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A Sufficient Condition for Graphs to Have Hamiltonian $[a, b]$-Factors

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Abstract—Let a and b be nonnegative integers with $2 \leq a < b$, and let G be a Hamiltonian graph of order n with $n \geq \frac{(a+b-4)(a+b-2)}{b-2}$. An $[a, b]$-factor F of G is called a Hamiltonian $[a, b]$-factor if F contains a Hamiltonian cycle. In this paper, it is proved that G has a Hamiltonian $[a, b]$-factor if $|N_G(X)| > \frac{(a-1)n+|X|-1}{a+b-3}$ for every non-empty independent subset X of $V(G)$ and $\delta(G) > \frac{(a-1)n+a+b-4}{a+b-3}$.

Keywords—graph, minimum degree, neighborhood, $[a, b]$-factor, Hamiltonian $[a, b]$-factor.

I. Introduction

In this paper we consider only finite undirected graphs without loops or multiple edges. In particular, a graph is said to be a Hamiltonian graph if it contains a Hamiltonian cycle. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, the neighborhood $N_G(x)$ of x is the set vertices of G adjacent to x, and the degree $d_G(x)$ of x is $|N_G(x)|$. We denote the minimum degree of G by $\delta(G)$. For $S \subseteq V(G)$, $N_G(S) = \cup_{x \in S} N_G(x)$ and $G[S]$ is the subgraph of G induced by S and $G - S$ is the subgraph obtained from G by deleting all the vertices in S together with the edges incident to vertices in S. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges.

Let g and f be two nonnegative integer-valued functions defined on $V(G)$ with $g(x) \leq f(x)$ for each $x \in V(G)$. A spanning subgraph F of G is called a (g, f)-factor if it satisfies $g(x) \leq d_F(x) \leq f(x)$ for each $x \in V(G)$. If $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$, then a (g, f)-factor is called an $[a, b]$-factor. A (g, f)-factor F of G is called a Hamiltonian (g, f)-factor if F contains a Hamiltonian cycle. If $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$, then we say a Hamiltonian (g, f)-factor to be a Hamiltonian $[a, b]$-factor. If $a = b = k$, then a Hamiltonian $[a, b]$-factor is simply called a Hamiltonian k-factor. The other terminologies and notations not given here can be found in [1].

Many authors have investigated factors [2–7]. Y. Gao, G. Li and X. Li [8] gave a degree condition for a graph to have a Hamiltonian k-factor. H. Matsuda [9] showed a degree condition for graphs to have Hamiltonian $[a, b]$-factors. S. Zhou and B. Pu [10] obtained a neighborhood condition for a graph to have a Hamiltonian $[a, b]$-factor.

The following results on Hamiltonian k-factors and Hamiltonian $[a, b]$-factors are known.

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Theorem 1. ([8]). Let $k \geq 2$ be an integer and let G be a graph of order $n > 12(k-2)^2 + 2(5-\alpha)(k-2) - \alpha$. Suppose that kn is even, $\delta(G) \geq k$ and

$$\max\{d_G(x), d_G(y)\} \geq \frac{n+\alpha}{2}$$

for each pair of nonadjacent vertices x and y in G, where $\alpha = 3$ for odd k and $\alpha = 4$ for even k. Then G has a Hamiltonian k-factor if for a given Hamiltonian cycle C, $G - E(C)$ is connected.

Theorem 2. ([9]). Let a and b be integers with $2 \leq a < b$, and let G be a Hamiltonian graph of order $n \geq \frac{(a+b-4)(2a+b-6)}{b-2}$. Suppose that $\delta(G) \geq a$ and

$$\max\{d_G(x), d_G(y)\} \geq \frac{(a-2)n}{a+b-4} + 2$$

for each pair of nonadjacent vertices x and y of $V(G)$. Then G has a Hamiltonian $[a, b]$-factor.

Theorem 3. ([10]). Let a and b be nonnegative integers with $2 \leq a < b$, and let G be a Hamiltonian graph of order n with $n \geq \frac{(a+b-3)(2a+b-6)-a+2}{b-2}$. Suppose for any subset $X \subset V(G)$, we have

$$N_G(X) = V(G) \qquad if \qquad |X| \geq \left\lfloor \frac{(b-2)n}{a+b-3} \right\rfloor; \quad or$$

$$|N_G(X)| \geq \frac{a+b-3}{b-2}|X| \qquad if \qquad |X| < \left\lfloor \frac{(b-2)n}{a+b-3} \right\rfloor.$$

Then G has a Hamiltonian $[a, b]$-factor.

G. Liu and L. Zhang [11] proposed the following problem.

Problem. Find sufficient conditions for graphs to have connected $[a, b]$-factors related to other parameters in graphs such as binding number, neighborhood and connectivity.

We now show our main theorem which partially solves the above problem.

Theorem 4. Let a and b be nonnegative integers with $2 \leq a < b$, and let G be a Hamiltonian graph of order n with $n \geq \frac{(a+b-4)(a+b-2)}{b-2}$. Suppose that

$$|N_G(X)| > \frac{(a-1)n + |X| - 1}{a+b-3}$$

for every non-empty independent subset X of $V(G)$, and

$$\delta(G) > \frac{(a-1)n + a + b - 4}{a+b-3}.$$

Then G has a Hamiltonian $[a, b]$-factor.

II. The Proof of Theorem 4

The proof of our main Theorem relies heavily on the following lemma. Lemma 2.1 is a well-known necessary and sufficient for a graph to have a (g, f)-factor which was given by Lovasz. The following result is the special case which we use to prove our main theorem.

Lemma 2.1. ([12]). Let G be a graph, and let g and f be two nonnegative integer-valued functions defined on $V(G)$ with $g(x) < f(x)$ for each $x \in V(G)$. Then G has a (g, f)-factor if and only if

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \geq 0$$

for any disjoint subsets S and T of $V(G)$.

Proof of Theorem 4. According to assumption, G has a Hamiltonian cycle C. Let $G' = G - E(C)$. Note that $V(G') = V(G)$.

Obviously, G has a Hamiltonian $[a, b]$-factor if and only if G' has an $[a-2, b-2]$-factor. By way of contradiction, we assume that G' has no $[a-2, b-2]$-factor. Then, by Lemma 2.1, there exist disjoint subsets S and T of $V(G')$ such that

$$\delta_{G'}(S,T) = (b-2)|S| + d_{G'-S}(T) - (a-2)|T| \leq -1. \quad (1)$$

We choose such subsets S and T so that $|T|$ is as small as possible.

If $T = \emptyset$, then by (1), $-1 \geq \delta_{G'}(S,T) = (b-2)|S| \geq |S| \geq 0$, which is a contradiction. Hence, $T \neq \emptyset$. Set

$$h = \min\{d_{G-S}(x) : x \in T\}.$$

We choose $x_1 \in T$ satisfying $d_{G-S}(x_1) = h$. Clearly,

$$\delta(G) \leq d_{G-S}(x_1) + |S| = h + |S|. \quad (2)$$

Now, we prove the following claims.

Claim 1. $d_{G'-S}(x) \leq a-3$ for all $x \in T$.

Proof. If $d_{G'-S}(x) \geq a-2$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (1). This contradicts the choice of S and T.

Claim 2. $d_{G-S}(x) \leq d_{G'-S}(x) + 2 \leq a-1$ for all $x \in T$.

Proof. Note that $G' = G - E(C)$. Thus, we get from Claim 1

$$d_{G-S}(x) \leq d_{G'-S}(x) + 2 \leq a-1$$

for all $x \in T$.

In terms of the definition of h and Claim 2, we have

$$0 \leq h \leq a-1.$$

We shall consider two cases according to the value of h and derive contradictions.

Case 1. $1 \leq h \leq a-1$.

Using (1), Claim 2, $|S| + |T| \leq n$ and $a - h \geq 1$, we get

$$\begin{aligned} -1 &\geq \delta_{G'}(S,T) = (b-2)|S| + d_{G'-S}(T) - (a-2)|T| \\ &\geq (b-2)|S| + d_{G-S}(T) - 2|T| - (a-2)|T| \\ &= (b-2)|S| + d_{G-S}(T) - a|T| \\ &\geq (b-2)|S| + h|T| - a|T| \\ &= (b-2)|S| - (a-h)|T| \\ &\geq (b-2)|S| - (a-h)(n-|S|) \\ &= (a+b-h-2)|S| - (a-h)n, \end{aligned}$$

that is,

$$|S| \leq \frac{(a-h)n-1}{a+b-h-2}. \quad (3)$$

In terms of (2), (3) and the assumption of the theorem, we obtain

$$\frac{(a-1)n+a+b-4}{a+b-3} < \delta(G) \leq |S|+h \leq \frac{(a-h)n-1}{a+b-h-2}+h. \quad (4)$$

Subcase 1.1. $h = 1$.

From (4), we get

$$\begin{aligned} \frac{(a-1)n+a+b-4}{a+b-3} &< \frac{(a-1)n-1}{a+b-3}+1 \\ &= \frac{(a-1)n+a+b-4}{a+b-3}. \end{aligned}$$

That is a contradiction.

Subcase 1.2. $2 \leq h \leq a-1$.

If the LHS and RHS of (4) are denoted by A and B respectively, then (4) says that

$$A - B < 0. \quad (5)$$

Multiplying $A - B$ by $(a+b-3)(a+b-h-2)$ and by $n \geq \frac{(a+b-4)(a+b-2)}{b-2}$ and $2 \leq h \leq a-1 < a+b-2$, we have

$$\begin{aligned} & (a+b-3)(a+b-h-2)(A-B) \\ = \; & (a+b-h-2)((a-1)n+a+b-4) \\ & -(a+b-3)((a-h)n-1) \\ & -(a+b-3)(a+b-h-2)h \\ = \; & (h-1)((b-2)n-(a+b-3)(a+b-h-2)) \\ & -(a+b-h-2) \\ \geq \; & (h-1)((a+b-4)(a+b-2) \\ & -(a+b-3)(a+b-h-2)) - (a+b-h-2) \\ = \; & (h-1)((a+b-3)h-(a+b-2)) \\ & -(a+b-h-2) \\ = \; & (h-1)((a+b-2)(h-1)-h) - (a+b-h-2) \\ \geq \; & (h-1)(a+b-2-h) - (a+b-h-2) \\ = \; & (h-2)(a+b-2-h) \geq 0, \end{aligned}$$

which implying

$$A - B \geq 0.$$

Which contradicts (5).

Case 2. $h = 0$.

Let $Y = \{x \in T : d_{G-S}(x) = 0\}$. Clearly, $Y \neq \emptyset$. Since Y is independent, we get from the assumption of the theorem

$$\frac{(a-1)n+|Y|-1}{a+b-3} < |N_G(Y)| \leq |S|. \quad (6)$$

Using (6) and $|S|+|T|\leq n$, we obtain

$$\begin{aligned}
\delta_{G'}(S,T) &= (b-2)|S|+d_{G'-S}(T)-(a-2)|T| \\
&\geq (b-2)|S|+d_{G-S}(T)-2|T| \\
&\quad -(a-2)|T| \\
&= (b-2)|S|+d_{G-S}(T)-a|T| \\
&\geq (b-2)|S|+|T|-|Y|-a|T| \\
&= (b-2)|S|-(a-1)|T|-|Y| \\
&\geq (b-2)|S|-(a-1)(n-|S|)-|Y| \\
&= (a+b-3)|S|-(a-1)n-|Y| \\
&> (a+b-3)\cdot\frac{(a-1)n+|Y|-1}{a+b-3} \\
&\quad -(a-1)n-|Y| \\
&= -1,
\end{aligned}$$

which contradicts (1).

From the above contradictions we deduce that G' has an $[a-2,b-2]$-factor. This completes the proof of Theorem 4.

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