive.

The necessary condition is obvious. Conversely, assume that the weights of the 2-cycles whose vertices in $\{1, 2, \cdots, n\} \setminus \{i_0, j_0\}$ are negative and $a_{i_0i_0} = a_{j_0j_0} = +$. We can assume, by permutation similarity, that $A$ is of the following form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & 0 & 0 \\ 0 & a_{23} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1,n} \\ a_{n,1} & 0 & 0 & \cdots & a_{n,n-1} & 0 \end{pmatrix} \quad (5)$$

Consider $B \in \mathcal{Q}(A)$ of the form

$$B = \begin{pmatrix} b_{11} & b_{12} & 0 & \cdots & 0 & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & 0 & 0 \\ 0 & b_{23} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b_{n-1,n} \\ b_{n,1} & 0 & 0 & \cdots & b_{n,n-1} & 0 \end{pmatrix} \quad (6)$$

Assume that $|b_{ii+1}| = |b_{i+1i}| = |b_{n1}| = |b_{1n}| = 1$ and $b_{11} = b_{22} = 2$ where $i = 1, 2, \cdots, n-1$. Since the weights of the 2-cycles whose vertices in $\{2, 3, \cdots, n\}$ are negative, we have that $w(C_i) = 1$ for $i = 2, 3, \cdots, n$.

First, we show that $B[\alpha] \geq 0$. All the even cycles whose lengths $|\alpha|$ are products of 2-cycles or products of the two loops and 2-cycles.

If $|\alpha|$ is even and there exists no cycle of length $|\alpha|$ whose vertex set equals to $\alpha$, then $\det B[\alpha] = 0$. Next we consider the case that $|\alpha|$ is even and there exists at least one cycle of length $|\alpha|$ whose vertex set equals to $\alpha$. If $\{1, 2\} \subseteq \alpha$, then $\det B[\alpha] \geq 3$, if $\{1, 2\} \nsubseteq \alpha$ then $\det B[\alpha] \geq 1$.

If $|\alpha|$ is odd, every cycle of length $|\alpha|$ is a product of a loop and some 2-cycles. If there exists no cycle of length $|\alpha|$ whose vertex set equals to $\alpha$, then $\det B[\alpha] = 0$. If there exists at least one cycle of length $|\alpha|$ whose vertex set equals to $\alpha$, then $\det B[\alpha] \geq 1$.

Next, we show that $\det B \geq 0$. Now we consider the the following two cases.

**Case 1.** When $n$ is even

If $n = 2$, it is clear. If $n \geq 4$, then there are 4 cycles $C^{'}$, $C^{''}$, $b_{11}b_{22}C_3C_5\cdots C_{n-1}$, $C_1C_3\cdots C_{n-1}$ and $C_2C_4\cdots C_n$ of lengths n. Since the $w(C^{'})w(C^{''}) = 1$ and $|w(C^{'})| = |w(C^{''})| = 1$, we have that $w(C^{'}) + w(C^{''}) \geq -2$, As $w(b_{11}b_{22}C_3C_5\cdots C_{n-1}) = 4$, $|w(C_1C_3\cdots C_{n-1})| = 1$ and $w(C_2C_4\cdots C_n) = 1$, we have that $\det B = w(C^{'}) + w(C^{''}) + w(b_{11}b_{22}C_3C_5\cdots C_{n-1}) + w(C_1C_3\cdots C_{n-1}) + w(C_2C_4\cdots C_n) \geq 2$.

**Case 2.** When $n$ is odd

There are 4 cycles $C^{'}$, $C^{''}$, $b_{11}C_2C_4\cdots C_{n-1}$ and $b_{22}C_3C_5\cdots C_n$ of lengths n. Since the $w(C^{'})w(C^{''}) = -1$ and $|w(C^{'})| = |w(C^{''})| = 1$, we have that $w(C^{'}) + w(C^{''}) = 0$, As $|w(b_{11}C_2C_4\cdots C_{n-1})| \geq -2$, $w(b_{22}C_3C_5\cdots C_n) = 2$, we have that $\det B = w(C^{'}) + w(C^{''}) + w(b_{11}C_2C_4\cdots C_{n-1}) + w(b_{22}C_3\cdots C_n) \geq 0$.

(1) If $i_0$ and $j_0$ are not consecutive.

The necessary condition is obvious. Conversely, assume that if $A$ has the 2-cycle property and $a_{i_0i_0} = a_{j_0j_0} = +$. We can assume, by permutation similarity, that $A$ is of the following form

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & 0 & 0 \\ 0 & a_{23} & a_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1,n} \\ a_{n,1} & 0 & 0 & \cdots & a_{n,n-1} & 0 \end{pmatrix} \quad (7)$$

Consider $B \in \mathcal{Q}(A)$ of the form

$$B = \begin{pmatrix} b_{11} & b_{12} & 0 & \cdots & 0 & b_{1n} \\ b_{21} & 0 & b_{23} & \cdots & 0 & 0 \\ 0 & b_{23} & b_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & b_{n-1,n} \\ b_{n,1} & 0 & 0 & \cdots & b_{n,n-1} & 0 \end{pmatrix} \quad (8)$$

In the digraph $D(B)$, there are $n$ 2-cycles, $C_i$ for $i = 1, 2, \cdots, n$ and 2 $n$-cycles $C^{'}$ and $C^{''}$ and two loops $b_{11}$ and $b_{33}$.

Assume that $|b_{ii+1}| = |b_{i+1i}| = |b_{n1}| = |b_{1n}| = b_{11} = b_{33} = 1$ where $i = 1, 2, \cdots, n-1$. Since $A$ has the 2-cycle property, we have that $w(C_i) = 1$, where $i = 1, 2, \cdots, n$.

First, we show that $B[\alpha] \geq 0$. All the even cycles whose lengths $|\alpha|$ are products of 2-cycles or products of the two loops and 2-cycles.

If $|\alpha|$ is odd, every cycle of length $|\alpha|$ is a product of a loop and some 2-cycles. If there exists no cycle of length $|\alpha|$ whose vertex set equals to $\alpha$, then $\det B[\alpha] = 0$. If there exists at least one cycle of length $|\alpha|$ whose vertex set equals to $\alpha$, then $\det B[\alpha] \geq 1$.

Next, we show that $\det B \geq 0$. Now we consider the the following two cases.

**Case 1.** When $n$ is even:

If $n = 2$, it is clear. If $n \geq 4$, Let $C_1^{**} = C_1C_3 \cdots C_{n-1}$ and $C_2^{**} = C_2C_4 \cdots C_n$. There are 2 composite cycles $C_1^{**}$ and $C_2^{**}$ of lengths $n$ and 2 $n$-cycles $C^{'}$ and $C^{''}$. Since $w(C_i) = 1$, $w(C^{''})w(C^{''}) = 1$ and $w(C_1^{**}) = w(C_2^{**}) = 1$. It follows that $w(C^{''}) + w(C^{''}) \geq -2$. Hence $\det B[\alpha] = w(C^{''}) + w(C^{''}) + w(C_1^{**}) + w(C_2^{**}) \geq 0$.

**Case 2.** When $n$ is odd:

In this case, there are 2 composite cycles $C^* = b_{11}C_2C_4 \cdots C_{n-1}$ and $C^{**} = b_{33}C_1C_4C_6 \cdots C_{n-1}$ of lengths $n$ and 2 $n$-cycles $C^{'}$ and $C^{''}$. Since $w(C_i) = -1$, we have $w(C^*) = w(C^{**}) = 1$ and $w(C^{'})w(C^{''}) = -1$. It follows that $w(C^{'}) + w(C^{''}) = 0$. Hence, $\det B[\alpha] = w(C^*) + w(C^{**}) + w(C^{'}) + w(C^{''}) = 2$. $\square$

Let $D_{s,n}$ denote the digraph consisting of a Hamilton cycle $D_n = 1 \to 2 \to \cdots \to n-1 \to n \to 1$ and a cycle $C_s = 1 \to 2 \to \cdots \to s-1 \to s \to 1$ of length $s$. $\square$

**Theorem 3.5.** *Let $A = (a_{ij})$ be an $n \times n$ sign pattern matrix, with $a_{ii} = 0$ for all $i$, whose associated graph $D(A)$ is a the digraph $D_{s,n}$. Then the following statements are equivalent:*

1. $The\ signs\ of\ the\ cycles\ D_n\ and\ D_s\ are\ (-1)^{n+1}\ and\ (-1)^{s+1},\ respectively.$
2. $There\ exists\ a\ P_0-matrix\ in\ Q(A).$
3. $All\ matrices\ in\ Q(A)\ are\ P_0-matrices.$

**Proof.** Without loss of generality, we may assume that any matrix $B \in \mathcal{Q}(A)$ is of the form

$$B = \begin{pmatrix} 0 & b_{12} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & b_{23} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & b_{s-1s} & 0 & 0 & \cdots & 0 \\ b_{s1} & 0 & 0 & \cdots & 0 & b_{ss+1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & b_{s+1s+2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & b_{n-1n} \\ b_{n1} & & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (9)$$

We have that $\det B = (-1)^{n+1}a_{12}a_{23} \cdots a_{n-1n}a_{n1}$. Let $\alpha \subset \{1, 2, \cdots, n\}$. Since there are only 2 cycles $C_s$ and $C_n$, for any $\alpha \neq \{1, 2, \cdots, s\}$, $\det B[\alpha] = 0$ and for any $\alpha = \{1, 2, \cdots, s\}$, $\det B[\alpha] = (-1)^{s+1}a_{12}a_{23} \cdots a_{s-1s}a_{s1}$. This implies $A$ is a $P_0$-matrix if and only if the signs of the cycles $D_s$ and $D_n$ are $(-1)^{s+1}$ and $(-1)^{n+1}$, respectively.$\square$

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