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The direct Ansäz method for finding exact multi-wave solutions to the (2+1)-dimensional extension of the Korteweg de-Vries equation

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Abstract—In this paper, the direct Ansäz method is used for constructing the multi-wave solutions to the (2+1)-dimensional extension of the Korteweg de-Vries (shortly EKdV) equation. A new breather type of three-wave solutions including periodic breather type soliton solution, breather type of two-solitary solution are obtained. Some cases with specific values of the involved parameters are plotted for each of the three-wave solutions. Mechanical features of resonance interaction among the multi-wave are discussed. These results enrich the variety of the dynamics of higher-dimensional nonlinear wave field.

Keywords—EKdV equation; Breather; Soliton; Bilinear form; The direct Ansäz method.

I. Introduction

NONLINEAR evolution equations (NLEEs) have an important role in many nonlinear science fields such as fluid dynamics, nonlinear optics, elasticity theory, plasma physics, the propagation of long internal waves and many other fields. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations [1]. Exact soliton solutions may help us to find and explain physical phenomena and experimental results. KdV equation is the earliest soliton equation which was firstly derived by Korteweg and de Vries to model the evolution of shallow water wave in 1895 ([1],[2],[3]]). In order to search for soliton solutions and study interaction of solitons for nonlinear partial differential equations, many effective methods have been developed, such as Inverse scattering transformation[1], Bäcklund transformation [2], Darboux transformation ([3],[4]), Painlevé expansion method [5], Hirota bilinear method [6], Painlevé analysis method [7], similarity reductions method ([8],[9]), homogeneous balance method [10], homotopy perturbation method [11] variational method [12], Adomian method [13], F-expansion method [14], Exp-function method [15], Extended Homoclinic test function ([16],[17]),The mapping approach [18],The improved mapping approach [19],Conditional Similarity Reduction Method [20], Projective equation method ([21],[22]) and so on.

In this paper, we use the direct Ansäz method to obtain the exact multi-wave solutions of the (2+1)-dimensional EKdV equation. The (2+1)-dimensional EKdV[23] can be shown in the from of

$$u_t + 3uu_y + u_{xxy} + 3u_x \int_{-\infty}^{x} u_y dx = 0. \tag{1}$$

In Ref.[23], a type of bell-shape soliton and exact two-soliton solution had been obtained for EKdV equation. Equation (1) now becomes an alternative form with $u = w_x$:

$$w_{xt} + 3w_x w_{xy} + w_{xxxy} + 3w_{xx} w_y = 0. \tag{2}$$

Integrating Eq. (2) ones with respect to x, we obtain

$$w_t + 3w_x w_y + w_{xxy} = \lambda(y, t). \tag{3}$$

where $\lambda(y,t)$ is an arbitrary function. For convenience, Eq. (3) is called (2+1)-dimensional potential EKdV (shortly PEKdV) equation with $\lambda(y,t) = 0$. In Ref.[24], Shen discussed Lie symmetries, Lie algebra of symmetry vector fields and similarity reductions, and found the PEKdV equation is not Painlevé integrable by means of the WTC-Painlevé analysis method. In Ref.[17], kinky periodic solitary-wave solutions, periodic soliton solutions, and cross kink-wave solutions of the (2+1)-dimensional potential EKdV equation were obtained.

II. The direct Ansäz method

We consider general form of higher dimensional nonlinear evolution equation

$$F(u, u_t, u_x, u_y, u_{xx}, u_{yy}, \cdots) = 0, \tag{4}$$

where $u = u(x, y, t)$ and F is a polynomial about u and its derivatives.

By Painlevé analysis, a transformation is made

$$u = T(f), \tag{5}$$

where f is a new unknown function. Then, the NLEE (4) is reduced to Hirota's bilinear equation

$$G(D_x, D_t, D_y; f, f) = 0, \tag{6}$$

where the D-operator [6] is defined by

$$D_x^m D_t^n f(x,t) \cdot g(x,t) =$$
$$(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})^m (\frac{\partial}{\partial t} - \frac{\partial}{\partial t'})^n [f(x,t) g(x', t')]|_{x'=x, t'=t},$$

In this section, we shall seek the multi-wave solution for a given partial differential equation in the following form:

$$f(x, y, t) = e^{-\xi_1} + \delta_1 \cos \xi_2 + \delta_2 \cosh \xi_3 + \delta_3 e^{\xi_1}, \tag{7}$$

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where $\xi_i = a_i x + b_i y + c_i t$ a_i, b_i, c_i, δ_i $(i = 1, 2, 3)$ are some constants to be determined later from the solution of (6).

Substituting Eq.(7) into Eq.(6) and setting the coefficients of the same power of $e^{j(\xi_1)}, \cos\xi_2, \sin\xi_2, \cosh\xi_3, \sinh\xi_3)$ $(j = -1, 1)$ equal to zero, we obtain algebraic equations. Solving the set of algebraic equations, we can find solutions a_i, b_i, c_i and $\delta_i (i = 1, 2, 3)$. Substituting solutions a_i, b_i, c_i and $\delta_i (i = 1, 2, 3)$ into Eq.(7) and Eq.(5) we can obtain exact multi-wave solutions of Eq.(4).

III. Application to the the (2+1)-dimensional extension of the Korteweg de-Vries equation

By the dependent variable transformation

$$u = 2(lnf)_{xx}, \tag{8}$$

where the function $f(x, y, t)$ is an unknown real function.

Eq. (1) is transformed into the bilinear differential equation

$$G(D_x, D_y, D_t)f \cdot f = D_x(D_t + D_x^2 D_y)f \cdot f = 0, \tag{9}$$

By using the simplest direct Ansäz method, we may choose the solution of (9) in the form

$$f(x, y, t) = e^{\xi_1} + \delta_1 \cos\xi_2 + \delta_2 \cosh\xi_3 + \delta_3 e^{-\xi_1}, \tag{10}$$

where $\xi_1 = a_1 x + b_1 y + c_1 t, \xi_2 = a_2 x + b_2 y + c_2 t, \xi_3 = a_3 x + b_3 y + c_3 t$ and a_i, b_i, c_i, δ_i $(i = 1, 2, 3)$ are some constants to be determined later from the solution of (9).

Substituting Eq.(10) into Eq.(9), and setting the coefficients of the same power of $e^{\pm\xi_1}, \cos\xi_2, \sin\xi_2, \cosh\xi_3, \sinh\xi_3$ equal to zero, we obtain algebraic equations about a_i, b_i, c_i, δ_i $(i = 1, 2, 3)$. Solving the set of algebraic equations, we can find solution

Case 1.

$$a_1 = 0, a_2 = ia_3, b_2 = -ib_3, c_1 = -b_1 {a_3}^2,$$
$$c_2 = ib_3 {a_3}^2, c_3 = -b_3 {a_3}^2, \delta_1 = \delta_2$$

under a transformation $a_3 \to a_3 i, b_3 \to b_3 i$ in the above relations, (10) can be represented by the following form

$$f(x, y, t) = e^{b_1\xi} + 2\delta_2 \cos(a_3 x)\cos(b_3\xi) + \delta_3 e^{-b_1\xi} \tag{11}$$

Where $\xi = y + {a_3}^2 t$.

Substituting Equation (11) into (8), then we obtain the x-periodic breather type soliton solution for EKdV equation as follows

$$u\,(x, y, t) = -\frac{2\delta_2 {a_3}^2 \cos(b_3\xi)\left(\sqrt{\delta_3}\cos(a_3 x)\cosh(b_1\xi+\theta_0)+\delta_2\cos(b_3\xi)\right)}{\left(\sqrt{\delta_3}\cosh(b_1\xi+\theta_0)+\delta_2\cos(a_3 x)\cos(b_3\xi)\right)^2}, \tag{12}$$

Where $\xi = y + {a_3}^2 t, \theta_0 = \ln\sqrt{\delta_3}$ and $\delta_2 \neq 0, \delta_3 > 0$.

Obviously, the solution (12) is a type of periodic breather solitons which is a periodic standing wave in the propagating direction x with period $\frac{2\pi}{a_3}$, and at the same time is also a breather solitary wave with $\xi = y + {a_3}^2 t$ and exponentially decay in y, so it is called the periodic breather-type soliton. Note that the denominator of expression (12) is greater than zero when ξ and x take arbitrarily values with $\sqrt{\delta_3} > \delta_2$. Therefore we see that (12) has no poles and should be well behaved everywhere. So the solution is a nonsingular periodic breather soliton solution. The periodic and breather behavior of the solution (12) are shown in Figs. 1 (a)-(b).

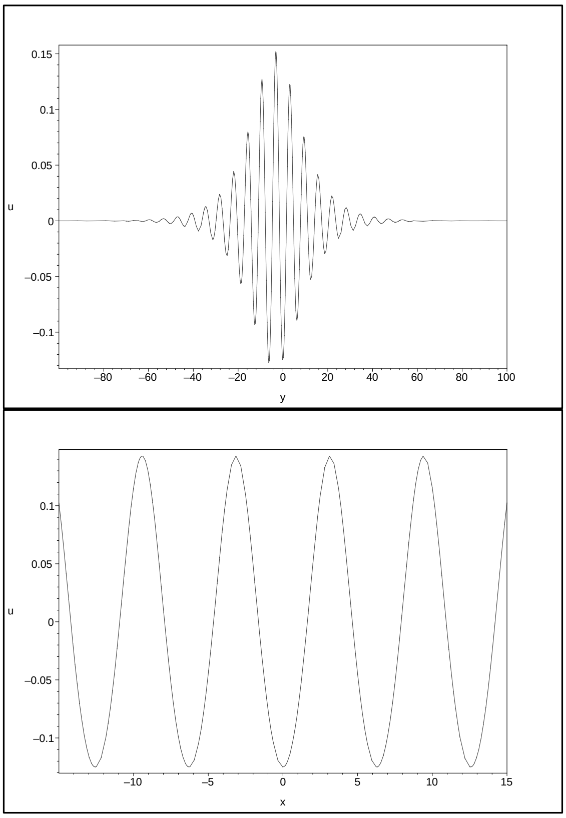

Fig. 1. (a)The breather behavior of solution (12) in y-direction. (b) The periodic behavior of solution (12) in x-direction.

Case 2.

$$a_1 = a_2 = c_3 = b_3 = 0, c_1 = -{a_3}^2 b_1, b_2 = -\frac{c_2}{{a_3}^2}. \tag{13}$$

Substituting Equation (10) into (8) with (13), we get the breather-type two-soliton solution of Equation (1) as follows:

$$u(x, y, t) = \frac{2\,\delta_2 {a_3}^2(\delta_1\cosh(a_3 x)\cos(\frac{c_2}{{a_3}^2}\xi)+2\sqrt{\delta_3}\cosh(a_3 x)\cosh(b_1\xi-\theta_0)+\delta_2)}{(\delta_1\cos(\frac{c_2}{{a_3}^2}\xi)+\delta_2\cosh(a_3 x)+2\sqrt{\delta_3}\cosh(b_1\xi-\theta_0))^2}, \tag{14}$$

Where $\xi = y - {a_3}^2 t, \theta_0 = \ln\sqrt{\delta_3}$, $\delta_3 > 0$ and $\delta_2 \neq 0$.

The expression (14) is the breather-type two-soliton solution of EKdV equation which is a breather solitary wave on the y-axis, and meanwhile is a soliton on the x-axis. In fact, the solution (14) reflects the interaction between two solitons. There are three types of resonance interactions in two soliton solutions, namely full resonance, partial resonance and non-resonance interactions. The value of δ_3 plays an important role in determining the type of resonance interactions occurrence. Resonance only occurs when the value of δ_3 approaches 0. Because when the value of δ_3 approaches 0, then $ln\sqrt{\delta_3}$ approaches ∞. If $\delta_3 = 0$ or $\delta_3 \to 0$, then the partial resonance and full resonance interactions will occur. For the partial resonance interaction, the length of the resonant breather soliton increases with $\delta_3 \to 0$. If the value of δ_3 is not equal to zero or approaches 0, then the resonance interaction will be not exist. Fig. 2 shows that the partial resonance interaction between two soliton solutions. It is is not completely elastic. That is, when two initial solitons come into interaction it will produce some particularly high and steep wave humps in the vicinity of the crossing point and later break up again two solitons which are actually the original soliton respectively(see Figs. 2 (a)). These particularly high and steep wave humps represent the localized oscillation, namely, they express a breather soliton solution. It is called the the resonance breather-soliton solution. The resonance breather-soliton solution is converted into the line-soliton solution accordingly as the value of $|\delta_1|$ becomes small.

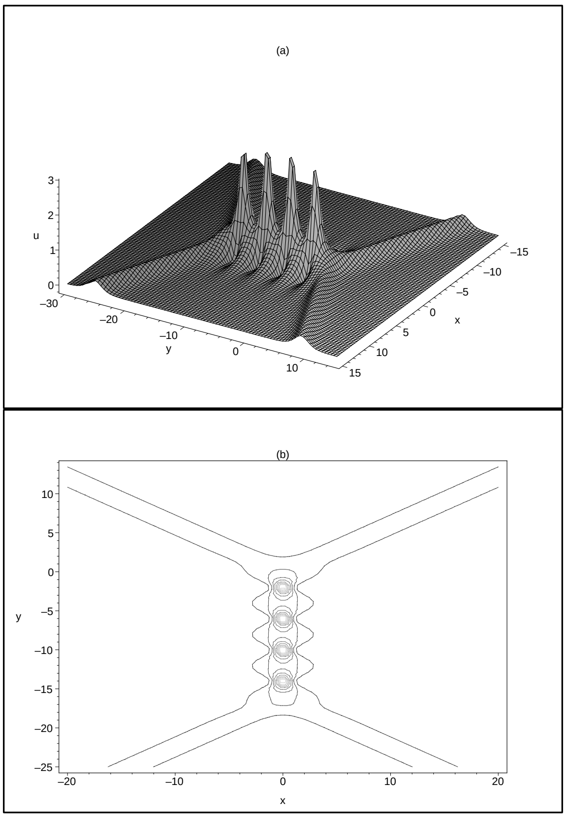

Fig. 2. The partial resonance interaction of two-soliton solution (14) and contourplot map of u in (x,y)-plane.

IV. Conclusion

In this letter, we have applied the direct Ansäz method to obtain the exact multi-wave solutions of the (2+1)-dimensional EKdV equation. The obtained solutions have very concise and explicit forms. It is also shown that the simplest direct Ansäz method is a direct, concise and effective method. The properties of the obtained solutions are discussed and shown in Figures 1 and 2. These obtained results enrich the variety of the dynamics of higher-dimensional nonlinear wave field. The direct Ansäz method can also be applied to solve other types of higher dimensional integrable and non-integrable systems.

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