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A novel approach to positive almost periodic solution of BAM neural networks with time-varying delays

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Abstract—In this paper, based on almost periodic functional hull theory and M-matrix theory, some sufficient conditions are established for the existence and uniqueness of positive almost periodic solution for a class of BAM neural networks with time-varying delays. An example is given to illustrate the main results.

Keywords—Delayed BAM neural networks; Hull theorem; M-matrix; Almost periodic solution; Global exponential stability.

I. INTRODUCTION

IN the past few years, BAM neural networks have been extensively studied and applied in many different fields such as signal processing, pattern recognition, solving optimization problems and automatic control engineering. They have been studied widely both in theory and applications. In [1-4], some sufficient conditions have been obtained for global stability of delayed BAM networks. Moreover, authors in [5-7] investigated the periodic oscillatory solution of BAM neural networks. But, as mentioned above, these researches and applications mainly rely on the existence and stability of equilibrium points or periodic solutions. However, upon considering long-term dynamical behaviors, the periodic parameters often turn out to experience certain perturbations, that is, parameters become periodic up to a small error. Thus, almost periodic oscillatory behavior is considered to be more accordant with reality.

Recently, existence of almost periodic solutions of BAM neural networks have received much attention, one can see [8-9] and the references cited therein. However, the results obtained in these papers by using the same method - Banach fixed point theorem. In this paper, we will study the existence and global exponential stability of almost periodic solution based on almost periodic functional hull theory and M-matrix theory.

Motivated by the above, we consider the following BAM neural networks:

$$\begin{cases} x_i'(t) \ = \ -a_i(t)x_i(t) + \sum\limits_{j=1}^{m} p_{ji}(t) f_j(y_j(t-\tau_{ji}(t))) \\ \qquad\qquad +I_i(t),\ i=1,2,\cdots,n, \\ y_j'(t) \ = \ -b_j(t)y_j(t) + \sum\limits_{i=1}^{n} q_{ij}(t) g_i(x_i(t-\vartheta_{ij}(t))) \\ \qquad\qquad +L_j(t),\ j=1,2,\cdots,m, \end{cases} \tag{1}$$

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where $x_i(t)$ and $y_j(t)$ are the activations of the ith neuron and the jth neuron, respectively. p_{ji}, q_{ij} are the connection weights at time t, $I_i(t)$ and $L_j(t)$ denote the external inputs at time t. g_i, f_j are the input-output functions (the activation functions). Time delays $\tau_{ji}(t), \vartheta_{ij}(t)$ correspond to finite speed of axonal transmission, $a_i(t), b_j(t)$ represent the rate with which the ith neuron and jth neuron will reset their potential to the resting state in isolation when they are disconnected from the network and the external inputs at time t. m, n correspond to the number of neurons in layers.

The system (1) is supplemented with initial values given by

$$\begin{cases} x_i(s) = \phi_i(s),\ s \in [-\vartheta, 0], \\ \vartheta = \max\limits_{1\le i\le n, 1\le j\le m} \sup\limits_{t\in\mathbb{R}} \{\vartheta_{ij}(t)\},\ i=1,2,\cdots,n, \\ y_j(s) = \varphi_j(s),\ s\in[-\tau,0], \\ \tau = \max\limits_{1\le i\le n, 1\le j\le m} \sup\limits_{t\in\mathbb{R}} \{\tau_{ji}(t)\},\ j=1,2,\cdots,m, \end{cases}$$

where $\phi_i(\cdot)$ and $\varphi_i(\cdot)$ denote real-valued continuous functions defined on $[-\tau, 0]$ and $[-\vartheta, 0]$. Let $\psi(s) = (\phi_1(s), \cdots, \phi_n(s), \varphi_1(s), \cdots, \varphi_m(s))^T$.

Throughout this paper, we make the following assumptions:

$(H1)$ $a_i(t)(i=1,2,\cdots,n,\ t\in\mathbb{R})$ and $b_j(t)(j=1,2,\cdots,m,\ t\in\mathbb{R})$ are positive, continuous and bounded functions.

$(H2)$ $p_{ji}(t), q_{ij}(t), I_i(t), L_j(t), 0 < \vartheta_{ij}(t) < \vartheta, 0 < \tau_{ji}(t) < \tau$ are all positive continuous bounded almost periodic functions on $\mathbb{R}$, $i=1,2,\cdots,n, j=1,2,\cdots,m$.

$(H3)$ $f_j, g_i \in C(\mathbb{R},\mathbb{R})$ and $f_j, g_i \ge 0 (i=1,2,\cdots,n, j=1,2,\cdots,m)$ are Lipschitzian with Lipschitz constants $\eta_j, \lambda_i > 0$, $|f_j(x)-f_j(y)| \le \eta_j|x-y|$, $|g_i(x)-g_i(y)| \le \lambda_i|x-y|$, $\forall\ x,y\in\mathbb{R}$.

For convenience, we denote $\bar{a} = \sup\limits_{t\in\mathbb{R}} |a(t)|$, $\underline{a} = \inf\limits_{t\in\mathbb{R}} |a(t)|$.

The organization of this paper is as follows: In Section 2, we introduce some notations and definitions and prove some preliminary results needed in the later sections. In Section 3, by using M-matrix theory, we shall derive sufficient conditions to ensure that the solution of (1) is global exponentially stable. In Section 4, by using almost periodic functional hull theory, we show that the almost periodic system (1) has a unique globally exponentially stable strictly positive almost periodic solution. In Section 5, an examples is given to illustrate the main result.

II. PRELIMINARIES

In this section, we shall first recall some basic definitions, lemmas which are used in what follows.

Definition 2.1 [10] Let $x(t) \in C(\mathbb{R}, \mathbb{R})$ be continuous in t. $x(t)$ is said to be almost periodic in the sense of Bohr on $\mathbb{R}$, if for any $\epsilon > 0$, the set $T(x, \epsilon) = \{\tau : |x(t+\tau) - x(t)| < \epsilon, \forall t \in \mathbb{R}\}$ is relatively dense, i.e., for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\tau = \tau(\epsilon)$ in this interval such that $|x(t+\tau) - x(t)| < \epsilon$, $\forall t \in \mathbb{R}$.

Definition 2.2 [10] Suppose that $f(t)$ be an almost periodic function, the function set $H(f) = \{g$: there exist real sequence α, such that $T_\alpha f = g$ uniformly on $\mathbb{R}\}$ is called the hull of $f(t)$, where $T_\alpha f(t) = g(t)$ denotes $\lim_{k\to\infty} f(t+\alpha_n) = g(t),\ t \in \mathbb{R}$, and T is called shifting operator.

Definition 2.3 The almost periodic solution $z^* = (x_1^*, x_2^*, \cdots, x_n^*, y_1^*, y_2^*, \cdots, y_m^*)^T$ of system (1) is said to be exponentially stable, if there exist constants $\gamma > 0$ and $\lambda > 0$ such that

$$\|z - z^*\| \le \gamma \|\psi - z^*\| e^{-\lambda t}$$

for all $t \ge 0$.

Lemma 2.1 [11] Let $z(t)$ be a solution of the differential inequality

$$D^+ z(t) \le C z(t) + D\bar{z}(t),\ t \ge 0,$$

where $\bar{z}(t) = (\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n, \bar{y}_1, \bar{y}_2, \cdots, \bar{y}_m)^T$, $\bar{x}_i(t) = \sup_{s\in[t-\vartheta,t]} x_i(s)$, $\bar{y}_j(t) = \sup_{s\in[t-\tau,t]} y_j(s)$. If the conditions

(1) $C = (c_{ij})_{(n+m)\times(n+m)}, c_{ij} \ge 0 (i \ne j), D = (d_{ij})_{(n+m)\times(n+m)}, d_{ij} \ge 0 (i \ne j)$,
(2) $-(C + D)$ is an M-matrix,

hold. Then there exists a constant $\lambda > 0$ and a constant vector $k \ge \bar{z}(0)$ such that $z(t) \le k e^{-\lambda t}, t \ge 0$.

Lemma 2.2 Let $y, f \in C(\mathbb{R}, \mathbb{R})$ and p is a constant, then

$$y'(t) \le p y(t) + f(t),\ \forall\ t \in \mathbb{R}$$

implies

$$y(t) \le y(t_0) e^{p(t-t_0)} + \int_{t_0}^{t} e^{p(t-s)} f(s) \mathrm{d}s,\ \forall\ t \in \mathbb{R}.$$

The lemma also holds, if " $\le$ " replaced by " $\ge$ ".

Proof: We only prove the " $\le$ " case, the " $\ge$ " case can be proved similarly. By using the product rule to calculate

$$\begin{aligned}[y(t)e^{-p(t-t_0)}]' &= y'(t)e^{-p(t-t_0)} - p y(t) e^{-p(t-t_0)} \\ &= (y'(t) - p y(t)) e^{-p(t-t_0)},\end{aligned}$$

then

$$\begin{aligned} y(t)e^{-p(t-t_0)} - y(t_0) &= \int_{t_0}^{t} (y'(s) - p y(s)) e^{-p(s-t_0)} \mathrm{d}s \\ &\le \int_{t_0}^{t} f(s) e^{-p(s-t_0)} \mathrm{d}s,\end{aligned}$$

that is

$$y(t) \le y(t_0) e^{p(t-t_0)} + \int_{t_0}^{t} e^{p(t-s)} f(s) \mathrm{d}s,\ \forall\ t \in \mathbb{R}.$$

The proof is completed. ■

Lemma 2.3 Assume that the assumptions (H_1)-(H_3) are satisfied. Any solution $z(t) = (x_1(t), \cdots, x_n(t), y_1(t), \cdots, y_m(t))$ of system (1) is uniformly bounded on $[0, \infty)$.

Proof: From system (1), for any $t \in [0, \infty)$, we have

$$\begin{aligned} x_i'(t) &\le -\underline{a}_i x_i(t) + \sum_{j=1}^{m} \bar{p}_{ji} \bar{f}_j + \bar{I}_i, \\ y_j'(t) &\le -\underline{b}_j y_j(t) + \sum_{i=1}^{n} \bar{q}_{ij} \bar{g}_i + \bar{L}_j, \end{aligned} \tag{2}$$

and

$$\begin{aligned} x_i'(t) &\ge -\bar{a}_i x_i(t) + \sum_{j=1}^{m} \underline{p}_{ji} \underline{f}_j + \underline{I}_i, \\ y_j'(t) &\ge -\bar{b}_j y_j(t) + \sum_{i=1}^{n} \underline{q}_{ij} \underline{g}_i + \underline{L}_j. \end{aligned} \tag{3}$$

Then, from (2), by Lemma 2.2, for some $t_0 \ge 0$, we have

$$\begin{aligned} x_i(t) &\le x_i(t_0) e^{-\underline{a}_i (t-t_0)} \\ &\quad + \int_{t_0}^{t} e_{-\underline{a}_i}(t, \sigma(s)) \Big[\sum_{j=1}^{m} \bar{p}_{ji} \bar{f}_j + \bar{I}_i \Big] \mathrm{d}s \\ &\le x_i(t_0) e^{-\underline{a}_i (t-t_0)} \\ &\quad + \Big[- \frac{\sum_{j=1}^{m} \bar{p}_{ji} \bar{f}_j + \bar{I}_i}{\underline{a}_i} \Big] (e^{-\underline{a}_i (t-t_0)} - 1) \\ &= e^{-\underline{a}_i (t-t_0)} \Big[x_i(t_0) - \frac{\sum_{j=1}^{m} \bar{p}_{ji} \bar{f}_j + \bar{I}_i}{\underline{a}_i} \Big] \\ &\quad + \frac{\sum_{j=1}^{m} \bar{p}_{ji} \bar{f}_j + \bar{I}_i}{\underline{a}_i} \\ &\le \frac{\sum_{j=1}^{m} \bar{p}_{ji} \bar{f}_j + \bar{I}_i}{\underline{a}_i},\ i = 1, 2, \cdots, n. \end{aligned}$$

Similarly, we can get

$$y_j(t) \le \frac{\sum_{i=1}^{n} \bar{q}_{ij} \bar{g}_i + \bar{L}_j}{\underline{b}_j},\ j = 1, 2, \cdots, m.$$

On another side, from (3), by Lemma 2.2, for some $t_0 \ge 0$, then

$$\begin{aligned} x_i(t) &\ge x_i(t_0) e^{-\bar{a}_i (t-t_0)} \\ &\quad + \int_{t_0}^{t} e_{-\bar{a}_i}(t, \sigma(s)) \Big[\sum_{j=1}^{m} \underline{p}_{ji} \underline{f}_j + \underline{I}_i \Big] \mathrm{d}s \\ &\ge x_i(t_0) e^{-\bar{a}_i (t-t_0)} \\ &\quad + \Big[- \frac{\sum_{j=1}^{m} \underline{p}_{ji} \underline{f}_j + \underline{I}_i}{\bar{a}_i} \Big] (e^{-\bar{a}_i (t-t_0)} - 1) \\ &= e^{-\bar{a}_i (t-t_0)} \Big[x_i(t_0) - \frac{\sum_{j=1}^{m} \underline{p}_{ji} \underline{f}_j + \underline{I}_i}{\bar{a}_i} \Big] \\ &\quad + \frac{\sum_{j=1}^{m} \underline{p}_{ji} \underline{f}_j + \underline{I}_i}{\bar{a}_i} \end{aligned}$$

$$\geq \frac{\sum_{j=1}^{m} \underline{p}_{ji}\underline{f}_j + \underline{I}_i}{\bar{a}_i},\ i = 1, 2, \cdots, n.$$

Similarly, we can get

$$y_j(t) \geq \frac{\sum_{i=1}^{n} \underline{q}_{ij}\underline{g}_i + \underline{L}_j}{\bar{b}_j},\ j = 1, 2, \cdots, m.$$

So, any solution $z(t) = (x_1(t), \cdots, x_n(t), y_1(t), \cdots, y_m(t))$ of system (1) is uniformly bounded on $[0, \infty)$. The proof is completed. ■

III. Global exponential stability

Suppose that $z^* = (x_1^*, x_2^*, \cdots, x_n^*, y_1^*, y_2^*, \cdots, y_m^*)^T = (z_1^*, z_2^*, \cdots, z_{n+m}^*)^T$ is a solution of system (1). In this section, we will construct some suitable differential inequality to study the global exponential stability of this solution. Hereafter, we will use the following norm:

$$\begin{aligned} \|z\| &= \max_{1\leq l\leq n+m} \sup_{t\in\mathbb{R}} |z_l(t)| \\ &= \max\left\{ \max_{1\leq i\leq n} \sup_{t\in\mathbb{R}} |x_i(t)|, \max_{1\leq j\leq m} \sup_{t\in\mathbb{R}} |y_j(t)| \right\}. \end{aligned}$$

Theorem 3.1 Assume that $(H_1) - (H_3)$ hold and if

$$\Upsilon := \begin{bmatrix} A & -PL \\ -Q\Lambda & B \end{bmatrix}_{(n+m)\times(n+m)}$$

is an M-matrix, where $A = \mathrm{diag}(\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_n)_{n\times n}$, $B = \mathrm{diag}(\underline{b}_1, \underline{b}_2, \cdots, \underline{b}_m)_{m\times m}$, $P = (\bar{p}_{ji})_{m\times n}, Q = (\bar{q}_{ij})_{n\times m}$, $L = \mathrm{diag}(\eta_1, \eta_2, \cdots, \eta_m), \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$, then system (1) is global exponential stability.

Proof: Suppose that $z^* = (x_1^*, x_2^*, \cdots, x_n^*, y_1^*, y_2^*, \cdots, y_m^*)^T$ is a solution of system (1), and $z = (x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m)^T$ is another arbitrary solution. Then, system (1) can be written as

$$\begin{cases} \big(x_i(t) - x_i^*(t)\big)' = -a_i(t)x_i(t) + a_i(t)x_i^*(t) \\ \qquad + \sum_{j=1}^{m} p_{ji}(t)(f_j(y_j(t - \tau_{ji}(t))) \\ \qquad - f_j(y_j^*(t - \tau_{ji}(t)))), \\ \big(y_j(t) - y_j^*(t)\big)' = -b_j(t)y_j(t) + b_j(t)y_j^*(t) \\ \qquad + \sum_{i=1}^{n} q_{ij}(t)(g_i(x_i(t - \vartheta_{ij}(t))) \\ \qquad - g_i(x_i^*(t - \vartheta_{ij}(t)))). \end{cases} \tag{4}$$

The initial condition of (4) is $\psi(s) = (\phi_1(s), \cdots, \phi_n(s), \varphi_1(s), \cdots, \varphi_m(s))^T$.

Let $V(t) = |z(t) - z^*(t)|$, the upper right derivative $D^+V(t)$ along the solutions of system (4) is as follows:

$$\begin{aligned} D^+V(t) &= \mathrm{sign}(z(t) - z^*(t))(z(t) - z^*(t))' \\ &\leq \begin{bmatrix} -A & 0 \\ 0 & -B \end{bmatrix} V(t) + \begin{bmatrix} 0 & PL \\ Q\Lambda & 0 \end{bmatrix} \bar{V}(t). \end{aligned}$$

According to Lemma 2.1, then there exist constants $\mu > 0$, $\gamma > 1$, such that

$$|z_l(t) - z_l^*| \leq \gamma \max\left\{ \max_{1\leq i\leq n} \sup_{-\vartheta\leq s\leq 0} |\phi_i(s) - x_i^*|, \max_{1\leq j\leq m} \sup_{-\tau\leq s\leq 0} |\varphi_j(s) - y_j^*| \right\} e^{-\mu t},$$

where $1 \leq l \leq n + m$. So

$$\|z - z^*\| = \gamma\|\psi - z^*\|e^{-\mu t}.$$

From Definition 2.3, the almost periodic solution $z^* = (x_1^*, x_2^*, \cdots, x_n^*, y_1^*, y_2^*, \cdots, y_m^*)^T$ is global exponentially stable. The proof is completed. ■

IV. Almost periodic solution

Suppose that $h(t)$ is an almost periodic function defined on $\mathbb{R}$. Let $H(h(t))$ denote the hull of $h(t)$.

Suppose that

$$\begin{aligned} &a_i^*(t) \in H(a_i(t)),\ b_j^*(t) \in H(b_j(t)),\ p_{ji}^*(t) \in H(p_{ji}(t)), \\ &q_{ij}^*(t) \in H(q_{ij}(t)), \tau_{ji}^*(t) \in H(\tau_{ji}(t)), \\ &\vartheta_{ij}^*(t) \in H(\vartheta_{ij}(t)), I_i^*(t) \in H(I_i(t)),\ L_j^*(t) \in H(L_j(t)) \end{aligned}$$

are selected such that there is a time sequence $\{t_k\}$:

$$\begin{aligned} &a_i(t + t_k) \to a_i^*(t),\ b_j(t + t_k) \to b_j^*(t), \\ &p_{ji}(t + t_k) \to p_{ji}^*(t),\ q_{ij}(t + t_k) \to q_{ij}^*(t), \\ &\tau_{ji}(t + t_k) \to \tau_{ji}^*(t),\ \vartheta_{ij}(t + t_k) \to \vartheta_{ij}^*(t), \\ &I_i(t + t_k) \to I_i^*(t),\ L_j(t + t_k) \to L_j^*(t) \end{aligned}$$

as $k \to \infty$ and $t_k \to \infty$ for all $t \in \mathbb{R}$. Then we get a hull equation of system (1) as follows:

$$\begin{cases} x_i'(t) = -a_i^*(t)x_i(t) + \sum_{j=1}^{m} p_{ji}^*(t)f_j(y_j(t - \tau_{ji}^*(t))) \\ \qquad + I_i^*(t),\ i = 1, 2, \cdots, n, \\ y_j'(t) = -b_j^*(t)y_j(t) + \sum_{i=1}^{n} q_{ij}^*(t)g_i(x_i(t - \vartheta_{ij}^*(t))) \\ \qquad + L_j^*(t),\ j = 1, 2, \cdots, m, \end{cases} \tag{5}$$

According to the almost periodic theory, we can conclude that if system (1) satisfies $(H_1) - (H_3)$, then the hull equation (5) also satisfies $(H_1) - (H_3)$.

For convenience, we write functional differential equation (1) as the following almost periodic functional differential equation

$$z'(t) = F(t, z_t), \tag{6}$$

where $z = (x_1, \cdots, x_n, y_1, \cdots, y_m),\ F(t, z_t) \in C(\mathbb{R} \times S, \Omega)$ is an almost periodic function, and Ω is compact subset of $\mathbb{R}^{n+m}$.

Lemma 4.1 If each of hull equation of system (6) has a unique strictly positive solution, then almost periodic system (1) has a unique strictly positive almost periodic solution.

Proof: Suppose $\varphi(t)$ is a strictly positive solution of system (6) for t on $\mathbb{R}$. There exist sequences of real values $\hat{\alpha}$ and $\hat{\beta}$ which have common subsequence $\alpha \subset \hat{\alpha}$ and $\beta \subset \hat{\beta}$ such that $T_{\alpha+\beta} = T_\alpha T_\beta F(t, z_t)$ for t on $\mathbb{R}$ and $z \in \mathbb{R}^{n+m}$, $T_{\alpha+\beta}\varphi(t)$ and $T_\alpha T_\beta\varphi(t)$ exist uniformly on compact set of $\mathbb{R}$. Then $T_{\alpha+\beta}\varphi(t)$ and $T_\alpha T_\beta\varphi(t)$ are solutions of the following common hull equation of system (6)

$$z'(t) = T_{\alpha+\beta}F(t, z_t).$$

Therefore, we have $T_{\alpha+\beta}\varphi(t) = T_\alpha T_\beta\varphi(t)$ then $\varphi(t)$ is an almost periodic solution of system (6). Since $\alpha \subset \hat{\alpha} = \{\hat{\alpha}\}$ and $\hat{\alpha} \to \infty$ as $k \to \infty$, $T_\alpha F(t, z_t) = F(t, z_t)$ is uniformly tenable with respect to t on $\mathbb{R}$ and $z \in \mathbb{R}^{n+m}$. For the sequences $\hat{\alpha}$ and $\alpha \subset \hat{\alpha}$, we conclude that $T_\alpha \varphi(t) = \psi(t)$ is uniformly tenable with respect to t on $\mathbb{R}$ and $\psi(t) \in \mathbb{R}^{n+m}$. According to the uniqueness of solution and $T_\alpha \psi(t) = \psi(t)$ one obtains that $\varphi(t) = \psi(t)$. The proof is completed. ■

Lemma 4.2 Suppose that conditions $(H_1) - (H_3)$ are satisfied, then there exists a bounded solution $z^*(t),\ t \in \mathbb{R}$ of system (1).

Proof: Since $a_i(t),\ b_j(t),\ p_{ji}(t),\ q_{ij}(t),\ \tau_{ji}(t),\ \vartheta_{ij}(t),\ I_i(t),\ L_j(t)$ are nonnegative almost periodic functions, and with same sequence $\{t_k\}$, as $k \to \infty$ and $t_k \to \infty$ for all t on $\mathbb{R}$, and

$$\begin{array}{l} a_i(t+t_k) \to a_i(t),\ b_j(t+t_k) \to b_j(t), \\ p_{ji}(t+t_k) \to p_{ji}(t),\ q_{ij}(t+t_k) \to q_{ij}(t), \\ \tau_{ji}(t+t_k) \to \tau_{ji}(t),\ \vartheta_{ij}(t+t_k) \to \vartheta_{ij}(t), \\ I_i(t+t_k) \to I_i(t),\ L_j(t+t_k) \to L_j(t) \end{array}$$

If $z(t) = (x_1(t), \cdots, x_n(t), y_1(t), \cdots, y_m(t))$ is a bounded solution of system (1) for $t \geq 0$ corresponding to the initial condition $\psi(t)$, then $z_k(t) = z(t+t_k)$ for $t \geq t_k$ satisfies

$$\left\{ \begin{array}{rcl} x_i'(t) &=& -a_i(t+t_k)x_i(t) \\ && +\sum\limits_{j=1}^{m} p_{ji}(t+t_k) f_j(y_j(t - \tau_{ji}(t+t_k))) \\ && +I_i(t+t_k), \\ y_j'(t) &=& -b_j(t+t_k)y_j(t) \\ && +\sum\limits_{i=1}^{n} q_{ij}(t+t_k) g_i(x_i(t - \vartheta_{ij}(t+t_k))) \\ && +L_j(t+t_k). \end{array} \right.$$

Since $z_k(t)$ is bounded uniformly on $[t_k, \infty),\ k = 1, 2, \cdots$, which implies that $z(t+t_k)$ is also bounded uniformly on $[t_k, \infty),\ k = 1, 2, \cdots$. Hence $z_k(t)$ is bounded uniformly and equicontinuous. So, there exists a subsequence $\{t_k^1\}$ of $\{t_k\}$ with $t_k^1 > t_2$ such that $z(t + t_k^1) \to z^1(t)(k \to \infty)$ and $z^1(t)(t \in [-t_1, \infty))$ satisfies system (1). Similarly, proceeding by induction we have subsequence $\{t_k^n\}$ of $\{t_k^{n-1}\}$ such that $z(t+t_k^n) \to z^n(t)(k \to \infty)$ and $z^n(t)(t \in [-t_k, \infty))$ satisfies system (1). According to the diagonal procedure we have $z(t+t_k^n) \to z^*(t)(k \to \infty)$ and $z^n(t)(t \in [-t_k, \infty))$ converges uniformly on any compact set of $\mathbb{R}$, and z^* satisfies system (1). ■

Theorem 4.1 If almost periodic system (1) satisfies $(H_1)-(H_3)$, then almost periodic system (1) has a unique strictly positive almost periodic solution which is global exponentially stable.

Proof: By Lemma 4.1, we only need to prove that each of hull equation of almost periodic system (1) has a unique strictly positive solution, hence we need firstly prove that each of hull equation of almost periodic system (1) has at least a strictly positive solution (the existence), then we further prove that each of hull equation of system (1) has a unique strictly positive solution (the uniqueness).

Now we prove the existence of strictly positive solution of any hull equation (5). According to the almost periodic hull basic theory, there exists a time sequence $\{t_k\}$:

$$\begin{array}{l} a_i(t+t_k) \to a_i^*(t),\ b_j(t+t_k) \to b_j^*(t), \\ p_{ji}(t+t_k) \to p_{ji}^*(t),\ q_{ij}(t+t_k) \to q_{ij}^*(t), \\ \tau_{ji}(t+t_k) \to \tau_{ji}^*(t),\ \vartheta_{ij}(t+t_k) \to \vartheta_{ij}^*(t), \\ I_i(t+t_k) \to I_i^*(t),\ L_j(t+t_k) \to L_j^*(t) \end{array}$$

as $k \to \infty$ and $t_k \to \infty$ for all t on $\mathbb{R}$. Suppose $z(t) = (x_1(t), \cdots, x_n(t), y_1(t), \cdots, y_m(t))$ is any positive solution of hull equation (5). By the proof of Lemma 2.3, we have

$$0 < \inf_{t \in [0,\infty)} x_i(t) \leq \sup_{t \in [0,\infty)} x_i(t) < \infty, \qquad (7)$$

$$0 < \inf_{t \in [0,\infty)} y_j(t) \leq \sup_{t \in [0,\infty)} y_j(t) < \infty. \qquad (8)$$

Let $z_k(t) = z(t+t_k)$ for all $t \geq -t_k,\ k = 1, 2, \ldots$ such that

$$\left\{ \begin{array}{rcl} x_i'(t) &=& -a_i(t+t_k)x_i(t) \\ && +\sum\limits_{j=1}^{m} p_{ji}(t+t_k) f_j(y_j(t - \tau_{ji}(t+t_k))) \\ && +I_i(t+t_k), \\ y_j'(t) &=& -b_j(t+t_k)y_j(t) \\ && +\sum\limits_{i=1}^{n} q_{ij}(t+t_k) g_i(x_i(t - \vartheta_{ij}(t+t_k))) \\ && +L_j(t+t_k). \end{array} \right. \qquad (9)$$

From inequality (7), (8) and assumptions $(H_1)-(H_3)$, there exists a positive constant vector K which is independent of n such that $z_k'(t) \leq K$ for all $t \geq -t_k,\ k = 1, 2, \ldots$. Therefore, for any positive integer r, sequence $\{z_k(t) : k \geq r\}$ is uniformly bounded and equicontinuous on $[-t_k, \infty)$. According to Ascoli-Arzela Theorem, one concludes that there exists a time subsequence $\{t_k\}$ of $\{t_k\}$ such that sequence $\{z_k(t)\}$ not only converges on t on $\mathbb{R}$, but also converges uniformly on any compact set of $\mathbb{R}$ as $k \to \infty$. Suppose $\lim\limits_{k \to \infty} z_k(t) = z^*(t) = (x_1^*(t), \cdots, x_n^*(t), y_1^*(t), \cdots, y_m^*(t))$, then $z^*(t)$ is continuous on $\mathbb{R}$, and by Lemma 4.2, we have

$$0 < \inf_{t \in (-\infty,\infty)} x_i^*(t) \leq \sup_{t \in (-\infty,\infty)} x_i^*(t) < \infty,$$

$$0 < \inf_{t \in (-\infty,\infty)} y_j^*(t) \leq \sup_{t \in (-\infty,\infty)} y_j^*(t) < \infty.$$

From differential equation (9) and assumptions $(H_1)-(H_3)$, we can easily see that $z^*(t)$ is a solution of hull equation (5), hence each of hull equation of almost periodic system (1) has at least a strictly positive solution.

In the following section, we will prove the uniqueness of strictly positive solution for any hull equation (5). Suppose that the hull equation (5) has two arbitrary strictly positive solutions $z_1^*(t) = (x_1^*(t), \cdots, x_n^*(t), y_1^*(t), \cdots, y_m^*(t))$ and $z_2^*(t) = (\hat{x}_1^*(t), \cdots, \hat{x}_n^*(t), \hat{y}_1^*(t), \cdots, \hat{y}_m^*(t))$. Now by using the method in section 3, then we can get

$$0 \leq \|z_1^* - z_2^*\| \leq \eta \|\psi - z^*\| e^{-\mu t} \to 0, \text{ as } t \to \infty.$$

so, it is proved that any hull equation of system (1) has a unique strictly positive solution.

Summarizing the inference above, we know that any hull equation of system (1) has a unique strictly positive solution. By Lemma 4.1 and Theorem 3.1, almost periodic system (1) has a unique strictly positive almost periodic solution which is global exponentially stable. The proof is completed. ■

V. An example

Consider the following BAM neural networks with time-varying delays

$$\begin{cases} x_i'(t) = -a_i(t)x_i(t) + \sum_{j=1}^{m} p_{ji}(t)f_j(y_j(t-\tau_{ji}(t))) \\ \qquad +I_i(t),\ i=1,2,\cdots,m, \\ y_j'(t) = -b_j(t)y_j(t) + \sum_{i=1}^{n} q_{ij}(t)g_i(x_i(t-\vartheta_{ij}(t))) \\ \qquad +L_j(t),\ j=1,2,\cdots,n, \end{cases} \tag{10}$$

where $I_i(t)=\sin t+1$, $L_j(t)=\cos t+1$, $g_i(x_i(t-\vartheta_{ij}(t)))=\frac{1}{2}\sin(x_i-\tau(t))+1,\ f_j(y_j(t-\tau_{ji}(t)))=\cos(y_j-\tau(t))+1$, $t\in\mathbb{R}$, $\lambda_i=\frac{1}{2},\ \eta_j=1,\ i=j=1,2$. $\tau(t)=3|\cos t|+1$, and

$$a_1(t)=b_2(t)=3-\sin t,\ a_2(t)=b_1(t)=3-\cos t,$$

then

$$\underline{a}_1=\underline{a}_2=\underline{b}_1=\underline{b}_2=2, \bar{a}_1=\bar{a}_2=\bar{b}_1=\bar{b}_2=4.$$

Let

$$\begin{aligned} &p_{11}(t)=0.05\sin t+1,\ p_{12}(t)=0.1\cos t+1, \\ &p_{21}(t)=0.15\cos t+1,\ p_{22}(t)=0.05\sin t+1. \\ &q_{11}(t)=0.25\sin t+1,\ q_{12}(t)=0.05\cos t+1, \\ &q_{21}(t)=0.05\cos t+1,\ q_{22}(t)=0.5\sin t+1. \end{aligned}$$

Then

$$\Upsilon=\begin{bmatrix} 2 & 0 & -1.05 & -1.1 \\ 0 & 2 & -1.15 & -1.05 \\ -0.625 & -0.525 & 2 & 0 \\ -0.525 & -0.75 & 0 & 2 \end{bmatrix}.$$

It is easy to see that $(H_1)-(H_3)$ hold and Υ is an M-matrix. According to Theorems 3.1 and 4.1, system (10) has exactly one positive almost periodic solution, which is globally exponentially stable.

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