

# Development Partitioning Intervalwise Block Method for Solving Ordinary Differential Equations

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**Abstract**—Solving Ordinary Differential Equations (ODEs) by using Partitioning Block Intervalwise (PBI) technique is our aim in this paper. The PBI technique is based on Block Adams Method and Backward Differentiation Formula (BDF). Block Adams Method only use the simple iteration for solving while BDF requires Newton-like iteration involving Jacobian matrix of ODEs which consumes a considerable amount of computational effort. Therefore, PBI is developed in order to reduce the cost of iteration within acceptable maximum error

**Keywords**—Adam Block Method, BDF, Ordinary Differential Equations, Partitioning Block Intervalwise

## I. INTRODUCTION

LET's consider the linear system of first order ODEs as follows:

$$y' = Ay + \phi(x), \quad y(a) = \eta, \quad a \leq x \leq b \quad (1)$$

where

$$y^T = (y_1, y_2, \dots, y_s), \quad \eta^T = (\eta_1, \eta_2, \dots, \eta_s)$$

Initially, the partitioning technique was developed by Enright and Kamel [1] and then had been extended by Watkins and Hansonsmith [5] which developed automatically partitioning into stiff systems and non-stiff systems. Other researchers that studied partitioning technique were Hall and Suleiman [2] and Suleiman and Baok [4]. Recently, Khairil et.al [3] has developed intervalwise partitioning technique using 2-point block method formula for solving ODEs. The development of partitioning technique in this paper is based on Khairil et.al [3] paper but we use 3-point block method formula.

## II. 3 POINT BLOCK BDF AND 3 POINT ADAMS BLOCK METHOD

In 3-point block method, 3 new values  $y_{n+1}$ ,  $y_{n+2}$  and  $y_{n+3}$  are generated simultaneously at each step using previous block. We use point  $x_{n-3}$ ,  $x_{n-2}$ ,  $x_{n-1}$ ,  $x_n$ ,  $x_{n+1}$ ,  $x_{n+2}$  and  $x_{n+3}$  and substitute into Lagrange polynomial and be as follow:

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$$\begin{aligned}
 P(x) = & \frac{(X - X_{n+2}) \dots (X - X_{n-3})}{(X_{n+3} - X_{n+2}) \dots (X_{n+3} - X_{n-3})} Y_{n+3} \\
 & + \frac{(X - X_{n+3}) \dots (X - X_{n-3})}{(X_{n+2} - X_{n+3}) \dots (X_{n+2} - X_{n-3})} Y_{n+2} \\
 & + \frac{(X - X_{n+3}) \dots (X - X_{n-3})}{(X_{n+1} - X_{n+3}) \dots (X_{n+1} - X_{n-3})} Y_{n+1} \\
 & + \frac{(X - X_{n+3}) \dots (X - X_{n-3})}{(X_n - X_{n+3}) \dots (X_n - X_{n-3})} Y_n \\
 & + \frac{(X - X_{n+3}) \dots (X - X_{n-3})}{(X_{n-1} - X_{n+3}) \dots (X_{n-1} - X_{n-3})} Y_{n-1} \\
 & + \frac{(X - X_{n+3}) \dots (X - X_{n-3})}{(X_{n-2} - X_{n+3}) \dots (X_{n-2} - X_{n-3})} Y_{n-2} \\
 & + \frac{(X - X_{n+3}) \dots (X - X_{n-2})}{(X_{n-3} - X_{n+3}) \dots (X_{n-3} - X_{n-2})} Y_{n-3} \quad (2)
 \end{aligned}$$

Eventually, 3-point block BDF with step size ratio  $r = 1$ ,  $r = \frac{1}{2}$  and  $r = 2$  are using for solving stiff system as

follow:  
 $r = 1$

$$\begin{aligned}
 Y_{n+1} = & -\frac{1}{35} Y_{n-3} + \frac{8}{35} Y_{n-2} - \frac{6}{7} Y_{n-1} + \frac{16}{7} Y_n \\
 & - \frac{24}{35} Y_{n+2} + \frac{2}{35} Y_{n+3} + \frac{12}{7} hf_{n+1} \\
 Y_{n+2} = & \frac{2}{77} Y_{n-3} - \frac{15}{77} Y_{n-2} + \frac{50}{77} Y_{n-1} - \frac{100}{77} Y_n \\
 & + \frac{150}{77} Y_{n+1} - \frac{10}{77} Y_{n+3} + \frac{60}{77} hf_{n+2} \\
 Y_{n+3} = & -\frac{10}{147} Y_{n-3} + \frac{24}{49} Y_{n-2} - \frac{75}{49} Y_{n-1} + \frac{400}{147} Y_n \\
 & - \frac{150}{49} Y_{n+1} + \frac{120}{49} Y_{n+2} + \frac{20}{49} hf_{n+3} \quad (3)
 \end{aligned}$$

$$r = \frac{1}{2}$$

$$\begin{aligned}
 Y_{n+1} &= -\frac{4}{21}Y_{n-3} + \frac{75}{64}Y_{n-2} - \frac{20}{7}Y_{n-1} + \frac{25}{8}Y_n \\
 &\quad - \frac{15}{56}Y_{n+2} + \frac{25}{1344}Y_{n+3} + \frac{15}{16}hf_{n+1} \\
 Y_{n+2} &= \frac{320}{957}Y_{n-3} - \frac{1225}{638}Y_{n-2} + \frac{1344}{319}Y_{n-1} - \frac{1225}{319}Y_n \\
 &\quad + \frac{735}{319}Y_{n+1} - \frac{175}{1914}Y_{n+3} + \frac{210}{319}hf_{n+2} \\
 Y_{n+3} &= -\frac{3584}{3265}Y_{n-3} + \frac{3969}{653}Y_{n-2} - \frac{41472}{3265}Y_{n-1} + \frac{7056}{653}Y_n \\
 &\quad - \frac{15876}{3265}Y_{n+1} + \frac{9072}{3265}Y_{n+2} + \frac{252}{653}hf_{n+3}
 \end{aligned} \tag{4}$$

$$r = 2$$

$$\begin{aligned}
 Y_{n+1} &= -\frac{25}{3552}Y_{n-3} + \frac{21}{296}Y_{n-2} - \frac{245}{592}Y_{n-1} + \frac{1225}{296}Y_n \\
 &\quad - \frac{3675}{1184}Y_{n+2} + \frac{35}{111}Y_{n+3} + \frac{210}{37}hf_{n+1} \\
 Y_{n+2} &= \frac{1}{525}Y_{n-3} - \frac{16}{875}Y_{n-2} + \frac{12}{125}Y_{n-1} - \frac{16}{25}Y_n \\
 &\quad + \frac{1536}{875}Y_{n+1} - \frac{512}{2625}Y_{n+3} + \frac{24}{25}hf_{n+2} \\
 Y_{n+3} &= -\frac{175}{46112}Y_{n-3} + \frac{405}{11528}Y_{n-2} - \frac{3969}{23056}Y_{n-1} + \frac{11025}{11528}Y_n \\
 &\quad - \frac{2835}{1441}Y_{n+1} + \frac{99225}{46112}Y_{n+2} + \frac{630}{1441}hf_{n+3}
 \end{aligned} \tag{5}$$

Meanwhile, for solving non stiff system, Zanariah [6] has derived the 3-point Adams Block Method Formula with step size ratio  $r = 1$ ,  $r = \frac{1}{2}$  and  $r = 2$  as follow:

$$r = 1$$

$$\begin{aligned}
 Y_{n+1} &= Y_n + \frac{h}{60480}(271f_{n+3} - 2760f_{n+2} + 30819f_{n+1} \\
 &\quad + 37504f_n - 6771f_{n-1} + 1608f_{n-2} - 191f_{n-3}) \\
 Y_{n+2} &= Y_n + \frac{h}{3780}(-37f_{n+3} + 1398f_{n+2} + 4863f_{n+1} \\
 &\quad + 1328f_n + 33f_{n-1} - 30f_{n-2} + 5f_{n-3}) \\
 Y_{n+3} &= Y_n + \frac{h}{2240}(685f_{n+3} + 3240f_{n+2} + 1161f_{n+1} \\
 &\quad + 2176f_n - 729f_{n-1} + 216f_{n-2} - 29f_{n-3})
 \end{aligned} \tag{6}$$

$$r = \frac{1}{2}$$

$$\begin{aligned}
 Y_{n+1} &= Y_n + \frac{h}{317520}(631f_{n+3} - 7794f_{n+2} + 133560f_{n+1} \\
 &\quad + 313026f_n - 175680f_{n-1} + 63441f_{n-2} - 9664f_{n-3}) \\
 Y_{n+2} &= Y_n + \frac{h}{39690}(-341f_{n+3} + 14274f_{n+2} + 52794f_{n+1} \\
 &\quad + 6594f_n + 11520f_{n-1} - 6741f_{n-2} + 1280f_{n-3}) \\
 Y_{n+3} &= Y_n + \frac{h}{11760}(3469f_{n+3} + 18090f_{n+2} + 1512f_{n+1} \\
 &\quad + 30534f_n - 29376f_{n-1} + 13419f_{n-2} - 2368f_{n-3})
 \end{aligned} \tag{7}$$

$$r = 2$$

$$\begin{aligned}
 Y_{n+1} &= Y_n + \frac{h}{2540160}(24160f_{n+3} - 200277f_{n+2} + 1527840f_{n+1} \\
 &\quad + 1229718f_n - 48132f_{n-1} + 7578f_{n-2} - 727f_{n-3}) \\
 Y_{n+2} &= Y_n + \frac{h}{158760}(-1600f_{n+3} + 59031f_{n+2} + 203328f_{n+1} \\
 &\quad + 57246f_n - 504f_{n-1} + 18f_{n-2} + f_{n-3}) \\
 Y_{n+3} &= Y_n + \frac{h}{94080}(30112f_{n+3} + 127197f_{n+2} + 73440f_{n+1} \\
 &\quad + 54642f_n - 3780f_{n-1} + 702f_{n-2} - 73f_{n-3})
 \end{aligned} \tag{8}$$

### III. IMPLEMENTATION OF PBI

In Partitioning Intervalwise Block technique, primarily the whole system of ODEs (1) was treated as non stiff system of ODEs and solving ODEs by using Adams Block Method. Once it indicates instability due to step failure, there may be presence of stiffness. Therefore, the system will be tested for stiffness by calculating the trace of Jacobian,  $\left(\frac{\partial f}{\partial y}\right)$ . If the

trace is negative, the whole system is switched to stiff system and solve by using BDF method. If the trace is positive, the whole system will be solved by using Adams Block Method with the half step size. Khairil et.al [3] has described that the error algorithm is control as follow:-

- i. If the error control is less than tolerance limit, the step size  $h$  is doubled to gain computation speed.
- ii. In case of step failure, the step size  $h$  is halved and the step is repeated.

IV. NUMERICAL RESULTS

The PBI technique will be tested with some test problems in order to validate the efficiency.

Problem 1:

$$y_1' = -2y_1 + y_2 + 2 \sin x \quad y_1(0) = 2 \quad 0 \leq x \leq 5$$

$$y_2' = 998y_1 - 999y_2 + 999(\cos - \sin x) \quad y_2(0) = 3$$

Solution:  $y_1(x) = 2e^{-x} + \sin x$   
 $y_2(x) = 2e^{-x} + \cos x$

Problem 2:

$$y_1' = -20y_1 - 0.25y_2 - 19.75y_3 \quad y_1(0) = 1 \quad 0 \leq x \leq 5$$

$$y_2' = 20y_1 - 20.25y_2 + 0.25y_3 \quad y_2(0) = 0$$

$$y_3' = 20y_1 - 19.75y_2 - 0.25y_3 \quad y_3(0) = -1$$

Solution:  $y_1(x) = 0.5(e^{-0.5x} + e^{-20x}(\cos 20x + \sin 20x))$   
 $y_2(x) = 0.5(e^{-0.5x} - e^{-20x}(\cos 20x - \sin 20x))$   
 $y_3(x) = -0.5(e^{-0.5x} + e^{-20x}(\cos 20x - \sin 20x))$

Table I and Table II show the numerical results for tested problems.

The notation in the table is defined as follow:

TOL	Tolerance
MAXE	Maximum Error
TIME	Time taken to execute the problem in seconds
IFST	Total number of failure step
IST	Total number of step accepted
TS	Total number of steps taken
Ode15s	BDFs in Matlab
PBI	Partitioning Block Intervalwise

TABLE I  
NUMERICAL RESULT FOR PROBLEM 1

TOL	METHO D	IFS T	IST	TS	MAXE	TIME
10 <sup>-3</sup>	Ode15s	1	28	29	2.4e-03	0.7938
	PBI	1	21	22	1.25e-04	0.0035
10 <sup>-4</sup>	Ode15s	0	38	38	2.41e-04	2.5669
	PBI	2	28	30	3.16e-05	0.0038
10 <sup>-5</sup>	Ode15s	1	54	55	5.30e-05	52.4463
	PBI	3	25	28	3.78e-07	0.0036

TABLE II  
NUMERICAL RESULT FOR PROBLEM 2

TOL	METHO D	IFS T	IST	TS	MAXE	TIME
10 <sup>-3</sup>	Ode15s	0	64	64	0.9011	1.0482
	PBI	1	37	38	8.49e-03	0.0040
10 <sup>-4</sup>	Ode15s	0	89	89	0.9004	2.5669
	PBI	1	41	42	3.88e-03	0.0040
10 <sup>-5</sup>	Ode15s	0	122	122	0.9004	444.5928
	PBI	3	25	28	1.71e-06	0.0044

V. CONCLUSION

The numerical results from Table I and Table II show that partitioning technique gives better accuracy and execution time compare Ode15s. As conclusion, the PBI is more efficient and applicable to solve ODEs.

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