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# Quasi-Permutation Representations for the Group $SL(2,q)$ when Extended by a certain Group of order Two

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***Abstract*—** A square matrix over the complex field with non-negative integral trace is called a quasi-permutation matrix. For a finite group $G$ the minimal degree of a faithful representation of $G$ by quasi-permutation matrices over the rationals and the complex numbers are denoted by $q(G)$ and $c(G)$ respectively. Finally $r(G)$ denotes the minimal degree of a faithful rational valued complex character of $G$. The purpose of this paper is to calculate $q(G), c(G)$ and $r(G)$ for the group $SL(2,q)$ when extended by a certain group of order two.

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In [10] Wong defined a quasi-permutation group of degree $n$ to be a finite group $G$ of automorphisms of an $n$-dimensional complex vector space such that every element of $G$ has non-negative integral trace. Also Wong studied the extent to which some facts about permutation groups generalize to the quasi-permutation group situation. In [3] the authors investigated further the analogy between permutation groups and quasi-permutation groups. They also worked over the rational field and found some interesting results.

By a quasi-permutation matrix we mean a square matrix over the complex field $C$ with non-negative integral trace. For a given finite group $G$, let $q(G)$ denote the minimal degree of a faithful representation of $G$ by quasi-permutation matrices over the rational field Q and let $c(G)$ be the minimal degree of a faithful representation of $G$ by complex quasi-permutation matrices and finally let $r(G)$ denote the minimal degree of a faithful rational valued character of $G$. In this paper we will apply the algorithms in [1] to the group $K_2^2(2^n)$ , where

$$K_n^2(q) =< SL(n,q), \theta | \theta^2 = 1, \theta^{-1} A \theta = (A^t)^{-1} > .$$

We will prove

*Theorem 1:* **A)** Let $G = K_2^2(2)$ then $r(G) = 2, c(G) = q(G) = 4$

**B)** Let $K_2^2(q)$ , where $q = 2^n$ . Then

**1)** If $q \equiv -1 \mod 3$ then $r(G) = q - 1$ , $c(G) = q(G) = 2(q-1)$

**2)** Otherwise : $r(G) = q$ , $c(G) = q(G) = 2q$

Let $SL(n,q)$ denote the special general linear group of a vector space of dimension $n$ over a field with $q$ elements . Let $\theta : SL(n,q) \longrightarrow SL(n,q)$ be the automorphism of $SL(n,q)$ given by $\theta(A) = (A^t)^{-1}$ , where $A^t$ denotes the transpose of the matrix $A \in SL(n,q)$ . In this case one can define the split extension $SL(n,q). < \theta >$ that following the notations used in [6] is denoted by $K_n^2(q)$ . Therefore we have $K_n^2(q) =< SL(n,q), \theta | \theta^2 = 1, \theta^{-1} A \theta = (A^t)^{-1} >$ , see[4] .

Now let $G$ denote the group $SL(n,q)$ and let the split extension of $G$ by the cyclic group $< \theta >$ of order 2 be denoted by $G^+$ . Since $[G^+ : G] = 2$ , we have $G^+ = G \cup \theta G$ , and elements of $G^+$ which lie in $G$ are called positive and those outside $G$ are called negative elements . A conjugacy class in $G^+$ is called positive if it lies in $G$ otherwise it is called negative . We may assume that using [7] one can obtain information about conjugacy classes and complex irreducible characters of $G$ , therefore so far as conjugacy classes of $G^+$ are concerned one must pay attention to negative conjugacy classes of $G^+$ .

One can show that there is a one-to-one correspondence between the set of negative conjugacy classes of $G^+$ and the set of equivalence classes of invertible matrices in $G$ .

Now we begin with a summary of facts relevant to the irreducible complex characters of $K_2^2(q)$.
Complex irreducible characters of $G^+$ are divided into two kinds . The group $< \theta >$ acts on the set of complex irreducible characters of $G$ as follows . If $\chi \in Irr(G)$ , then $\chi^\theta(A) := \chi(\theta^{-1} A \theta)$. If $\chi^\theta = \chi$ , then we say that $\chi$ is invariant under $< \theta >$ and in this case $\chi$ forms an orbit of $G^+$ acting on $Irr(G)$ . Now by standard results that can be found in [8] there exists an irreducible character $\varphi$ of $G^+$ such that $\varphi \downarrow_G = \chi$ . Since $G^+/G \cong Z_2$ has two linear characters , therefore multiplication of $\varphi$ with the non-trivial character of $G^+/G$ gives another irreducible character $\varphi'$ of $G^+$ such that $\varphi \downarrow_G = \chi$ . In this case we say that $\chi$ extends to $\varphi$ and $\varphi'$ and it is enough to calculate one of them on the negative conjugacy classes of $G^+$ .
As we mentioned earlier we have $K_2^2(q) = SL(2,q). < \theta > =< SL(2,q), \theta | \theta^2 = 1, \theta^{-1} A \theta = (A^t)^{-1}, \forall A \in SL(2,q) >$ . In the following Lemma we give the structure of $K_2^2(q)$ .

*Lemma 1:* Let $G = K_2^2(q)$ . If $q$ is even , then $K_2^2(q) \cong SL(2,q) \times < \theta >$and if $q$ is odd , then $K_2^2(q) \cong SL(2,q) \circ 4$ a central product of $SL(2,q)$ with the cyclic group of order 4 .

*Proof.* The automorphism $\theta : SL(2,q) \rightarrow SL(2,q)$ is

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given by $\theta(A) = (A^{-1})^t$ for all $A \in SL(2,q)$ . If we set $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ , then it is easy to verify that for $A \in SL(2,q)$ we have $J^{-1}AJ = (A^{-1})^t$ and therefore $\theta$ is equal to an inner automorphism $i_J$ of $SL(2,q)$ .We have $K_2^2(q) =< SL(2,q), \theta >=< SL(2,q), \theta J >= SL(2,q). < \theta J >$ and since $\theta J \in Z(K_2^2(q))$ hence $K_2^2(q) \cong \frac{SL(2,q)\times<\theta J>}{\{(I,I),(-I,-I)\}}$. If the characteristic of $GF(q)$ is even , then we get $K_2^2(q) = SL(2,q)\times < \theta J >\cong SL(2,q)\times < \theta >$ and if the characteristic of $GF(q)$ is odd we obtain $K_2^2(q) \cong SL(2,q) \circ < \theta J >$ the central product of $SL(2,q)$ with a cyclic group of order 4 .

By [5] we have two important lemmas as follows

*Lemma 2: a)* Let $V_i (i = 1,2)$ be $KG$-modules . Then the tensor product $V_1 \otimes_K V_2$ over $K$ obviously becomes a $K[G_1 \times G_2]$ module by

$$(v_1 \otimes v_2)(g_1, g_2) = v_1 g_1 \otimes v_2 g_2$$

For $v_i \in V_i, g_i \in G$ .
If $\chi_i$ is the character of $G_i$ on $V_i$ , then the character $\tau$ of $G_1 \times G_2$ on $V_1 \otimes V_2$ is given by

$$\tau((g_1, g_2)) = \chi_1(g_1)\chi_2(g_2)$$

For $g_i \in G_i$ .

*b)* Let $\chi_1, ..., \chi_h$ be the irreducible characters of $G_i$ over $C$ and $\psi_1, ..., \psi_k$ the irreducible characters of $G_2$ over $C$ . Then the $t_{ij}$ defined by $t_{ij}((g_1, g_2)) = \chi_i(g_1)\psi_j(g_2)$ where $i = 1, ..., h$ and $j = 1, ..., k$ are all the irreducible characters of $G_1 \times G_2$ .

*Lemma 3:* Let $F$ be the finite field of $q = 2^n$ elements , and let $\nu$ be a generator of the cyclic group $F^* = F - 0$ . Denote

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, a = \begin{pmatrix} \nu & 0 \\ 0 & \nu^{-1} \end{pmatrix}$$

in $G = SL(2, F)$ . $G$ contains an element $b$ of order $q + 1$ . For any $x \in G$ , let $(x)$ denote the conjugacy class of $G$ containing $x$ . Then $G$ has exactly $q + 1$ conjugacy classes $(1), (c), (a), (a^2), ..., (a^{(q-2)/2}), (b), ..., (b^{q/2})$ , where

**Table** (1)
Conjugacy Classes of $SL(2, 2^n)$

| $x$ | $1$ | $c$ | $a^l$ | $b^m$ |
|---|---|---|---|---|
| $\|(x)\|$ | $1$ | $q^2-1$ | $q(q+1)$ | $q(q-1)$ |

for $1 \le l \le (q-2)/2, 1 \le m \le q/2$.
let $\rho \in C$ be a primitive $(q-1)$-th root of 1 , table of $G$ over $C$ is

**Table** (2)
Character Table of $SL(2, 2^n)$

| | $1$ | $c$ | $a^l$ | $b^m$ |
|---|---|---|---|---|
| $1_G$ | $1$ | $1$ | $1$ | $1$ |
| $\psi$ | $q$ | $0$ | $l$ | $-1$ |
| $\chi_i$ | $q+1$ | $1$ | $\rho^{il} + \rho^{-il}$ | $0$ |
| $\theta_j$ | $q-1$ | $-1$ | $0$ | $-(\sigma^{jm} + \sigma^{-jm})$ |

for $1 \le i \le (q-2)/2, 1 \le j \le q/2, 1 \le l \le (q-2)/2, 1 \le m \le q/2$.

*Remark 1:* In the case of $q$ even $< \theta J >$ has order 2 and its irreducible characters are denoted by $\mu_0$ and $\mu_1$ where $\mu_0$ is the identity character . Regarding the structure of $K_2^2(q)$ and Lemmas $2, 3$ the irreducible characters of $K_2^2(q)$ in the case of $q$ even are $\mu_k 1_G, \mu_k\psi, \mu_k\chi_i$ and $\mu_k\theta_j$ where $k = 0, 1$ and $1 \le i \le \frac{q-2}{2}, 1 \le j \le \frac{q}{2}$.

*Lemma 4:* Let $G = SL(2,q)$ , if $q$ is a power of 2 then the Schur index of any irreducible character of $G$ over the rational numbers $Q$ is 1 .

*Proof.* See [9] .

By [9] it is easy to see that :

*Lemma 5:* Let $G = H \times K$ and $\psi \in Irr(H)$ and $\theta \in Irr(K)$. Let $\chi = \psi \times \theta$ and let $F \subseteq C$.
**a)** $m_F(\chi)$ divides $m_F(\psi)m_F(\theta)$ .
**b)** Equality occurs in $(a)$ provided $(m_F(\psi), \theta(1)|F(\theta) : F|) = 1$ and
$(m_F(\theta), \psi(1)|F(\psi) : F|) = 1$

*Lemma 6:* Let $G$ be a finite group . If the Schur index of each non-principal irreducible character is equal to $m$ , then $q(G) = mc(G)$ .
*Proof.* See [1],Corollary 3.15 .

We can see all the following statements in [1],[2].

*Definition 1:* Let $\chi$ be a complex charater of $G$, such that, $\ker\chi = 1$. Then define

**1)** $d(\chi) = |\Gamma(\chi)|\chi(1)$
**2)** $m(\chi) = \begin{cases} 0 & \chi = 1_G \\ |\min\{\sum_{\alpha\in\Gamma(\chi)}\chi^\alpha(g) : g \in G\}| & otherwise \end{cases}$
**3)** $c(\chi) = \sum_{\alpha\in\Gamma(\chi)}\chi^\alpha + m(\chi)1_G$.

Now according to Corollary 3.11 of [1] and above statements the following lemma is useful for calculation of $r(G), c(G)$ and $q(G)$.

*Lemma 7:* Let $G$ be a finite group with a unique minimal normal subgroup. Then

**1)** $r(G) = \min\{d(\chi) : \chi$ is a faithful irreducible complex character of $G\}$

**2)** $c(G) = \min\{c(\chi)(1) : \chi$ is a faithful irreducible complex character of $G\}$

**3)** $q(G) = \min\{m_Q(\chi)c(\chi)(1) : \chi$ is a faithful irreducible complex character of $G\}$.

By [2] we have the following lemmas .

*Lemma 8:* Let $\varepsilon$ be a primitive $n$-th root of unity in $C$. Then $\varepsilon + \varepsilon^{-1}$ is rational if and only if $n = 1, 2, 3, 4, 6$. The values which occur are as follows:

**Table** (3)

| $n$ | 1 | 2 | 3 | 4 | 6 |
|---|---|---|---|---|---|
| $+^{-1}$ | 2 | $-2$ | $-1$ | 0 | 1 |

*Lemma 9:* Let $\varepsilon$ be a primitive $n$-th root of unity in $C$ and $m \in Z$ . If $\varepsilon + \varepsilon^{-1}$ is rational , then so $\varepsilon^m + \varepsilon^{-m}$ .

*Lemma 10:* Let $\varepsilon$ be a primitive $n$-th root of unity. Then $\varepsilon^j + \varepsilon^{-j}, 1 \leq j \leq n$ is rational if and only if $n = j, 2j, 3j, 4j, 6j, \frac{3}{2}j, \frac{4}{3}j, \frac{6}{5}j$.

*Lemma 11:* Let $\chi \in Irr(G), \chi \neq 1_G$ . Then $c(\chi)(1) \geq d(\chi) + 1 \geq \chi(1) + 1$ .

*Proof.* From Definition 1 it follows that $c(\chi)(1)$ is a non-negative rational valued character of $G$ so by [1], Lemma 3.2 , $m(\chi) \geq 1$ . Now the result follows from Definition 1 .

*Lemma 12:* Let $\chi \in Irr(G)$. Then

(1) $c(\chi)(1) \geq d(\chi) \geq \chi(1)$ ;

(2) $c(\chi)(1) \leq 2d(\chi)$ . Equality occurs if and only if $Z(\chi)/ker\chi$ is of even order .

*Proof.* (1) follows from the definition of $c(\chi)(1)$ and $d(\chi)$.(2) follows from [1] Lemma 3.13 .

*Lemma 13:* Let $G = SL(2, q)$ where $q = 2^n$ and $n \geq 2$ . Then for each $j$ , $1 \leq j \leq q/2$ ,

(1) $\theta_j$ is rational if and only if $q \equiv -1$ mod 3 and $j = \frac{q+1}{3}$;

(2) $d(\theta_j) \geq q - 1$ and equality holds if $\theta_j$ is rational ;

(3) $c(\theta_j) \geq q + 1$ and equality holds if $\theta_j$ is rational .

*Proof* . As $1 \leq j \leq \frac{q}{2} < \frac{q+1}{2}$ and as $\sigma$ is a primitive $(q + 1)$-th root of unity , Lemmas 9 and 10 implies that $\theta_j$ is rational if and only if $j = \frac{q+1}{6}, \frac{q+1}{4}, \frac{q+1}{3}$. Since $q + 1$ is odd , $\frac{q+1}{6}$ and $\frac{q+1}{4}$ are not integers . Thus , $\sigma^j + \sigma^{-j} \in Q$ if and only if $3|q + 1$ and $j = \frac{q+1}{3}$ . This proves (1) . If $\theta_j$ is not rational , then $|\Gamma| \geq 2$ where $\Gamma = \Gamma(Q(\theta_j) : Q)$ so that $c(\theta_j)(1) \geq d(\theta_j) \geq 2(q - 1) > q + 1$ by Lemma 12 . On the other hand if $3|q + 1$ , then $8 \leq q$, so that $3 \leq \frac{q}{2}$ ; but $\theta_{\frac{q+1}{3}}(b_3) = -2 \leq \theta_{\frac{q+1}{3}}(g)$ for all $g \in G$ so that $m(\theta_{\frac{q+1}{3}}) = 2$ . Thus $d(\theta_{\frac{q+1}{3}}) = q - 1$ and $c(\theta_{\frac{q+1}{3}})(1) = q + 1$ . This completes the proofs of (2) and (3) .

*Theorem 2:* Let $G = K_2^2(2)$ then $r(G) = 2, c(G) = q(G) = 4$

*Proof.* By Lemmas 4, 5 Schur index of each irreducible characters is 1 and so by Lemma 6 we have $c(G) = q(G)$ . Since the only faithful irreducible character of $G$ is $\mu_1\psi$ so by the character table of the group $K_2^2(2)$ the result follows .

*Theorem 3:* Let $G = K_2^2(q)$ , where $q = 2^n$ . Then

**1)** If $q \equiv -1$ mod 3 then $r(G) = q - 1$ , $c(G) = q(G) = 2(q - 1)$

**2)** Otherwise $r(G) = q$ , $c(G) = q(G) = 2q$.

*Proof.* By Lemmas 4, 5 Schur index of each irreducible characters is 1 and so by Lemma 6 we have $c(G) = q(G)$ .

By Lemma 3 we have the irreducible characters of $SL(2, q)$ and by Remark 1 we have the irreducible characters of $K_2^2(2^n)$ , now by definition of $d(\chi)$ , $c(\chi)$ and character table of $K_2^2(q)$ we obtain the Table (4) as follows

**Table (4)**

| $\chi$ (faithful) | $d(\chi)$ | $c(\chi)(1)$ |
|---|---|---|
| $\mu_1\psi$ | $q$ | $2q$ |
| $\mu_1\chi_i$ | $\geq q + 1$ | $\geq 2(q + 1)$ |
| $\mu_1\theta_j$ | $\geq (q - 1)$ | $\geq 2(q - 1)$ |

Now by Lemma 7 , Lemma 13 and Table (4) when $q \equiv -1$ mod3 we have

min $\{d(\chi) : \chi$ is a faithful irreducible complex character of $G\} = q - 1$ and

min $\{c(\chi)(1) : \chi$ is a faithful irreducible complex character of $G\} = 2(q - 1)$.

Otherwise $d(\mu_1\theta_j) > q-1$ and so in this case min $d(\chi) = q$ and min $c(\chi) = 2q$.

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