

# The ratios between the spectral norm, the numerical radius and the spectral radius

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*Abstract*—Recently, Uhlig [Numer. Algorithms, 52(3):335-353, 2009] proposed open questions about the ratios between the spectral norm, the numerical radius and the spectral radius of a square matrix. In this note, we provide some observations to answer these questions.

*Keywords*—Spectral norm, Numerical radius, Spectral radius, Ratios

## I. INTRODUCTION

THE numerical radius  $w(A)$  of an  $n \times n$  matrix  $A$  is the real number

$$w(A) = \max_{z \in F(A)} |z|,$$

where  $F(A)$  denotes the field of values (or numerical range) of  $A$ , defined by

$$F(A) = \{x^*Ax : x \in \mathbb{C}^n, \|x\|_2 = 1\}.$$

The spectral radius  $\rho(A)$  of  $A$  is the real number

$$\rho(A) = \max_{z \in \sigma(A)} |z|,$$

where  $\sigma(A)$  denotes the spectrum of  $A$ . The spectral norm of  $A$  is defined by

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2.$$

In this note, we consider the ratios

$$s(A) = \|A\|_2/w(A)$$

and

$$\tau(A) = w(A)/\rho(A).$$

It is well known that

$$0 \leq \rho(A) \leq w(A) \leq \|A\|_2 \leq 2w(A).$$

Thus,

$$1 \leq s(A) \leq 2$$

and

$$1 \leq \tau(A) \leq \infty.$$

Here we employ the convention that  $\tau(A) = \infty$  for  $\rho(A) = 0$ . Obviously,  $s(zA) = s(A)$  and  $\tau(zA) = \tau(A)$  for all  $z \neq 0$ . It follows from  $\rho(A^m) = [\rho(A)]^m$  and  $w(A^m) \leq [w(A)]^m$  that  $\tau(A^m) \leq [\tau(A)]^m$ .

Recently, Uhlig [13] proposed the questions: What do the ratios  $s(A)$  and  $\tau(A)$  indicate about the matrix  $A$ ? What can one conclude about  $A$  when these ratios are large or very different, or when they are nearly equal? In this note, we provide some observations to answer these questions.

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## II. THE RATIOS BETWEEN THE SPECTRAL NORM, THE NUMERICAL RADIUS AND THE SPECTRAL RADIUS

A. The extreme cases  $\tau(A) = 1$ ,  $s(A) = 1$  and  $s(A) = 2$

In this subsection, we review the existing results for the extreme cases  $\tau(A) = 1$ ,  $s(A) = 1$  and  $s(A) = 2$ , respectively. We focus on the relation between  $s(A)$  and  $\tau(A)$ .

A matrix  $A$  is said to be *spectral* if  $w(A) = \rho(A)$ , i.e.,  $\tau(A) = 1$ . The spectral matrices have been investigated by several researchers. We have the following results (see [5] and [9, p.60]).

**Proposition 1.** Let  $A \in \mathbb{C}^{n \times n}$  such that  $\tau(A) = 1$ .

- If  $n \leq 2$ , then  $s(A) = 1$ , and  $A$  is a normal matrix.
- If  $n > 2$ , then  $A$  is unitarily similar to a triangular matrix of the form

$$\begin{bmatrix} \Lambda_k & 0 \\ 0 & B \end{bmatrix}, \quad (1)$$

where  $1 \leq k \leq n$ ,

$$\Lambda_k = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix},$$

$$B = \begin{bmatrix} \lambda_{k+1} & * & * \\ & \ddots & * \\ & & \lambda_n \end{bmatrix},$$

and

$$\rho(B) < \rho(A) = |\lambda_1| = \dots = |\lambda_k|,$$

$$w(B) \leq \rho(A).$$

Furthermore, if  $\rho(A) < \|B\|_2$ ,

$$1 < s(A) \leq 2;$$

otherwise,  $s(A) = 1$ .

A matrix  $A$  is said to be *radial* if  $w(A) = \|A\|_2$ , i.e.,  $s(A) = 1$ . We have the following results (see [9, p.45]).

**Proposition 2.** Let  $A \in \mathbb{C}^{n \times n}$  such that  $s(A) = 1$ . Then  $\tau(A) = 1$ .

- If  $n \leq 2$ , then  $A$  is a normal matrix.
- If  $n > 2$ , then  $A$  is unitarily similar to a block diagonal matrix of the form (1) such that  $\rho(B) < \rho(A)$  and  $\|B\|_2 \leq \rho(A)$ .

*Remark 3.* Note that  $s(A) \approx 1$  does not imply that  $\tau(A) \approx 1$ . See Example 4.

**Example 4** (A scaled Jordan block). Let

$$J_n^\alpha(\lambda) = \begin{bmatrix} \lambda & \alpha & & \\ & \lambda & \ddots & \\ & & \ddots & \alpha \\ & & & \lambda \end{bmatrix}_{n \times n} = \lambda I + N \quad (2)$$

be a matrix of order  $n > 1$ . Then  $F(J_n^\alpha(\lambda))$  is a disk centered at  $\lambda$  with radius  $|\alpha| \cos \frac{\pi}{n+1}$  (see [10, Theorem 2.1]). We have  $\rho(J_n^\alpha(\lambda)) = |\lambda|$ ,  $w(N) = |\alpha| \cos \frac{\pi}{n+1}$ ,  $w(J_n^\alpha(\lambda)) = |\lambda| + |\alpha| \cos \frac{\pi}{n+1}$  and  $\|J_n^\alpha(\lambda)\|_2 \leq |\lambda| + |\alpha|$ . Then

$$\tau(J_n^\alpha(\lambda)) = 1 + \frac{|\alpha|}{|\lambda|} \cos \frac{\pi}{n+1},$$

$$s(J_n^\alpha(\lambda)) \leq \frac{1 + |\alpha/\lambda|}{1 + |\alpha/\lambda| \cos \frac{\pi}{n+1}} \leq \frac{1}{\cos \frac{\pi}{n+1}}.$$

Thus, when  $n \rightarrow \infty$  and  $|\alpha|/|\lambda| \rightarrow \infty$ ,  $s(J_n^\alpha(\lambda)) \rightarrow 1$ . However,  $\tau(J_n^\alpha(\lambda)) \rightarrow \infty$ .

Propositions 1 and 2 give the answers of the questions ((1)(2)) of [13, p.352]. When  $s(A) = 2$ , we have the following result (see [8, p.18-7]).

**Proposition 5.** Let  $A \in \mathbb{C}^{n \times n}$  such that  $s(A) = 2$ . Then  $A$  is unitarily similar to a block diagonal matrix of the form

$$\begin{bmatrix} \|A\|_2 J_2(0) & \\ & B \end{bmatrix},$$

where  $J_2(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  and  $w(B) \leq \frac{\|A\|_2}{2}$ .

By Proposition 5, it is easy to show that in the case  $s(A) = 2$ ,  $1 \leq \tau(A) \leq \infty$ .

**B. Upper bounds for  $s(A)$  and  $\tau(A)$**

Let

$$A = U \Sigma V^*, \quad \Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n) \quad (3)$$

be a singular value decomposition of  $A$ , where  $U$  and  $V$  are unitary and  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n \geq 0$ . Denote the 2-norm condition number of a nonsingular matrix  $A$  by

$$\kappa(A) := \|A\|_2 \|A^{-1}\|_2 = \sigma_1 / \sigma_n.$$

**Proposition 6.** Let  $A \in \mathbb{C}^{n \times n}$  be as in (3) such that  $\sigma_n > 0$ . Then  $s(A) \leq \kappa(A)$  and  $\tau(A) \leq \kappa(A)$ . In particular, the following statements are equivalent:

- (i)  $s(A) = \kappa(A)$ .
- (ii)  $\tau(A) = \kappa(A)$ .
- (iii)  $\kappa(A) = 1$ .
- (iv)  $A$  is a nonzero multiple of a unitary matrix.

**Proposition 7.** Let  $A \in \mathbb{C}^{n \times n}$  be as in (3) such that  $\sigma_n > 0$  and  $s(A) = \tau(A)$ . Then  $s(A) = \tau(A) \leq \sqrt{\kappa(A)}$  and the equality holds if and only if  $\kappa(A) = 1$ .

Note that  $\sigma_n \leq \rho(A) \leq w(A) \leq \|A\|_2 = \sigma_1$  and if  $\rho(A) = \sigma_n > 0$  then  $\sigma_1 = \dots = \sigma_n$ . The proofs of Propositions 6 and 7 are trivial.

By Proposition 6, a large  $\tau(A)$  implies that  $A$  is ill conditioned and if  $A$  is well conditioned, i.e.,  $\kappa(A) \approx 1$ , then  $s(A) \approx 1$  and  $\tau(A) \approx 1$ . For a diagonalizable (singular or nonsingular) nonzero matrix  $A = X \Lambda X^{-1}$ , we have  $s(A), \tau(A) \leq \|A\|_2 / \|\Lambda\|_2 \leq \kappa(X)$ . Thus, a large  $\tau(A)$  implies that any eigenvector basis of the diagonalizable matrix  $A$  is ill conditioned.

**Remark 8.** Let  $A \in \mathbb{C}^{n \times n}$  be singular. The generalized 2-norm condition number is defined by  $\kappa^\dagger(A) = \sigma_1 / \sigma_r$ , where  $r = \text{rank}(A)$  is the rank of  $A$ . In general, we do not have  $s(A) \leq \kappa^\dagger(A)$  and  $\tau(A) \leq \kappa^\dagger(A)$ . For example, let

$$A = \begin{bmatrix} \epsilon & 1 & 0 \\ 0 & \epsilon & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

We have  $s(A) \rightarrow \sqrt{2}$ ,  $\tau(A) \rightarrow \infty$  and  $\kappa^\dagger(A) \rightarrow 1$  as  $\epsilon \rightarrow 0$ .

Let

$$A = U(\Lambda + N)U^* \quad (4)$$

be a Schur decomposition of  $A$ , where  $U$  is a unitary matrix,  $\Lambda$  is a diagonal matrix whose diagonal elements are the eigenvalues of  $A$  and  $N$  is a strictly upper triangular matrix.

**Proposition 9.** Let  $A \in \mathbb{C}^{n \times n}$  be as in (4). Then

$$w(A) \leq \rho(A) + w(N),$$

and if  $\rho(A) \neq 0$ ,

$$\tau(A) \leq 1 + w(N)/\rho(A).$$

*Proof:* Since  $U$  is unitary  $F(A) = F(\Lambda + N)$  [9, p.11].

Then

$$\begin{aligned} w(A) &= \max_{\|x\|_2=1} |x^*(\Lambda + N)x| \\ &\leq \max_{\|x\|_2=1} |x^*\Lambda x| + \max_{\|x\|_2=1} |x^*Nx| \\ &= \rho(A) + w(N). \end{aligned}$$

The proof of the second inequality is trivial. ■

The bound in Proposition 9 is attainable. For example, let  $J_n^\alpha(\lambda)$  be as in (2). Assume  $\lambda \neq 0$ . We have

$$\tau(J_n^\alpha(\lambda)) = 1 + \frac{w(N)}{\rho(J_n^\alpha(\lambda))}.$$

Another obvious bound for  $\tau(A)$  is  $\tau(A) \leq \|A\|_2 / \rho(A)$ . When  $\tau(A) = \|A\|_2 / \rho(A)$ , i.e.,  $s(A) = 1$ , we have  $\tau(A) = 1$  (see Proposition 2).

### III. A SUFFICIENT CONDITION FOR $0 \in F(A)$

For the matrix  $J_n^\alpha(\lambda)$  in Example 4 with  $\lambda \neq 0$ , if  $|\alpha|$  is sufficiently small, then  $0 \notin F(J_n^\alpha(\lambda))$ . And when  $\tau(J_n^\alpha(\lambda)) \geq 2$ ,  $0 \in F(J_n^\alpha(\lambda))$ . So a natural question is: Does there exist a constant  $c > 1$  s.t., if  $\tau(A) \geq c$ , then  $0 \in F(A)$ ? The answer to this question is positive. It was proved in [2, Lemma 2.6] that for any  $A \in \mathbb{C}^{n \times n}$  if 0 is not an interior point of  $F(A)$ , then  $\tau(A) \leq n$ . Here we give a slightly different version. For completeness we include its proof, which is similar to that of Lemma 2.6 of [2].

**Theorem 10.** Let  $A \in \mathbb{C}^{n \times n}$  such that  $\tau(A) \geq n$ . Then  $0 \in F(A)$ .

*Proof:* It is sufficient to prove if  $0 \notin F(A)$  then  $\tau(A) < n$ . It is well known that if  $0 \notin F(A)$  then there exists a real number  $\theta$  such that the Hermitian matrix  $H(e^{i\theta}A) = (e^{i\theta}A + e^{-i\theta}A^*)/2$  is positive definite; see, e.g., [9, p.21]. By rotating  $A$ , we assume that the Hermitian part  $H(A) = (A + A^*)/2$  of  $A$  is positive definite. Since  $F(A)$  is unitary similarity invariant [9, p.11], we also assume  $A$  is in upper triangular form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \ddots & \ddots & \vdots \\ & & \ddots & a_{n-1,n} \\ & & & a_{nn} \end{bmatrix}$$

The positive definiteness of  $H(A)$  implies that for all  $i, j = 1, \dots, n$ ,

$$\frac{1}{2} \begin{bmatrix} 2\text{Re}(a_{ii}) & a_{ij} \\ \bar{a}_{ij} & 2\text{Re}(a_{jj}) \end{bmatrix}$$

are positive definite. Then

$$|a_{ij}| < 2\sqrt{\text{Re}(a_{ii})\text{Re}(a_{jj})} \leq 2\rho(A).$$

Let  $|A| = (|a_{ij}|)$ . By Gershgorin circle theorem,  $\rho(|A| + |A|^T) < 2n\rho(A)$ . Then  $\tau(A) < n$  follows from  $w(A) \leq w(|A|) = \rho(|A| + |A|^T)/2$  (see [9, p.44]). ■

**Remark 11** (A geometric interpretation of the  $2 \times 2$  case). If  $A \in \mathbb{C}^{2 \times 2}$  has eigenvalues  $\lambda_1$  and  $\lambda_2$ , then  $F(A)$  is an (possibly degenerate) elliptical disk with foci  $\lambda_1$  and  $\lambda_2$ . Since any elliptical disk can be covered by two circular disks centered at  $\lambda_1$  and  $\lambda_2$  with radius  $a$ , where  $a$  is the length of the semi-major axis of the elliptical disk (see Figure 1), we have  $w(A) \leq \rho(A) + a$ . Thus,  $\tau(A) \geq 2$  means  $\rho(A) \leq a$ . We have  $|\lambda_1| + |\lambda_2| \leq 2\rho(A) \leq 2a$ . Therefore,  $0 \in F(A)$ .

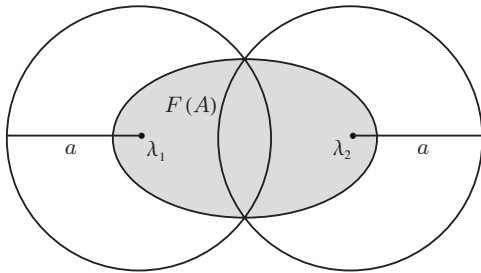


Fig. 1. Any elliptical disk can be covered by two circular disks centered at foci with radius  $a$ , where  $a$  is the length of the semi-major axis of the elliptical disk.

By Remark 11, if the numerical range of an  $n \times n$  matrix  $A$  is an elliptical disk with foci  $c_1, c_2 \in \sigma(A)$ , the condition in Theorem 10 can be reduced to  $\tau(A) \geq 2$ .

**Corollary 12.** Let  $A \in \mathbb{C}^{n \times n}$  such that  $\tau(A) \geq n$ . Then the Hermitian matrix  $H(e^{i\theta}A)$  is neither positive nor negative definite. Furthermore, if  $\tau(A) > n$  then  $H(e^{i\theta}A)$  is indefinite.

Note that  $w(A) = w(e^{i\theta}A)$  and  $\rho(A) = \rho(e^{i\theta}A)$ . The proof of Corollary 12 is easy. The following example shows that  $H(A)$  may be semi-definite when  $\tau(A) = n$ .

**Example 13** (An upper triangular Toeplitz matrix  $Z_n$  satisfying  $\tau(Z_n) = n$ ). Let

$$Z_n = \begin{bmatrix} 1 & 2 & \cdots & 2 \\ & 1 & \ddots & \vdots \\ & & \ddots & 2 \\ & & & 1 \end{bmatrix}_{n \times n}$$

The matrix  $Z_n$  is given in [2] to show that the bound in Lemma 2.6 of [2] is sharp. We have  $w(Z_n) = \rho(H(Z_n)) = \rho(Z_n) = 1$  and  $\tau(Z_n) = n$ . The origin is on the boundary of  $F(Z_n)$ . Obviously,  $H(Z_n)$  is the matrix with all the entries being 1 and is positive semidefinite.

Let  $\mathbb{P}_k$  denote the set of (complex) polynomials of degree at most  $k$ . The polynomial numerical hull of  $A$  of degree  $k$ ,

$$\mathcal{V}^k(A) := \{z \in \mathbb{C} : |p(z)| \leq \|p(A)\|_2, \forall p \in \mathbb{P}_k\}$$

is introduced by Nevanlinna [11, p.41]. We have (see [11, (7)])

$$\mathcal{V}^1(A) = F(A). \tag{5}$$

By (5), we have

$$F(A) = \bigcap_{z \in \mathbb{C}} \{\lambda \in \mathbb{C} : |\lambda - z| \leq \|A - zI\|_2\}.$$

Then  $0 \in F(A)$  implies  $|z| \leq \|A - zI\|_2$  for all  $z \in \mathbb{C}$ . We have the following corollary.

**Corollary 14.** Let  $A \in \mathbb{C}^{n \times n}$  such that  $\tau(A) \geq n$ . Then  $\min_{z \in \mathbb{C}} \|I - zA\|_2 = 1$ .

*Proof:* Since  $\tau(A) \geq n$ , we have  $|z| \leq \|A - zI\|_2$  for all  $z \in \mathbb{C}$ . Then  $\|I - A/z\|_2 \geq 1$  for all  $z \neq 0$ . Note that  $\|I - zA\|_2 = 1$  when  $z = 0$ . The proof is completed. ■

#### IV. CONCLUDING REMARKS

In this note, we discuss the existing results for the ratios  $s(A)$  and  $\tau(A)$ . We also provide several upper bounds for them. If no further known conditions for  $A$  are given, there is no obvious relation between  $s(A)$  and  $\tau(A)$  except that  $s(A) = 1$  implies  $\tau(A) = 1$ . If  $\tau(A) \gg 1$ , the matrix  $A$  is extremely ill conditioned and highly non-normal.

We complete this note by discussing the convergence of GMRES [12] for the linear system

$$Ax = b, \quad A \in \mathbb{C}^{n \times n}, \quad b \in \mathbb{C}^n. \tag{6}$$

Given an initial guess  $x_0$  for the solution of (6), at the iteration step  $m (\geq 1)$ , GMRES yields the approximate solution  $x_m$  in the affine subspace  $x_0 + \mathcal{K}_m(A, r_0)$  such that

$$\|r_m\|_2 := \|b - Ax_m\|_2 = \min_{y \in x_0 + \mathcal{K}_m(A, r_0)} \|b - Ay\|_2,$$

where

$$\mathcal{K}_m(A, r_0) := \text{span}\{r_0, Ar_0, \dots, A^{m-1}r_0\}$$

is the  $m$ th Krylov subspace generated by the matrix  $A$  and the initial residual vector  $r_0 := b - Ax_0$ . For a diagonalizable matrix  $A = X\Lambda X^{-1}$ , one has the estimate (see, e.g., [6, p.54])

$$\frac{\|r_m\|_2}{\|r_0\|_2} \leq \kappa(X) \min_{\substack{\phi_m \in \mathbb{P}_m \\ \phi_m(0)=1}} \max_{1 \leq i \leq n} |\phi_m(\lambda_i)|. \quad (7)$$

Obviously, this bound (7) does not always yield satisfactory results when  $\tau(A) \gg 1$  due to  $\kappa(X) \geq \tau(A)$ . For a general matrix, one has the estimate (see [3, Corollary 6.2])

$$\frac{\|r_m\|_2}{\|r_0\|_2} \leq [1 - \nu(A)\nu(A^{-1})]^{m/2}, \quad (8)$$

where  $\nu(A) = \min\{|z| : z \in F(A)\}$  is the distance from the origin to  $F(A)$ . Thus, this bound (8) is useless when  $\tau(A) \geq n$ . The same conclusion also applies to the bound (3.3) in [4] and the bound (2.1) in [1].

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