

Estimating of the Renewal Function with heavy-tailed claims

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Abstract—We develop a new estimator of the renewal function for heavy-tailed claims amounts. Our approach is based on the peak over threshold method for estimating the tail of the distribution with a generalized Pareto distribution. The asymptotic normality of an appropriately centered and normalized estimator is established, and its performance illustrated in a simulation study.

Keywords—Renewal function, peak-over-threshold, POT method, extremes value, generalized Pareto distribution, heavy-tailed distribution.

I. INTRODUCTION AND MOTIVATION

RENEWAL processes have many applications in the practice, as the warranty control, the reliability analysis of technical systems and in telecommunication networks such as high speed packet-switched networks like the Internet. It is important for planning and control purposes to estimate the related traffic load in terms of the mean numbers of counted events and their variances in these intervals. In such applications the renewal function (rf) constitutes the basic characteristic of an underlying renewal process since by means of this function the expectation of the number of arrivals of the relevant events before a fixed time instant can be calculated. To estimate the rf, several realizations of the counting process may be required, e.g. the observations of the number of calls within several days. We estimate here the rf using inter-arrival times between events of only one realization of the process.

We will now introduce some notations in order to formulate the problem rigorously. Let X_1, X_2, \dots, X_n be independent and identically distributed (i.i.d) non-negative random variables with not arithmetic distribution functions (df) F , finite means μ , and possibly infinite variances σ^2 . Let

$$S_0 = 0, S_n = X_1 + \dots + X_n \text{ for all } n \geq 1,$$

and

$$N(t) = \max \{n \geq 0, S_n \leq t\},$$

the number of renewal before time t , let $H_F(t)$ be the renewal function defined by

$$H_F(t) = \mathbb{E}N(t) = \sum_{n=1}^{\infty} \mathbf{P}(S_n < t) = \sum_{n=1}^{\infty} F^{*n}(t), \quad (1)$$

where F^{*k} is the k th convolution of the distribution function F . The renewal function $H_F(t)$ depends on the df F and is therefore unknown.

Most nonparametric estimators of $H_F(t)$ have been proposed in the literature, and we refer to Frees [8], Schneider et al.

[15], Grübel and Pitts [7], Markovitch and Krieger [11], and references therein.

When the mean and the variance of the df of F are finites, Feller [5], shown that

$$H_F(t) - \frac{t}{\mu} \sim \frac{\sigma^2 + \mu - \mu^2}{2\mu^2} \text{ as } t \rightarrow \infty. \quad (2)$$

Sgibnev [17] has proved that when $\mathbb{E}[X^2] = \infty$, then

$$H_F(t) - \frac{t}{\mu} \sim \frac{1}{\mu^2} \int_0^t \left(\int_y^\infty \bar{F}(x) dx \right) dy, \quad (3)$$

when $t \rightarrow \infty$, where $\bar{F} = 1 - F$ is the survival function. Bebbington, Davydov and Zitikis [2] explore a large sample to proposed an empirical estimator of $H_F(t)$ as follow.

$$\tilde{H}_F(t) = \frac{t}{\bar{X}} + \frac{1}{\bar{X}^2} \left(\frac{1}{2n} \sum_{i=1}^n (X_i \wedge t) + \frac{t}{n} \sum_{i=1}^n (X_i - X_i \wedge t) \right), \quad (4)$$

where the \bar{X} is the empirical mean of μ . The estimator $\tilde{H}_F(t)$ is good to use when we are not willing to assume any assumption on the tail of F , that is, we work in a completely non-parametric situation.

In this paper, we are interested in the renewal function $H_F(t)$ of the tail of F are described by heavy-tailed distributions.

Our estimation procedures are based on the result of Balkema and de Haan (1974) and Pickands (1975), that is, the distribution of excess F_t over a threshold t , are approximated by the GPD function, in the sense that

$$\sup_{y>0} |F_t(y) - \mathbb{G}_{\xi, \beta}(y)| = O(t^{-\delta} \mathbb{L}(t)), \text{ as } t \rightarrow \infty, \quad (5)$$

where $t^{-\delta} \mathbb{L}(t) \rightarrow 0$, for any $\delta > 0$. Moreover, under suitable assumptions, we study the asymptotic behavior of this estimator.

The remainder of this paper is organized as follows. In section 2, we introduce some assumptions on the regular variation distribution, the GPD's approximation and the well known POT method and the mean of an heavy-tailed distribution The estimation of renewal function and the limiting behavior of the proposed estimator are given in Section 3. In section 4, we are presents some simulation results showing the results of previous sections. Finally, the proofs of our results are given in Section 5.

II. MAIN ASSUMPTIONS, NOTATIONS, AND THE POT METHOD

A. Distribution functions

We deal only with losses X that are heavy tailed. More specifically, we work within the class of regularly varying cdf's. This class includes such popular distributions as Pareto, Burr, Student, Lévy-stable, log-gamma, among many others known to be appropriate models for fitting large insurance claims, large fluctuations of prices, log-returns, etc. (e.g., Beirlant et al., 2001). For more details on this type of distributions, we refer, for example, to Bingham *et al.* (1987) and Rolski et al. (1999). Namely, the tail of cdf F is said to be with regularity varying at infinity, that is

$$\bar{F}(x) = cx^{-1/\xi} (1 + x^{-\delta}\mathbb{L}(x)) \text{ when } x \rightarrow \infty, \quad (6)$$

for $\xi \in (0, 1)$, $\delta > 0$ and some real constant c , where \mathbb{L} a slowly varying function.

B. The POT Method

Let X_1, \dots, X_n be a i.i.d r.v.'s with the same cdf F , and let t_n be some a large number, 'high level,' which we later let tend to infinity when $n \rightarrow \infty$. With the notation

$$\bar{F}_{t_n}(t) = \mathbf{P}[X_1 - t_n > y \mid X_1 > t_n],$$

we have, $\bar{F}_{t_n}(y) = \bar{F}(t_n + y)/\bar{F}(t_n)$ and thus

$$\bar{F}_{t_n}(y) = \left(1 + \frac{y}{t_n}\right)^{-1/\xi} \frac{1 + (t_n + y)^{-\delta}\mathbb{L}(t_n + y)}{1 + t_n^{-\delta}\mathbb{L}(t_n)}. \quad (7)$$

Upon recalling the definition of the generalised Pareto distribution, we have that, for all parameter values $\beta > 0$ and $\xi > 0$,

$$\mathbb{G}_{\xi, \beta}(y) = 1 - \left(1 + \xi \frac{y}{\beta}\right)^{-1/\xi}, \quad 0 \leq y < \infty, \quad (8)$$

we see that, the right-hand side of equation (7) is a perturbed version of $\mathbb{G}_{\xi, \beta}(y)$, with the notation $\beta = t_n \xi$. Balkema and de Haan (1974), and Pickands (1975) have shown that F_{t_n} is approximated by a generalized Pareto distribution GPD function $\mathbb{G}_{\xi, \beta}$ with shape parameter $\xi \in \mathbb{R}$ and scale parameter $\beta = \beta(t_n)$, in the following sense:

$$\sup_{y>0} |F_{t_n}(y) - \mathbb{G}_{\xi, \beta}(y)| = O(t_n^{-\delta}\mathbb{L}(t_n)), \quad (9)$$

where, for any $\delta > 0$, we have $t_n^{-\delta}\mathbb{L}(t_n) \rightarrow 0$ when $t_n \rightarrow \infty$.

Approximation (9) suggests to define an estimator of $\bar{F}_{t_n}(y)$ as follows:

$$\hat{\bar{F}}_{t_n}(y) = \bar{\mathbb{G}}_{\hat{\xi}_n, \hat{\beta}_n}(y), \quad (10)$$

for appropriate estimates $\hat{\xi}_n$ and $\hat{\beta}_n$ of ξ and β , respectively. Note that β will be estimated separately, i.e. $\beta = \xi t_n$ will not be used. The reason for this is to achieve greater flexibility in the parameter fitting, compensating for the underlying distribution not being an exact GPD. Theorem 3.2 in Smith

[18] gives us the asymptotic distribution of the tail parameters $(\hat{\xi}_n, \hat{\beta}_n)$ as

$$\sqrt{np_n} \begin{pmatrix} \hat{\beta}_n/\beta - 1 \\ \hat{\xi}_n - \xi \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_2(0, \Sigma^{-1}) \text{ when } n \rightarrow \infty, \quad (11)$$

provided that $\sqrt{np_n}t_n^{-\delta}\mathbb{L}(t_n) \rightarrow 0$ when $n \rightarrow \infty$ and the function $x \mapsto x^{-\delta}\mathbb{L}(x)$ is non-increasing for all sufficiently large x , where

$$\Sigma^{-1} = (1 + \xi) \begin{pmatrix} 2 & -1 \\ -1 & 1 + \xi \end{pmatrix}. \quad (12)$$

We note that when $\sqrt{np_n}t_n^{-\delta}\mathbb{L}(t_n) \not\rightarrow 0$, then the limiting distribution in (11) is biased.

Next, we define an estimator of $\bar{F}(t_n)$. For this, let $N \equiv N(t_n)$ be defined by

$$N = \#\{X_i : X_i > t_n : 1 \leq i \leq n\},$$

which is the number of those X_i 's that exceed t_n . Since N follows the binomial distribution $\text{Bin}(p_n, n)$ with the parameter $p_n = \mathbf{P}[X_1 > t_n]$, which is equal to $\bar{F}(t_n)$, we have a natural estimator of $\bar{F}(t_n)$ defined by $\hat{p}_n = \frac{N}{n}$. From the definition of $\bar{F}_{t_n}(y)$ we have $\bar{F}(t_n + y) = \bar{F}(t_n)\bar{F}_{t_n}(y)$. Hence, with the above defined estimators for $\bar{F}_{t_n}(y)$ and $\bar{F}(t_n)$, we have the following estimator of $\bar{F}(t_n + y)$:

$$\hat{\bar{F}}(t_n + y) = \hat{\bar{F}}(t_n)\hat{\bar{F}}_{t_n}(y) = \hat{p}_n \bar{\mathbb{G}}_{\hat{\xi}_n, \hat{\beta}_n}(y). \quad (13)$$

C. Estimating of the Mean

For $\xi \in (0, 1/2]$, X_1 has finite variance, in this case $\mu = \mathbf{E}X_1$ is naturally estimated by the sample mean $\bar{X} = \frac{1}{n}(X_1 + X_2 + \dots + X_n)$, which by the Central Limit Theorem (TCL) is asymptotically normal. Whereas for $\xi \in (1/2, 1)$, X_1 has infinite variance and therefore the TCL is not valid anymore. This case is frequently met in real insurance data (see for instance, Beirlant *et al.*, 2001).

An alternative way to estimate μ may be made by GPD's approximation (see Johansson [8]).

Indeed, for each $n \geq 1$, we have

$$\mu = \int_0^{t_n} v dF(v) + \int_{t_n}^{\infty} v dF(v) = \mu_n^* + \tau_n^*,$$

an empirical estimator of μ_n^* by using the empirical distribution

$$\hat{\mu}_n^* = \frac{1}{n} \sum_{i=1}^n X_i \mathbf{1}_{\{X_i \leq t_n\}},$$

here $\mathbf{1}_K$ denotes the indicator function of set K , and an estimator of τ_n^* by using the GPD distribution and after integrate, we obtain

$$\hat{\tau}_n^* = \hat{p}_n \left(t_n + \frac{\hat{\beta}_n}{1 - \hat{\xi}_n} \right).$$

Finally, the estimator of the mean μ is

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^n X_i \mathbf{1}_{\{X_i \leq t_n\}} + \hat{p}_n \left(t_n + \frac{\hat{\beta}_n}{1 - \hat{\xi}_n} \right). \quad (14)$$

III. THE NEW ESTIMATOR AND THE MAIN RESULT

Next, with the representation of the double integral

$$\begin{aligned} & \int_0^{t_n} \int_s^\infty \bar{F}(v) dv ds \\ &= \int_0^{t_n} v \bar{F}(v) dv + t_n \int_{t_n}^\infty \bar{F}(v) dv \\ &= \frac{t_n^2}{2} \bar{F}(t_n) + \frac{1}{2} \int_0^{t_n} v^2 dF(v) + t_n \int_{t_n}^\infty \bar{F}(v) dv, \end{aligned} \tag{15}$$

since $F_n(x) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{\{X_i \leq x\}}$ is an empirical estimate of F , and $\widehat{\mathbb{G}}_{\hat{\xi}_n, \hat{\beta}_n}$ is an estimator of the tail of distribution \bar{F} , with a simple integrate, we obtain the corresponding estimator for $H_F(t_n)$ as follows

$$\widehat{H}_n(t_n) = \frac{t_n}{\hat{\mu}_n} + \frac{t_n}{\hat{\mu}_n^2} \left[\frac{t_n}{2} \hat{p}_n + \frac{1}{2nt_n} \sum_{i=1}^n X_i^2 \mathbf{1}_{\{X_i \leq t_n\}} + \frac{\hat{p}_n \hat{\beta}_n}{1 - \hat{\xi}_n} \right] \tag{16}$$

The consistency and asymptotic behavior of $\widehat{H}_n(t_n)$ is established in the following theorems.

Theorem 1: Let F be a df fulfilling (6) with $\xi \in (1/2, 1)$. Suppose that \mathbb{L} is locally bounded in $[x_0, +\infty)$ for $x_0 \geq 0$ and $x \rightarrow x^{-\delta} \mathbb{L}(x)$ is non-increasing near infinity, for some $\delta > 0$. For any $t_n = O(n^{\alpha\xi/4})$ with $\alpha \in (0, 1)$, we have

$$\widehat{H}_n(t_n) - H_F(t_n) = o_{\mathbb{P}}(1), \text{ as } n \rightarrow \infty.$$

Theorem 2: Let F be as in theorem (1). Then, for any $t_n = O(n^{\alpha\xi/4})$ with $\alpha \in (4/(1+2\xi\delta), 1)$, we have

$$\frac{\sqrt{n}}{t_n \sigma_n} \left(\widehat{H}_n(t_n) - H_F(t_n) \right) \xrightarrow{D} \mathcal{N}(0, 1), \text{ as } n \rightarrow \infty,$$

where

$$\begin{aligned} \sigma_n^2 &= \gamma_n^2 \theta_1^2 + p_n(1-p_n) \left(\theta_2 + \theta_1 \left(t_n + \frac{\beta_n}{1-\xi} \right) \right)^2 \\ &+ \frac{(1+\xi)^2}{p_n} \left(\theta_3 + \theta_1 \frac{p_n \beta_n}{(1-\xi)^2} \right)^2 \\ &+ \frac{2(1+\xi)\beta_n^2}{p_n} \left(\theta_4 + \theta_1 \frac{p_n}{1-\xi} \right)^2 \\ &- \frac{(1+\xi)\beta_n}{p_n} \left(\theta_3 + \theta_1 \frac{p_n \beta_n}{(1-\xi)^2} \right) \left(\theta_4 + \theta_1 \frac{p_n}{1-\xi} \right), \end{aligned}$$

with

$$\begin{aligned} \theta_1 &= - \left(\frac{1}{\mu^3} \right) \left(\mu + p_n t_n + \frac{1}{t_n} \mathbf{E}(X^2 \mathbf{1}_{\{X_1 \leq t_n\}}) + \frac{2p_n \beta_n}{(1-\xi)} \right) \\ \theta_2 &= \left(\frac{1}{\mu^2} \right) \left(\frac{t_n}{2} + \frac{\beta_n}{(1-\xi)} \right), \quad \theta_3 = \frac{1}{\mu^2} \frac{p_n \beta_n}{(1-\xi)^2}, \\ \theta_4 &= \frac{1}{\mu^2} \frac{p_n}{(1-\xi)}, \end{aligned}$$

where $\beta_n = t_n \xi$, $p_n = \bar{F}(t_n)$ and $\gamma_n^2 = \mathbf{Var}(X_1 \mathbf{1}_{\{X_1 \leq t_n\}})$.

IV. SIMULATION STUDY

As an example for simulated the renewal function, we consider a random variable X with standard Pareto df

$$\bar{F}(x) = x^{-1/\xi}, x \geq 1,$$

and the selected threshold t_n . We generate 200 independent replicates of sizes 500, 1000 and 2000 from the selected parent distribution with values index $\xi = 2/3$ and $\xi = 3/4$. Using the **R** program package *evir*, for made a differents estimates of the renewal function. The overall estimated renewal function is then taken as the empirical mean of the values in the 200 repetitions. To this end, and in all cases, we calculate the bias and the root mean square error (rmse). We summarize the results in the Table (1).

TABLE I
POINT ESTIMATES OF THE RENEWAL FUNCTION BASED ON 200 SAMPLES OF PARETO-DISTRIBUTED RV'S WITH TAIL INDEX 2/3 AND 3/4.

ξ	2/3			3/4		
H_F	2.222			1.709		
n	$\widehat{H}_n(t_n)$	rmse	bias	$\widehat{H}_n(t_n)$	rmse	bias
500	2.296	0.307	0.074	1.749	0.368	0.041
1000	2.256	0.267	0.034	1.657	0.258	-0.039
2000	2.237	0.231	0.015	1.671	0.247	-0.036

V. PROOFS

The following proposition will be instrumental for our needs.

Proposition 3: Let F be a distribution function not arithmetic and fulfilling (6) with $\xi \in (0, 1)$, $\delta > 0$ and some real c . Suppose that \mathbb{L} is locally bounded in $[x_0, +\infty)$ for $x_0 \geq 0$. Then for n large enough, for any $t_n = O(n^{\alpha\xi/4})$ with $\alpha \in (0, 1)$, we have

$$\begin{aligned} p_n &= \mathbf{P}(X_1 > t_n) = c(1 + o(1))n^{-\alpha/4}, \\ \gamma_n^2 &= \mathbf{Var}(X_1 \mathbf{1}_{\{X_1 \leq t_n\}}) = O(n^{(\alpha/2)(\xi-1/2)}), \end{aligned}$$

and

$$\sqrt{np_n} t_n^{-\delta} \mathbb{L}(t_n) = O(n^{-\alpha/8 - \alpha\xi\delta/4 + 1/2}).$$

Proof of the proposition: We will only prove the second result, the other ones are straightforward from (6). Let $x_0 > 0$ be such that $\bar{F}(x) = cx^{-1/\xi} (1 + x^{-\delta} \mathbb{L}(x))$, for $x > x_0$, then for n large enough, we have

$$\mathbf{E}X_1 \mathbf{1}_{\{X_1 \leq t_n\}} = \int_0^{t_n} x dF(x) = \int_0^{x_0} x dF(x) + \int_{x_0}^{t_n} x dF(x).$$

Recall that $\mu < \infty$, hence $\int_0^{x_0} x dF(x) < \infty$. Making use of the proposition assumptions, we get

$$\mathbf{E}(X_1 \mathbf{1}_{\{X_i \leq t_n\}}) = O(1),$$

and

$$\mathbf{E}(X_1^2 \mathbf{1}_{\{X_i \leq t_n\}}) = O\left(t_n^{2-\frac{1}{\xi}}\right),$$

and therefore

$$\gamma_n^2 = O(n^{(\alpha/2)(\xi-1/2)}).$$

This achieves the proof of the proposition.

Proof of theorem 1: Let

$$D_n(t_n) = \widehat{H}_n(t_n) - H_F(t_n),$$

then

$$D_n(t_n) = \left[\frac{t_n}{\widehat{\mu}_n} - \frac{t_n}{\mu} \right] + \frac{t_n}{\widehat{\mu}_n^2} \left[\frac{t_n \widehat{p}_n}{2} + \frac{1}{2nt_n} \sum_{i=1}^n X_i^2 \mathbf{1}_{\{X_i \leq t_n\}} + \frac{\widehat{p}_n \widehat{\beta}_n}{1 - \widehat{\xi}_n} \right] - \frac{1}{\mu^2} \int_0^{t_n} \int_s^\infty \overline{F}(v) dv ds.$$

Using the representation

$$\int_0^{t_n} \int_s^\infty \overline{F}(v) dv ds = \int_0^{t_n} v \overline{F}(v) dv + t_n \int_{t_n}^\infty \overline{F}(v) dv = \left[\frac{u_n^2 p_n}{2} + \frac{1}{2} \mathbf{E}(X^2 \mathbf{1}_{\{X_1 \leq t_n\}}) \right] + t_n \int_{t_n}^\infty \overline{F}(v) dv.$$

Therefore

$$\frac{1}{t_n} D_n(t) = \Delta_{n,1} + \Delta_{n,2} + \Delta_{n,3}$$

where

$$\begin{aligned} \Delta_{n,1} &= -\frac{1}{\mu \widehat{\mu}_n} (\widehat{\mu}_n - \mu), \\ \Delta_{n,2} &= \frac{1}{2\widehat{\mu}_n^2} \left(t_n \widehat{p}_n + \frac{1}{t_n n} \sum_{i=1}^n X_i^2 \mathbf{1}_{\{X_i \leq t_n\}} \right) - \frac{1}{2\mu^2} \left(t_n p_n + \frac{1}{t_n} \mathbf{E}(X^2 \mathbf{1}_{\{X_1 \leq t_n\}}) \right), \\ \Delta_{n,3} &= \frac{1}{\widehat{\mu}_n^2} \frac{\widehat{p}_n \widehat{\beta}_n}{1 - \widehat{\xi}_n} - \frac{1}{\mu^2} \int_{t_n}^\infty \overline{F}(v) dv. \end{aligned}$$

First, we show that $\Delta_{n,1} \xrightarrow{\mathbb{P}} 0$ as $n \rightarrow \infty$, we have from Johansson (2003)

$$\widehat{\mu}_n - \mu = O\left(\frac{\gamma_n}{\sqrt{n}}\right), \text{ as } n \rightarrow \infty. \quad (17)$$

By the proposition, we have $\gamma_n^2 = O(n^{(\alpha/2)(\xi-1/2)})$, then

$$\widehat{\mu}_n - \mu = O\left(n^{(\alpha/4)(\xi-1/2)-1/2}\right) = o_{\mathbb{P}}(1) \text{ as } n \rightarrow \infty.$$

It follows that

$$-\frac{1}{\mu \widehat{\mu}_n} (\widehat{\mu}_n - \mu) = o_{\mathbb{P}}(1) \text{ as } n \rightarrow \infty.$$

On the other hand, $\Delta_{n,2}$ may be rewrite as follows

$$\begin{aligned} \Delta_{n,2} &= \frac{t_n}{2\widehat{\mu}_n^2} (\widehat{p}_n - p_n) - \left(\frac{p_n t_n}{2} + \frac{1}{2t_n} \mathbf{E}(X^2 \mathbf{1}_{\{X_1 \leq t_n\}}) \right) \left(\frac{\widehat{\mu}_n + \mu}{\mu^2 \widehat{\mu}_n^2} \right) (\widehat{\mu}_n - \mu) \\ &+ \frac{1}{2t_n \widehat{\mu}_n^2} \left(\frac{1}{n} \sum_{i=1}^n X_i^2 \mathbf{1}_{\{X_i \leq t_n\}} - \mathbf{E}(X^2 \mathbf{1}_{\{X_1 \leq t_n\}}) \right). \end{aligned}$$

■ Moreover, we have

$$\widehat{p}_n - p_n = O_{\mathbb{P}}\left(\sqrt{p_n/n}\right) \text{ as } n \rightarrow \infty, \quad (18)$$

then by the proposition, we have $\widehat{p}_n - p_n = O(n^{-\alpha/8-1/2})$ and

$$t_n (\widehat{p}_n - p_n) = O\left(n^{(\alpha/4)(\xi-1/2)-1/2}\right) = o_{\mathbb{P}}(1) \text{ as } n \rightarrow \infty,$$

further

$$p_n t_n (\widehat{\mu}_n - \mu) = O\left(n^{(\alpha/4)(2\xi-3/2)-1/2}\right) = o_{\mathbb{P}}(1) \text{ as } n \rightarrow \infty.$$

The value of $\frac{1}{n} \sum_{i=1}^n X_i^2 \mathbf{1}_{\{X_i \leq t_n\}} - \mathbf{E}(X^2 \mathbf{1}_{\{X_1 \leq t_n\}})$ is finite.

Then we infer that $\Delta_{n,2} \xrightarrow{\mathbb{P}} 0$ as $n \rightarrow \infty$. Finally, we have

$$\begin{aligned} \Delta_{n,3} &= \frac{1}{\mu^2} \left(\frac{p_n \beta_n}{1 - \xi} - \int_{t_n}^\infty \overline{F}(v) dv \right) \\ &+ \left(\frac{1}{\widehat{\mu}_n^2} \frac{\widehat{p}_n \widehat{\beta}_n}{1 - \widehat{\xi}_n} - \frac{1}{\mu^2} \frac{p_n \beta_n}{1 - \xi} \right), \end{aligned}$$

where

$$\begin{aligned} \int_{t_n}^\infty \overline{F}(v) dv &= \frac{p_n \beta_n}{1 - \xi} (1 - t_n^{-\delta} \mathbb{L}(t_n) + O(t_n^{-\delta} \mathbb{L}(t_n))) (1 + O(t_n^{-\delta})) \\ &= \frac{p_n \beta_n}{1 - \xi} [1 - t_n^{-\delta} \mathbb{L}(t_n) + O(t_n^{-\delta} \mathbb{L}(t_n))], \end{aligned}$$

then

$$\frac{1}{\mu^2} \left(\frac{p_n \beta_n}{1 - \xi} - \int_{t_n}^\infty \overline{F}(v) dv \right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

On the other hand, we may rewrite

$$\begin{aligned} \frac{1}{\widehat{\mu}_n^2} \frac{\widehat{p}_n \widehat{\beta}_n}{1 - \widehat{\xi}_n} - \frac{1}{\mu^2} \frac{p_n \beta_n}{1 - \xi} &= -\frac{p_n \beta_n}{1 - \xi} \left(\frac{\widehat{\mu}_n + \mu}{\mu^2 \widehat{\mu}_n^2} \right) (\widehat{\mu}_n - \mu) \\ &+ \frac{1}{\widehat{\mu}_n^2} \frac{\beta_n}{(1 - \xi)} (\widehat{p}_n - p_n) \\ &+ \frac{1}{\widehat{\mu}_n^2} \frac{\widehat{p}_n}{(1 - \xi)} (\widehat{\beta}_n - \beta_n) \\ &+ \frac{1}{\widehat{\mu}_n^2} \left(\frac{\widehat{p}_n \widehat{\beta}_n}{(1 - \xi)(1 - \widehat{\xi}_n)} \right) (\widehat{\xi}_n - \xi). \end{aligned}$$

where $\beta_n = t_n \xi$, then we have shown that

$$p_n \beta (\widehat{\mu}_n - \mu) = o_{\mathbb{P}}(1) \text{ as } n \rightarrow \infty,$$

too

$$\beta (\widehat{p}_n - p_n) = O\left(n^{(\alpha/4)(\xi-1/2)-1/2}\right) = o_{\mathbb{P}}(1) \text{ as } n \rightarrow \infty.$$

Moreover, from Smith (1987) we have as $n \rightarrow \infty$

$$\widehat{\beta}_n / \beta = 1 + O_{\mathbb{P}}(t_n^{-\delta} \mathbb{L}(t_n)) \text{ and } \widehat{\xi}_n - \xi = O_{\mathbb{P}}(t_n^{-\delta} \mathbb{L}(t_n)). \quad (19)$$

Then

$$\widehat{p}_n (\widehat{\beta}_n - \beta) = O\left(n^{-\alpha/4 - \alpha \xi \delta / 4}\right) = o_{\mathbb{P}}(1) \text{ as } n \rightarrow \infty,$$

and

$$\widehat{p}_n \widehat{\beta}_n (\widehat{\xi}_n - \xi) = O\left(n^{-\alpha/4 - \alpha \xi \delta / 2}\right) = o_{\mathbb{P}}(1) \text{ as } n \rightarrow \infty.$$

Finally, we infer that $\Delta_{n,3} \xrightarrow{P} 0$ as $n \rightarrow \infty$. Therefore

$$\widehat{H}_n(t_n) - H_F(t_n) = O(n^{(\alpha/2)(\xi-1/4)-1/2}) = o_{\mathbb{P}}(1) \text{ as } n \rightarrow \infty,$$

because $(\alpha/2)(\xi-1/4)-1/2 < 0$. ■

Proof of Theorem 2: In proof of theorem 1 we have shown that $\widehat{H}_n(t_n) = H_F(t_n) + o_{\mathbb{P}}(1)$. It follows that, for all large n , $\{\Delta_{n,i}\}_{i=1,3}$ may be rewritten into

$$\Delta_{n,1} = -\frac{1}{\mu^2} (1 + o_{\mathbb{P}}(1)) (\widehat{\mu}_n - \mu),$$

$$\begin{aligned} \Delta_{n,2} &= \frac{t_n}{2\mu^2} (1 + o_{\mathbb{P}}(1)) (\widehat{p}_n - p_n) \\ &\quad - \frac{p_n t_n}{\mu^3} (1 + o_{\mathbb{P}}(1)) (\widehat{\mu}_n - \mu) \\ &\quad - \frac{1}{\mu^3 t_n} \mathbf{E}(X^2 \mathbf{1}_{\{X_1 \leq t_n\}}) (1 + o_{\mathbb{P}}(1)) (\widehat{\mu}_n - \mu), \end{aligned}$$

and

$$\begin{aligned} \Delta_{n,3} &= -\frac{2p_n \beta}{(1-\xi)\mu^3} (1 + o_{\mathbb{P}}(1)) (\widehat{\mu}_n - \mu) \\ &\quad + \frac{1}{\mu^2} \frac{\beta}{(1-\xi)} (1 + o_{\mathbb{P}}(1)) (\widehat{p}_n - p_n) \\ &\quad + \frac{1}{\mu^2} \frac{p_n}{(1-\xi)} (1 + o_{\mathbb{P}}(1)) (\widehat{\beta}_n - \beta) \\ &\quad + \frac{1}{\mu^2} \frac{p_n \beta}{(1-\xi)^2} (1 + o_{\mathbb{P}}(1)) (\widehat{\xi}_n - \xi). \end{aligned}$$

Therefore

$$\frac{1}{t_n} D_n(t) = \Delta_{n,1} + \Delta_{n,2} + \Delta_{n,3}.$$

Then

$$\begin{aligned} \frac{\sqrt{n}}{t_n} D_n(t) &= \theta_1 (1 + o_{\mathbb{P}}(1)) \sqrt{n} (\widehat{\mu}_n - \mu) \\ &\quad + \theta_2 (1 + o_{\mathbb{P}}(1)) \sqrt{n} (\widehat{p}_n - p_n) \\ &\quad + \theta_3 (1 + o_{\mathbb{P}}(1)) \sqrt{n} (\widehat{\xi}_n - \xi) \\ &\quad + \theta_4 (1 + o_{\mathbb{P}}(1)) \sqrt{n} (\widehat{\beta}_n - \beta). \end{aligned}$$

where $\{\theta_i\}_{i=1,4}$ are those defined in Theorem 2. On the other hand, from Johansson (2003), we have

$$\begin{aligned} \sqrt{n} (\widehat{\mu}_n - \mu) &= \sqrt{n} (\widehat{\mu}_n^* - \mu_n^*) \\ &\quad + \left(t_n + \frac{\beta}{1-\xi} \right) \sqrt{n} (\widehat{p}_n - p_n) \\ &\quad + \frac{p_n \beta}{(1-\xi)^2} \sqrt{n} (\widehat{\xi}_n - \xi) \\ &\quad + \frac{p_n}{1-\xi} \sqrt{n} (\widehat{\beta}_n - \beta) + o_{\mathbb{P}}(\gamma_n) \end{aligned}$$

This enables us to rewrite, as $n \rightarrow \infty$, $\frac{\sqrt{n}}{t_n} D_n(t)$ into

$$\begin{aligned} \frac{\sqrt{n}}{t_n} D_n(t) &= \theta_1 (1 + o_{\mathbb{P}}(1)) \sqrt{n} (\widehat{\mu}_n^* - \mu_n^*) \\ &\quad + (1 + o_{\mathbb{P}}(1)) \left(\theta_2 + \theta_1 \left(t_n + \frac{\beta}{1-\xi} \right) \right) \sqrt{n} (\widehat{p}_n - p_n) \\ &\quad + \left(\theta_3 + \theta_1 \frac{p_n \beta}{(1-\xi)^2} \right) (1 + o_{\mathbb{P}}(1)) \sqrt{n} (\widehat{\xi}_n - \xi) \\ &\quad + (1 + o_{\mathbb{P}}(1)) \left(\theta_4 + \theta_1 \frac{p_n}{1-\xi} \right) \sqrt{n} (\widehat{\beta}_n - \beta) + o_{\mathbb{P}}(\gamma_n). \end{aligned}$$

Then

$$\begin{aligned} \frac{\sqrt{n}}{t_n \gamma_n} D_n(t) &= \theta_1 (1 + o_{\mathbb{P}}(1)) \frac{\sqrt{n}}{\gamma_n} (\widehat{\mu}_n^* - \mu_n^*) \\ &\quad + (1 + o_{\mathbb{P}}(1)) \left(\theta_2 + \theta_1 \left(t_n + \frac{\beta}{1-\xi} \right) \right) \frac{\sqrt{n}}{\gamma_n} (\widehat{p}_n - p_n) \\ &\quad + \left(\theta_3 + \theta_1 \frac{p_n \beta}{(1-\xi)^2} \right) (1 + o_{\mathbb{P}}(1)) \frac{\sqrt{n}}{\gamma_n} (\widehat{\xi}_n - \xi) \\ &\quad + (1 + o_{\mathbb{P}}(1)) \left(\theta_4 + \theta_1 \frac{p_n}{1-\xi} \right) \frac{\sqrt{n}}{\gamma_n} (\widehat{\beta}_n - \beta) + o_{\mathbb{P}}(1), \end{aligned}$$

where γ_n^2 is defined in the Proposition.

We observe that, $\frac{\sqrt{n}}{t_n \gamma_n} D_n(t)$ may be rewritten as follows

$$\begin{aligned} \frac{\sqrt{n}}{t_n \gamma_n} D_n(t) &= \theta_1 (1 + o_{\mathbb{P}}(1)) \mathcal{W}_1 \\ &\quad + (1 + o_{\mathbb{P}}(1)) \frac{\sqrt{p_n(1-p_n)}}{\gamma_n} \left(\theta_2 + \theta_1 \left(t_n + \frac{\beta}{1-\xi} \right) \right) \mathcal{W}_2 \\ &\quad + \frac{(1+\xi)}{\gamma_n \sqrt{p_n}} \left(\theta_3 + \theta_1 \frac{p_n \beta}{(1-\xi)^2} \right) (1 + o_{\mathbb{P}}(1)) \mathcal{W}_3 \\ &\quad + (1 + o_{\mathbb{P}}(1)) \frac{\beta \sqrt{2(1+\xi)}}{\gamma_n \sqrt{p_n}} \left(\theta_4 + \theta_1 \frac{p_n}{1-\xi} \right) \mathcal{W}_4 + o_{\mathbb{P}}(1). \end{aligned}$$

where $(\mathcal{W}_i)_{i=1,4}$ are standard normal rv's with $\mathbf{E}[\mathcal{W}_i \mathcal{W}_j] = 0$ for every $i, j = 1, \dots, 4$ except

$$\mathbf{E}[\mathcal{W}_3 \mathcal{W}_4] = -\frac{1}{\sqrt{2(1+\xi)}}.$$

From Lemma A-2 of Johansson 2003, under the assumptions of theorem 1, for any real numbers, s_1, s_2, s_3 and s_4 , we have

$$\begin{aligned} \mathbf{E} \left[\exp \left\{ \begin{aligned} &is_1 \frac{\sqrt{n}}{\gamma_n} (\widehat{\mu}_n^* - \mu_n^*) + i\sqrt{np_n}(s_2, s_3) \begin{pmatrix} \widehat{\beta}_n/\beta - 1 \\ \widehat{\xi}_n - \xi \end{pmatrix} \\ &+ is_4 \frac{\sqrt{n}(\widehat{p}_n - p_n)}{\sqrt{p_n(1-p_n)}} \end{aligned} \right\} \right] \\ \rightarrow \exp \left\{ -\frac{s_1^2}{2} - \frac{1}{2} (s_2, s_3) \Sigma^{-1} \begin{pmatrix} s_2 \\ s_3 \end{pmatrix} - \frac{s_4^2}{2} \right\} (1 + o_{\mathbb{P}}(1)), \end{aligned}$$

as $n \rightarrow \infty$, where Σ^{-1} is that in (12), $\gamma_n^2 = \mathbf{Var}(X_1 \mathbf{1}_{\{X_1 \leq t_n\}})$ and $i^2 = -1$. It follows that, with this result that

$$\frac{\sqrt{n}}{t_n \gamma_n \sigma_n} \left(\widehat{H}_{\rho, n} - H_{\rho} \right) \xrightarrow{D} \mathcal{N}(0, 1), \text{ as } n \rightarrow \infty,$$

with $\rho_n^2 = \text{Var} \left(\frac{\sqrt{n}}{t_n \gamma_n} D_n(t) \right)$, we have

$$\begin{aligned} \rho_n^2 &= \theta_1^2 + \frac{p_n(1-p_n)}{\gamma_n^2} \left(\theta_2 + \theta_1 \left(t_n + \frac{\beta}{1-\xi} \right) \right)^2 \\ &+ \frac{(1+\xi)^2}{p_n \gamma_n^2} \left(\theta_3 + \theta_1 \frac{p_n \beta}{(1-\xi)^2} \right)^2 \\ &+ \frac{2(1+\xi)\beta^2}{p_n \gamma_n^2} \left(\theta_4 + \theta_1 \frac{p_n}{1-\xi} \right)^2 \\ &- \frac{(1+\xi)\beta}{p_n \gamma_n^2} \left(\theta_3 + \theta_1 \frac{p_n \beta}{(1-\xi)^2} \right) \left(\theta_4 + \theta_1 \frac{p_n}{1-\xi} \right), \end{aligned}$$

where $\sigma_n^2 = \gamma_n^2 \rho_n^2$. This completes the proof of Theorem 2. ■

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