International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:6, 2013

A Family of Zero Stable Block Integrator for the Solutions of Ordinary Differential Equations

A. M. Sagir

Abstract**—**In this paper, linear multistep technique using power series as the basis function is used to develop the block methods which are suitable for generating direct solution of the special second order ordinary differential equations with associated initial or boundary conditions. The continuous hybrid formulations enable us to differentiate and evaluate at some grids and off – grid points to obtain two different four discrete schemes, each of order $(5,5,5,5)^T$, which were used in block form for parallel or sequential solutions of the problems. The computational burden and computer time wastage involved in the usual reduction of second order problem into system of first order equations are avoided by this approach. Furthermore, a stability analysis and efficiency of the block methods are tested on linear and non-linear ordinary differential equations and the results obtained compared favorably with the exact solution.

Keywords**—**Block Method, Hybrid, Linear Multistep Method, Self – starting, Special Second Order.

I. Introduction

LET us consider the numerical solution of the special second order ordinary differential equation of the form

$$y'' = f(x, y), \quad a \le x \le b \tag{1}$$

with associated initial or boundary conditions. The mathematical models of most physical phenomena especially in mechanical systems without dissipation leads to special second order initial value problem of type (1). Solutions to initial value problem of type (1) according to Fatunla [1], [2] are often highly oscillatory in nature and thus, severely restrict the mesh size of the conventional linear multistep method. Such system often occurs in mechanical systems without dissipation, satellite tracking, and celestial mechanics.

Lambert [3] and several authors such as Onumanyi *et al* [4], Awoyemi [5], and Fudziah *et al*. [6], have written on conventional linear multistep method

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j}, k \ge 2 \tag{2}$$

or compactly in the form

$$\rho(E) y_n = h^2 \delta(E) f_n \tag{3}$$

A.M. Sagir is with the Department of Basic Studies, College of Basic and Remedial Studies, Hassan Usman Katsina Polytechnic, P.M.B. 2052 Katsina. Katsina State. Nigeria (phone: 08039474856, e-mail: amsagir@yahoo.com).

where E is the shift operator specified by $E^j y_n = y_{n+j}$ while ρ and δ are characteristics polynomials and are given as

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j, \ \delta(\xi) = \sum_{j=0}^{k} \beta_j \xi^j \tag{4}$$

y_n is the numerical approximation to the theoretical solution y(x) and $f_n = f(x_n, y_n)$.

In the present consideration, our motivations for the study of this approach is a further advancement in efficiency, i.e obtaining the most accuracy per unit of computational effort, that can be secured with the group of methods proposed in this paper over Awoyemi [5] and Adeboye [7].

A. Definition: Consistent, Lambert [3]

The linear multistep method (2) is said to be consistent if it has order $p \ge 1$, that is if

$$\sum_{j=0}^{k} \alpha_j = 0 \text{ and } \sum_{j=0}^{k} j \alpha_j - \sum_{j=0}^{k} \beta_j = 0 \tag{5}$$

Introducing the first and second characteristics polynomials (4), we have from (5) LMM type (2) is consistent if

$$\rho(1) = 0, \ \rho^1(1) = \delta(1)$$

B. Definition: Zero stability, Lambert [3]

A linear multistep method type (2) is zero stable provided the roots $\xi_j, j = 0(1)k$ of first characteristics polynomial $\rho(\xi)$ specified as $\rho(\xi) = \det\left|\sum_{j=0}^{k} A(i)\xi^{(k-i)}\right| = 0$ satisfies $|\xi_j| \le 1$ and for those roots with $|\xi_j| = 1$ the multiciplicity must not exceed two. The principal root of $\rho(\xi)$ is denoted by

$$\xi_1 = \xi_2 = 1$$

C. Definition: Convergence, Lambert [3]

The necessary and sufficient conditions for the linear multistep method type (2) is said to be convergent if it is consistent and zero stable.

D. Definition: Order and Error Constant, Lambert [3]

The linear multistep method type (2) is said to be of order p if $c_0 = c_1 = \cdots c_{P+1} = 0$ but $c_{p+2} \ne 0$ and c_{p+2} is called the error constant, where

$c_0 = \sum_{j=0}^{k} \alpha_j = \alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_k$

$c_1 = \sum_{j=0}^{k} j\, \alpha_j = (\alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + k\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$

$c_2 = \sum_{j=0}^{k} \frac{1}{2!} j^2 \alpha_j - \sum_{j=0}^{k} \beta_j$

$$= \begin{Bmatrix} \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + 3^3\alpha_3 + \ldots + k^2\alpha_k) \\ -(\beta_1 + 2\beta_2 + 3\beta_3 + \ldots + k\beta_k) \end{Bmatrix}$$

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$c_q = \sum_{j=1}^{k} \{\frac{1}{q!} j^q \alpha_j - \frac{1}{(q-2)!} j^{q-2} \beta_j\}$

$$= \begin{Bmatrix} \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + 3^q\alpha_3 + \ldots + k^q\alpha_k) \\ -\frac{1}{(q-1)!}(\beta_1 + 2^{(q-1)}\beta_2 + 3^{(q-1)}\beta_3 + \cdots + k^{(q-1)}\beta_k) \end{Bmatrix} \quad (6)$$

E. Theorem: Lambert, [3]

Let $f(x, y)$ be defined and continuous for all points (x, y) in the region D defined by $\{(x, y) : a \le x \le b, -\infty < y < \infty\}$ where *a and b* finite, and let there exist a constant L such that for every x, y, y^* such that (x, y) *and* (x, y^*) are both in D:

$$|f(x, y) - f(x, y^*)| \le L|y - y^*| \quad (7)$$

Then if η is any given number, there exist a unique solution $y(x)$ of the initial value problem (1), where $y(x)$ is continuous and differentiable for all (x, y) *in D*. The inequality (7) is known as a Lipschitz condition and the constant L as a Lipschitz constant.

II. Derivation of the Proposed Methods

We proposed an approximate solution to (1) in the form

$$y(x) = \sum_{j=0}^{t+m-1} a_j x^i = y_{n+j},\ i = 0(1)m + t - 1 \quad (8)$$

$$y''(x) = \sum_{j=0}^{t+m-1} i(i-1) a_j x^{i-2} = f_{n+j} \quad (9)$$

$i = 2(3)m + t - 1$ with $m = 5$, $t = 2$ and $p = m+t-1$

where the $a_j, j = 0,1,(m + t - 1)$ are the parameters to be determined, t and m are points of interpolation and collocation respectively. Where P, is the degree of the polynomial interpolant of our choice.

Specifically, we collocate (9) at $x = x_{n+j}, j = 0(1)k$ and interpolate (8) at $x = x_{n+j},\ j = 0(1)k - 2$ using the method described above. Putting in the matrix equation form and then solved to obtain the values of parameters α_j's, j = 0, 1, … which is substituted in (8) yields, after some algebraic manipulation, the new continuous form for the solution

$$y(x) = \sum_{j=0}^{k-2} \alpha_j(x) y_{n+j} + \sum_{j=0}^{k} \beta_j(x) f_{n+j} \quad (10)$$

A. Derivation of First Block Method

Let us consider the numerical solution of the second order differential system of type (1). Put (8) and (9) in matrix equation form, which when solved either by matrix inversion techniques or Gaussian elimination method to obtain the values of the parameters $\alpha_j, j = 0,1, m + t - 1$ and then substituting them into (8) to give the continuous form:

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + h^2[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_{4/3}(x)f_{n+4/3} + \beta_2(x)f_{n+3}] \quad (11)$$

We set $\rho = (x - x_{n+1})$

If we let k = 3, after some algebraic manipulations we obtain a continuous form of solution

$$y(x) = \{-(\rho)\}y_n + \left\{\left(\frac{h+\rho}{h}\right)\right\}y_{n+1}$$
$$+ \left\{\frac{6(\rho)^6 - 30h(\rho)^5 + 45h^2(\rho)^4 - 20h^3(\rho)^3 + 101h^5(\rho)}{1440h^4}\right\} f_n$$
$$+ \left\{\frac{-6(\rho)^6 + 21h(\rho)^5 + 5h^2(\rho)^4 - 70h^3(\rho)^3 + 60h^4(\rho)^2 + 108h^5(\rho)}{120h^4}\right\} f_{n+1}$$
$$+ \left\{\frac{-6(\rho)^6 + 12h(\rho)^5 + 25h^2(\rho)^4 - 20h^3(\rho)^3 + 27h^5(\rho)}{240h^4}\right\} f_{n+2}$$
$$+ \left\{\frac{54(\rho)^6 - 162h(\rho)^5 - 135h^2(\rho)^4 + 540h^3(\rho)^3 - 459h^5(\rho)}{800h^4}\right\} f_{n+\frac{4}{3}}$$
$$+ \left\{\frac{6(\rho)^6 - 3h(\rho)^5 - 15h^2(\rho)^4 + 10h^3(\rho)^3 - 16h^5(\rho)}{1800h^4}\right\} f_{n+3} \quad (8)$$
(12)

Evaluating (12) at $x = x_{n+4/3}$, $x = x_{n+2}$ and $x = x_{n+3}$, yield the following schemes:

(a). $y_{n+\frac{4}{3}} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{h^2}{437400}\{10135f_n + 146580f_{n+1} + 15690f_{n+2} - 73953f_{n+4/3} - 1252f_{n+3}\}$

(b). $y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{1200}\{85f_n + 1180f_{n+1} + 190f_{n+2} - 243f_{n+4/3} - 12f_{n+3}\}$

(c). $y_{n+3} - 3y_{n+1} + 2y_n = \frac{h^2}{1200}\{155f_n + 2640f_{n+1} + 1470f_{n+2} - 729f_{n+\frac{4}{3}} + 64f_{n+3}\}$

(13)

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:6, 2013

Taking the first derivative of (12), thereafter, evaluate the resulting continuous polynomial solution at $x = x_0$ yields

$$\text{(d). } hz_0 - y_{n+1} + y_n = \frac{h^2}{7200}\left\{-1625f_n - 6060f_{n+1} - 1110f_{n+2} + 5103f_{n+\frac{4}{3}} + 92f_{n+3}\right\} \quad (14)$$

B. Derivation of Second Block Method

Using the same procedure as in first block method, but with one off grid point at interpolation, we obtain another

$$y(x) = \alpha_0(x)y_n + \alpha_1(x)y_{n+1} + \alpha_{4/3}(x)y_{n+4/3} + h^2[\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_2(x)f_{n+3}] \quad (15)$$

We set $\delta = (x - x_{n+1})$

$$\begin{aligned} y(x) &= \left\{\frac{-486(\delta)^6 + 1458h(\delta)^5 + 1215h^2(\delta)^4 - 4860h^3(\delta)^3 + 479h^5(\delta)}{3652h^6}\right\}y_n \\ &+ \left\{\frac{486(\delta)^6 - 1458h(\delta)^5 - 1215h^2(\delta)^4 + 4860h^3(\delta)^3 - 3218h^5(\delta) + 913h^6}{913h^6}\right\}y_{n+1} \\ &+ \left\{\frac{-1458(\delta)^6 + 4374h(\delta)^5 + 3645h^2(\delta)^4 - 14580h^3(\delta)^3 + 12393h^5(\delta)}{3652h^6}\right\}y_{n+\frac{4}{3}} \\ &+ \left\{\frac{4410(\delta)^6 - 15969h(\delta)^5 + 2670h^2(\delta)^4 + 25840h^3(\delta)^3 - 2791h^5(\delta)}{328680h^4}\right\}f_n \\ &+ \left\{\frac{9180(\delta)^6 - 2480h(\delta)^5 - 32080h^2(\delta)^4 + 82670h^3(\delta)^3 + 54780h^4(\delta)^2 - 25989h^5(\delta)}{109560h^4}\right\}f_{n+1} \\ &+ \left\{\frac{-1170(\delta)^6 + 771h(\delta)^5 + 7490h^2(\delta)^4 + 6560h^3(\delta)^3 - 1011h^5(\delta)}{109560h^4}\right\}f_{n+2} \\ &+ \left\{\frac{720(\delta)^6 + 579h(\delta)^5 - 1800h^2(\delta)^4 - 1930h^3(\delta)^3 + 271h^5(\delta)}{328680h^4}\right\}f_{n+3} \end{aligned} \quad (16)$$

Evaluating (16) at a certain point and taking the first and its second derivatives w.r.t x at some selected points yield the following schemes:

$$\text{(a). } y_{n+2} - \frac{2187}{1826}y_{n+\frac{4}{3}} - \frac{368}{913}y_{n+1} + \frac{1097}{1826}y_n = \frac{h^2}{2739}\left\{118f_n + 1594f_{n+1} + 316f_{n+2} - 18f_{n+3}\right\}$$

$$\text{(b). } y_{n+3} - \frac{6561}{1826}y_{n+\frac{4}{3}} + \frac{1635}{913}y_{n+1} + \frac{1465}{1826}y_n = \frac{h^2}{10956}\left\{503f_n + 10911f_{n+1} + 12009f_{n+2} + 697f_{n+3}\right\}$$

$$\text{(c). } \frac{5400}{913}y_{n+\frac{4}{3}} - \frac{7200}{913}y_{n+1} + \frac{1800}{913}y_n = \frac{h^2}{73953}\left\{10135f_n + 146580f_{n+1} - 73953f_{n+\frac{4}{3}} + 15690f_{n+2} - 1252f_{n+3}\right\}$$

$$\text{(d). } hz_0 + \frac{15309}{3652}y_{n+\frac{4}{3}} - \frac{6016}{913}y_{n+1} + \frac{8755}{3652}y_n = \frac{h^2}{41085}\left\{-5282f_n + 23136f_{n+1} - 156f_{n+2} + 32f_{n+3}\right\} \quad (17)$$

Equations (13), (14), and (17) constitute the member of a zero stable block integrators each of order $(5,5,5,5)^T$ with

$$c_7 = \left(\frac{2351}{3936600}, \frac{7}{3600}, \frac{1}{600}, -\frac{143}{50400}\right)$$

and

$$c_7 = \left(\frac{101}{82170}, -\frac{7}{14608}, \frac{2351}{665577}, -\frac{32}{95865}\right)$$

respectively. The application of the block integrators with n = 0 gives the accurate values of unknown as shown in tables I – IV of forth section of this paper. To start the IVP integration on the sub interval $[X_0, X_3]$, we combine (13) and (14), when n = 0 i.e the 1-block 4-point method.

III. Stability Analysis

Recall, that, it is a desirable property for a numerical integrator to produce solution that behave similar to the theoretical solution to a problem at all times. Thus, several definitions, which call for the method to posses some "adequate" region of absolute stability, can be found in several literatures. See Lambert [3], Fatunla [1], [2] etc.

The Fatunla's approach states that the block method is presented as a single block r-point multi-step method of the form:

$$A^{(0)}Y_m = \sum_{i=1}^{k} A^i Y_{m-i} + h^2 \sum_{i=0}^{k} B^{(i)} F_{m-i} \quad (18)$$

where, h is a fixed mesh size within a block, A^i, B^i, i = 0 (1)k are r x r matrix coefficients, and A^0 is r by r identity matrix, Y_m, Y_{m-i}, F_m and F_{m-i} are vectors of numerical estimates.

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:7, No:6, 2013

Following Fatunla [1], [2]; the four integrators proposed in this report in (13) and (14) are put in the matrix equation form and for easy analysis the result was normalized to obtain;

A. Convergence Analysis of the First Block Method with One Off-Grid Point at Collocation

The method is expressed in the form of type (18) to give

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+\frac{4}{3}}\\y_{n+2}\\y_{n+3}\end{bmatrix}=\begin{bmatrix}0&0&-\frac{1}{3}&1\\0&0&-1&3\\0&0&-2&6\\0&0&-1&3\end{bmatrix}\begin{bmatrix}y_{n-2}\\y_{n-\frac{5}{3}}\\y_{n-1}\\y_n\end{bmatrix}$$

$$+h^2\left\{\begin{bmatrix}-\frac{913}{5400}&\frac{523}{14580}&-\frac{313}{109350}&-\frac{14}{81}\\-\frac{81}{400}&\frac{19}{120}&-\frac{1}{100}&-1\\-\frac{243}{400}&\frac{49}{40}&\frac{4}{75}&-4\\\frac{567}{800}&-\frac{37}{240}&\frac{23}{1800}&\frac{1}{6}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+\frac{4}{3}}\\f_{n+2}\\f_{n+3}\end{bmatrix}+\begin{bmatrix}0&0&0&\frac{2027}{87480}\\0&0&0&\frac{17}{240}\\0&0&0&\frac{31}{240}\\0&0&0&-\frac{65}{288}\end{bmatrix}\begin{bmatrix}f_{n-2}\\f_{n-\frac{5}{3}}\\f_{n-1}\\f_n\end{bmatrix}\right\} \quad (19)$$

with $y_0=\begin{pmatrix}y_0\\hz_0\end{pmatrix}$ usually giving along the initial value problem. The first characteristics polynomial of the proposed 1- block 4 – point method is given by

$$\rho(\lambda)=\det[\lambda I-A_1^{(1)}] \quad (20)$$

$$\rho(\lambda)=\det\begin{bmatrix}\lambda&0&\frac{1}{3}&-1\\0&\lambda&1&-3\\0&0&\lambda+2&-6\\0&0&1&\lambda-3\end{bmatrix} \quad (21)$$

Solving the determinant of (18), yields

$$\rho(\lambda)=\lambda^3(\lambda-1)$$

which implies,

$$\lambda_1=\lambda_2=\lambda_3=0 \text{ or } \lambda_4=1$$

By definition of zero stable and (21), the 1 - block 4 - point method is zero stable and is also consistent as its order $(5,5,5,5)^T>1$, thus, it is convergent following Henrici [8] and Fatunla [2].

B. Convergence Analysis of the Block Method with One Off-Grid Point at Interpolation

The method expressed in the form of (18) to give:

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+\frac{4}{3}}\\y_{n+2}\\y_{n+3}\end{bmatrix}=\begin{bmatrix}0&-\frac{1097}{1826}&-\frac{2399139}{1460800}&\frac{3291}{1826}\\0&-\frac{1465}{1826}&-\frac{640791}{292160}&\frac{4395}{1826}\\0&-\frac{1800}{913}&-\frac{19683}{3652}&\frac{5400}{913}\\0&-\frac{8755}{3652}&-\frac{3829437}{584320}&\frac{26265}{3652}\end{bmatrix}\begin{bmatrix}y_{n-2}\\y_{n-\frac{5}{3}}\\y_{n-1}\\y_n\end{bmatrix}$$

$$+h^2\left\{\begin{bmatrix}\frac{316}{2739}&-\frac{6}{913}&\frac{26541}{45650}&-\frac{724}{913}\\\frac{4003}{3652}&\frac{697}{10956}&\frac{387909}{146080}&-\frac{3085}{913}\\\frac{5230}{24651}&-\frac{1252}{73953}&\frac{72491}{123255}&-\frac{2800}{2739}\\-\frac{52}{13695}&\frac{32}{41085}&\frac{9018}{22825}&-\frac{1528}{2739}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+\frac{4}{3}}\\f_{n+2}\\f_{n+3}\end{bmatrix}+\begin{bmatrix}0&\frac{118}{2739}&\frac{43011}{365200}&-\frac{118}{913}\\0&\frac{503}{10956}&\frac{366687}{2921600}&-\frac{503}{3652}\\0&\frac{10135}{73953}&\frac{54729}{146080}&-\frac{10135}{24651}\\0&-\frac{5282}{41085}&-\frac{641763}{1826000}&\frac{5282}{13695}\end{bmatrix}\begin{bmatrix}f_{n-2}\\f_{n-\frac{5}{3}}\\f_{n-1}\\f_n\end{bmatrix}\right\}$$

(22)

IV. Implementation of the Methods

This section deals with numerical experiments by considering the derived discrete schemes in block form for solution of second order initial value problems. The idea is to enable us see how the proposed methods performs when compared with exact solutions. The results are summarized in Table I to IV.

A. Numerical Experiment

From Awoyemi [5];

Consider a Non-Linear IVP;

$$y''=2y^3; y(1)=1, y'(1)=-1,$$

whose exact solution is

$$y(x)=1/x$$

The first characteristics polynomial of the proposed method (17) is given as in (18).

Substituting the values of λI and $A_1^{(1)}$ in (20), gives, $\rho(\lambda)=\lambda^3(\lambda-1)$, which implies, $\lambda_1=\lambda_2=\lambda_3=0$ or $\lambda_4=1$ which implies, $\lambda_1=\lambda_2=0$ or $\lambda_3=1$

From (18) and (21), the block hybrid method (17) is zero stable and is also consistent as its order $(5,5,5,5)^T>1$, thus, it is convergent as in [8] and [2].

B. Numerical Experiment

From Adeboye [7]; Consider the BVP

$$y''-y=4x-5; y(0)=y(1)=0, h=0.1,$$

whose exact solution is

$$y = \frac{7}{4(e^2 - e^{-2})}\left[e^{2x} - e^{-2x}\right] - \frac{3}{4}x$$

TABLE I
RESULTS FOR THE PROPOSED METHOD PRESENTED IN (13) AND (14) WITH ONE OFF - GRID POINT AT COLLOCATION

N	x	Exact Value	Approximate Value	Awoyemi [5]	Error of Proposed Method
0	1	1	1	0	0
1	1.1	0.909090109	0.9090914826	2.8483722E-03	1.37360E-06
2	1.2	0.833333333	0.8333348875	2.26883436E-01	1.55450E-06
3	1.3	0.769230769	0.7692330259	7.3968630E+00	2.25690E-06
4	1.4	0.714285714	0.7142880945	2.1168783E-01	2.38050E-06
5	1.5	0.666666667	0.6666693006	3.3156524E-01	2.63360E-06
6	1.6	0.625	0.6250029040	4.3968593E-01	2.90400E-06
7	1.7	0.588235294	0.5882382492	5.3903097E-01	2.95520E-06
8	1.8	0.555555556	0.5555586357	6.3121827E-01	3.07970E-06
9	1.9	0.526315789	0.5263190397	7.1723621E-01	3.25070E-06
10	2.0	0.5	0.5000032814	7.9776590E-01	3.28140E-06

TABLE II
RESULTS FOR THE PROPOSED METHOD PRESENTED IN (13) AND (14) WITH ONE OFF - GRID POINT AT INTERPOLATION

N	x	Exact Value	Approximate Value	Awoyemi [5]	Error of Proposed Method
0	1	1	1	0	0
1	1.1	0.909090109	0.9090914832	2.8483722E-03	1.37420E-06
2	1.2	0.833333333	0.8333348886	2.26883436E-01	1.55560E-06
3	1.3	0.769230769	0.7692330281	7.3968630E+00	2.25910E-06
4	1.4	0.714285714	0.7142880973	2.1168783E-01	2.38330E-06
5	1.5	0.666666667	0.6666693038	3.3156524E-01	2.63680E-06
6	1.6	0.625	0.6250029082	4.3968593E-01	2.90820E-06
7	1.7	0.588235294	0.5882382539	5.3903097E-01	2.95990E-06
8	1.8	0.555555556	0.5555586407	6.3121827E-01	3.08470E-06
9	1.9	0.526315789	0.5263190456	7.1723621E-01	3.25660E-06
10	2.0	0.5	0.5000032878	7.9776590E-01	3.28780E-06

TABLE III
RESULTS FOR THE PROPOSED METHOD PRESENTED IN (17) WITH ONE OFF - GRID POINT AT COLLOCATION

x	Exact Solution	Approximate Value	Adeboye [7]	Error of Proposed Method
0.00	0.00000000000	0.0000000000	3.379500000E-06	0.000000000E+00
0.10	0.14735784232	0.1473578284	6.598600000E-06	1.390000000E-08
0.20	0.25015214537	0.2501521164	9.454000000E-06	2.890000000E-08
0.30	0.31341504348	0.3134150000	1.156300000E-05	4.340000000E-08
0.40	0.34178302747	0.3417825591	1.204180000E-05	4.680000000E-07
0.50	0.33954334810	0.3395424500	8.902600000E-06	8.981000000E-07
0.60	0.31067692433	0.3106755871	1.922800000E-06	1.337200000E-06
0.70	0.25889818576	0.2588965200	2.803580000E-05	1.665700000E-06
0.80	0.18769224781	0.1876902363	8.259870000E-05	2.011500000E-06
0.90	0.10034979197	0.1003474152	1.870490000E-04	2.376700000E-06
1.00	0.00000000000	-0.0000023895	3.379500000E-06	2.389500000E-06

TABLE IV
RESULTS FOR THE PROPOSED METHOD PRESENTED IN (17) WITH ONE OFF - GRID POINT AT INTERPOLATION

x	Exact Solution	Approximate Value	Adeboye [7]	Error of Proposed Method
0.00	0.00000000000	0.0000000000	3.379500000E-06	0.000000000E+00
0.10	0.14735784232	0.1473578412	6.598600000E-06	1.100000000E-09
0.20	0.25015214537	0.2501521422	9.454000000E-06	3.100000000E-09
0.30	0.31341504348	0.3134150392	1.156300000E-05	4.200000000E-09
0.40	0.34178302747	0.3417830695	1.204180000E-05	4.210000000E-08
0.50	0.33954334810	0.3395434361	8.902600000E-06	8.800000000E-08
0.60	0.31067692433	0.3106770602	1.922800000E-06	1.359000000E-07
0.70	0.25889818576	0.2588981266	2.803580000E-05	5.910000000E-08
0.80	0.18769224781	0.1876919924	8.259870000E-05	2.554000000E-07
0.90	0.10034979197	0.1003493386	1.870490000E-04	4.533000000E-07
1.00	0.00000000000	-0.000004362	3.379500000E-06	4.362000000E-07

V. CONCLUSION

In this paper, new block methods with uniform integrators each of order $(5,5,5,5)^T$ were developed. The resultant numerical integrators posses the following desirable properties:

i. Zero stability
ii. Convergent schemes
iii. An addition of equation from the use of first derivative
iv. Being self – starting as such it eliminate the use of predictor – corrector method

v. Facility to generate solutions at 4 points simultaneously
vi. Produce solution over sub intervals that do not overlaps
vii. Apply uniformly to both IVP_s and BVP_s with adjustment to the boundary conditions

In addition, the new schemes compares favourably with the theoretical solution and the results are more accurate than Awoyemi [5] and Adeboye [7], see Table I - IV. Hence, the present work is an improvement over other cited works.

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