International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:5, No:12, 2011

Existence and global exponential stability of periodic solutions of cellular neural networks with distributed delays and impulses on time scales

Daiming Wang

Abstract—In this paper, by using Mawhin's continuation theorem of coincidence degree and a method based on delay differential inequality, some sufficient conditions are obtained for the existence and global exponential stability of periodic solutions of cellular neural networks with distributed delays and impulses on time scales. The results of this paper generalized previously known results.

Keywords—periodic solutions, global exponential stability, coincidence degree, M-matrix

I. Introduction

STABILITY and periodicity of cellular neural networks have been paid much attention in the past decades[1-10], due to its applicability in the image processing, pattern recognition and associative memories and so on.

It is well known that most widely studied and used neural networks can be classified as either continuous or discrete. However, there has been a somewhat new category of neural networks, which displays a combination of characteristics of both the continuous-time and discrete-time systems, these are called impulsive neural networks[11-14]. To our knowledge, not many authors discuss stability and periodicity of cellular neural networks with delays and impulses. Recently, Yongkun Li and Zhiwei Xing have studied the existence and global exponential stability of the periodic solution of the following cellular neural networks with time delays and impulses [15]:

$$\begin{cases} \frac{dx_i(t)}{dt} = -a_i(t)x_i(t) + \sum\limits_{j=1}^{n}[b_{ij}(t)f_j(x_j(t)) \\ \qquad +c_{ij}(t)f_j(x_j(t-\tau_j(t)))] + I_i(t), \\ \qquad t \geq 0,\ t \neq t_k,\ i = 1,2,\dots,n, \\ \triangle x_i(t_k) = J_i(x_i(t_k)) = -\gamma_{ik}x_i(t_k), i = 1,2,\dots,n, \\ \qquad k = 1,2,\dots, \end{cases}$$

However, in most situations, delays are variable, and in fact unbounded. So, in this paper, we will study the existence and global exponential stability of the periodic solution of cellular neural networks of the following with mixed delays and impulses:

$$\begin{cases} \frac{dx_i(t)}{dt} = -a_i(t)x_i(t) + \sum\limits_{j=1}^{n}[a_{ij}(t)f_j(x_j(t)) \\ \qquad +b_{ij}(t)f_j(x_j(t-\tau_{ij}(t))) \\ \qquad +c_{ij}(t)\int_{-\infty}^{t}k_{ij}(t-s)f_j(x_j(s))ds] + I_i(t), \\ \qquad t \geq 0,\ \ t \neq t_k,\ \ i = 1,2,\dots,n, \\ \triangle x_i(t_k) = J_i(x_i(t_k)) = -\gamma_{ik}x_i(t_k), \\ \qquad i = 1,2,\dots,n,\ \ k = 1,2,\dots,n. \end{cases} \tag{1}$$

Daiming Wang is with the Department of Mathematics, Fuyang Normal College, Fuyang 236041, China, e-mail: (15855588446@139.com)

where $\triangle x_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$, $i = 1,2,\dots,n$ are the impulses at moments t_k and $0 < t_1 < t_2 < \dots$ is a strictly increasing sequence such that $\lim_{t\to\infty} t_k = +\infty; x_i(t)$ $(i = 1,2,\dots,n)$ is the state of neuron and n is the number of neurons; $A(t) = (a_{ij}(t))_{n\times n}$,
$B(t) = (b_{ij}(t))_{n\times n}$ and $C(t) = (c_{ij}(t))_{n\times n}$ are connection matrix functions,
$I(t) = (I_1(t), I_2(t),\dots,I_n(t))^T : R^+ \mapsto R^n$ is continuous periodic function with period $\omega > 0$, $f(x) = (f_1(x_1), f_2(x_2)),\dots,f_n(x_n))^T$ is the activation function of the neurons,$F(t) = diag(a_1(t), a_2(t),\dots,a_n(t))$ with $a_i(t) > 0\,(i = 1,2,\dots,n)$. The delays $0 \leq \tau_{ij}(t) \leq \tau(i,j = 1,2,\dots,n)$ are bounded functions. Kernel function $k_{ij} : [0,\infty) \longrightarrow [0,\infty)(i,j = 1,2,\dots,n)$ are all piecewise continuous functions on $[0,\infty)$, and satisfy $\int_0^\infty k_{ij}(s)ds = 1, i,j = 1,2,\dots,n.$

As usual in the theory of impulsive differential equations, at the points of discontinuity t_k of the solution $t \mapsto x_i(t)$ we assume that $x_i(t_k) \equiv x_i(t_k^-)$. It is clear that, in general, the derivatives $x_i^{'}(t_k)$ do not exist. On the other hand, according to the first equality of (1) there exists the limits $x_i^{'}(t_k^{\pm})$. So, we assume that $x_i^{'}(t_k) \equiv x_i^{'}(t_k^-)$, $\ i = 1,2,\dots,n;\ \ k = 1,2,\dots$

The initial conditions of system (1) are of the form $x_i(s) = \phi_i(s) \neq 0, s \leq 0, i = 1,2,\dots,n.$ where ϕ_i is bounded and continuous function on $(-\infty, 0]$.

Throughout this paper, we impose the following assumptions:

(H_1) The delays $0 \leq \tau_{ij}(t) \leq \tau(i,j = 1,2,\dots,n)$ are bounded continuous ω-periodic functions.

(H_2) $a_i(t), i = 1,2,\dots,n$ are positive and bounded continuous ω-periodic functions, and $0 \leq \underline{a}_i \leq a_i(t) \leq \overline{a}_i$.

(H_3) kernel function $k_{ij}, i,j = 1,2,\dots,n$ are all piecewise continuous functions, and satisfy $\int_0^\infty k_{ij}(s)ds = 1$.

(H_4) There exist positive constants $M_j > 0$ such that $| f_j(x) |\leq M_j$ for $j = 1,2,\dots,n, x \in R$.

(H_5) $a_{ij}(t), b_{ij}(t), c_{ij}(t), i,j = 1,2,\dots,n$ are bounded continuous ω-periodic functions.

(H_6) There exists a positive integer q such that $t_{k+q} = t_k + \omega, \gamma_{i(k+q)} = \gamma_{ik}$, for $k = 1,2,\dots, i = 1,2,\dots,n$.

(H_7) $\prod\limits_{0\leq t_k<t}(1-\gamma_{ik}), i = 1,2,\dots,n$ are ω-periodic functions.

(H_8) $f_i \in C(R,R), j = 1,2,\dots,n$ are Lipschitzian with Lipschitz constants $L_j > 0$,
$| f_j(x) - f_j(y) |\leq L_j | x - y |$ for all $x, y \in R$.

For convenience, we introduce the following notations:

$\overline{a}_{ij} = \sup\{| a_{ij}(t) |, t \in [0,\omega]\}$, $\overline{b}_{ij} = \sup\{| b_{ij}(t) |, t \in [0,\omega]\}$, $\overline{c}_{ij} = \sup\{| c_{ij}(t) |, t \in [0,\omega]\}$, $\overline{I}_i = \sup\{| I_i(t) |, t \in [0,\omega]\}$ $N_i = (\int_0^\omega \prod_{0\le t_k<t}(1-\gamma_{ik})^{-2}dt)^{\frac{1}{2}}, i,j = 1,2,\dots,n.$

The organization of this paper is as follows. In Section II, we introduce some notations and definitions, and state some preliminary results needed in later sections. In Section III, we then study the existence of periodic solutions of system (1) by using the continuation theorem of coincidence degree proposed by Gains and Mawhin [16]. In Section IV, we shall derive sufficient conditions to ensure that the periodic solution of (1) is globally exponentially stable.

II. Preliminaries

In this section, we shall introduce some notations and definitions, and state some preliminary results.

Consider the impulsive system[15]:

$$\begin{cases} x'(t) = f(t, x(t), x(t-\tau_1(t), \dots, x(t-\tau_n(t))), \\ \qquad t \ne t_k, k = 1,2,\dots, \\ \triangle x(t) |_{t=t_k} = J_k(x(t_k^-)) = -\gamma_{ik}x_i(t_k), \end{cases} \tag{2}$$

where $x \in R^n, f : R \times R^n \to R^n$ is continuous and $f(t+\omega, x(t-\tau_1(t)), \dots, x(t-\tau_n(t))) = f(t, x(t-\tau_1(t)), \dots, x(t-\tau_n(t))), J_k : R^n \to R^n, k = 1,2,\dots,n$ are continuous; $\tau_i \in C(R,[0,\tau]), i = 1,2,\dots,n$ are ω-periodic functions and $t - \tau_i(t) \to \infty$, as $t \to \infty, i = 1,2,\dots,n$, and there exists a positive integer q such that $t_{k+q} = t_k + \omega, J_{k+q}(x) = J_k(x)$ with $t_k \in R, t_{k+1} > t_k, \lim_{k\to\infty} t_k = \infty, \triangle x(t) |_{t=t_k} = x(t_k^+) - x(t_k^-)$. For $t_k \ne 0$ $(k = 1,2,\dots), [0,\omega] \cap \{t_k\} = \{t_1, t_2, \dots, t_q\}$. As we know, $\{t_k\}$ are called points of jump.

Definition2.1.[15] A function $x \in ([0,\infty), R)$ is said to be a solution of system (2) in $[0,\infty)$ satisfying the initial value condition $x(s) = \phi(s) \ne 0, s \in (-\infty,0]$, where $\phi \in C((-\infty,0], R^n)$, if the following conditions are satisfied

(i) $x(t)$ is absolutely continuous in each interval $(t_k, t_{k+1}) \subset [0,\infty)$;

(ii) for any $t_k \in [0,\infty), k = 1,2,\dots, x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$;

(iii) x(t) satisfies (2) for almost everywhere in $[0,\infty)$ and at impulsive points $\{t_k\}$ situated in $[0,\infty)$ may have discontinuity of the first kind.

Definition 2.2.[14] The periodic solution of system (2) is said to be globally exponentially stable (GES), if there exist constants $\alpha > 0$ and $\beta > 0$ such that

$| x_i(t) - x_i^* | \le \beta \| \phi - x^* \| e^{-\alpha t}$
for all $t \ge 0$, where

$\| \phi - x^* \| = \sup_{s\in(-\infty,0]} (\sum_{i=1}^n | \phi_i(s) - x_i^* |).$

Consider the nonimpulsive delay differential system

$$\begin{aligned} \frac{dy_i(t)}{dt} =& - a_i(t)y_i(t) + \prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}\sum_{j=1}^n[a_{ij}(t) \\ &\times f_j(\prod_{0\le t_k<t}(1-\gamma_{jk})y_j(t)) \\ &+ b_{ij}(t)f_j(\prod_{0\le t_k<t-\tau_{ij}(t)}(1-\gamma_{jk})y_j(t-\tau_{ij}(t))) \\ &+ c_{ij}(t)\int_{-\infty}^t k_{ij}(t-s)f_j(\prod_{0\le t_k<s}(1-\gamma_{jk})y_j(s))ds] \\ &+ \prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}I_i(t), \quad t \ge 0,\ i = 1,2,\dots,n. \end{aligned} \tag{3}$$

with initial conditions $y_i(s) = \phi_i(s) \ne 0, \quad s \in (-\infty,0],\ i = 1,2,\dots,n.$

Lemma 2.1. Assume (H_7) holds, then

(i) if $y = (y_1,\dots,y_n)$ is a solution of (3), then $x = (\prod_{0\le t_k<t}(1-\gamma_{1k})y_1, \dots, \prod_{0\le t_k<t}(1-\gamma_{nk})y_n)$ is a solution of (1);

(ii) if $x = (x_1,\dots,x_n)$ is a solution of (1), then $y = (\prod_{0\le t_k<t}(1-\gamma_{1k})^{-1}x_1, \dots, \prod_{0\le t_k<t}(1-\gamma_{nk})^{-1}x_n)$ is a solution of (3).

Proof. The proof is similar to that of Theorem 2.1 in [14] and will be omitted here.

Let X, Y be real Banach spaces, $L : DomL \subset X \to dimY$ be a linear mapping, and $N : X \to Y$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $dimKerL = codimImL < +\infty$ and ImL is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $ImP = KerL, KerQ = Im(I-Q)$, it follows that mapping $L |_{DomL\cap kerP}: (I-P)X \to ImL$ is invertible. We denote the inverse of that mapping by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$. if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact. Since ImQ is isomorphic to $KerL$, there exists an isomorphism $J : ImQ \to KerL$.

Now, we introduce Mawhin's continuation theorem as follows.

Lemma 2.2.[16] Let $\Omega \subset X$ be an open bounded set and let $N : X \to Y$ be a continuous operator which is L-compact on $\overline{\Omega}$. Assume

(a) for each $\lambda \in (0,1), x \in \partial\Omega \bigcap DomL, Lx \ne \lambda Nx$,

(b) for each $x \in \partial\Omega \bigcap KerL, QNx \ne 0$, and $deg(JQN, \Omega \bigcap kerL, 0) \ne 0$.

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \bigcap DomL$.

Definition 2.3.[15] Let the $n \times n$ matrix $A = (a_{ij})_{n\times n}$ have nonpositive off-diagonal elements and all principal minors of A are positive, then A is said to be an M-matrix.

Lemma 2.3.[17] Let $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ be a solution of the differential inequality:

$x'(t) \le Ax(t) + B\overline{x}(t), t \ge t_0,$
where

$\overline{x}(t) = (\sup_{t-\tau\le s\le t}\{x_1(s)\}, \sup_{t-\tau\le s\le t}\{x_2(s)\}, \ldots, \sup_{t-\tau\le s\le t}\{x_n(s)\})^T, A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n}.$
If

$$(A_1) a_{ij} \ge 0 (i \ne j), b_{ij} \ge 0, i, j = 1, 2, \ldots, n; \sum_{j=1}^{n} \overline{x}_j(t_0) > 0;$$

(A_2) The matrix $-(A + B)$ is an M-matrix.

Then there always exist constants $\lambda > 0, r_i > 0$ ($i = 1, 2, \ldots, n$) such that
$x_i(t) \le r_i \sum_{j=1}^{n} \overline{x}_j(t_0) e^{\lambda(t-t_0)}.$

III. Existence of periodic solutions

In this section, based on the Mawhin's continuation theorem, we shall study the existence of at least one periodic solution of (1). For convenience, we introduce the following notations:

$$\begin{aligned} G_i^y =& G_i(t, y_1(t), \ldots, y_n(t)) \\ =& -a_i(t)y_i(t) + \prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}\sum_{j=1}^{n}[a_{ij}(t) \\ &\times f_j(\prod_{0\le t_k<t}(1-\gamma_{jk})y_j(t)) \\ &+ b_{ij}(t)f_j(\prod_{0\le t_k<t-\tau_{ij}(t)}(1-\gamma_{jk})y_j(t-\tau_{ij}(t))) \\ &+ c_{ij}(t)\int_{-\infty}^{t} k_{ij}(t-s)f_j(\prod_{0\le t_k<s}(1-\gamma_{jk})y_j(s))ds] \\ &+ \prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}I_i(t), \quad t \ge 0, \end{aligned}$$

where $y = (y_1, y_2, \ldots, y_n)^T$ is ω-periodic function, $i = 1, 2, \ldots, n$. Our main result of this section is as follows.

Theorem 3.1. Suppose (H_1)-(H_7) hold, then the system (1) has at least one ω-periodic solution.

Proof. According to Lemma 2.1, we need only to prove that the nonimpulsive delay differential system (3) has an ω-periodic solution. In order to use the continuation theorem of coincidence degree theory to establish the existence of a solution of (3), we take
$Y = Z = \{y(t) \in C(R, R^n) : y(t+\omega) = y(t), t \in R, y = (y_1, y_2, \ldots, y_n)^T\}$
with the norm
$\| y \| = \sum_{k=1}^{n} | y_k |_0, \quad | y_k |_0 = \sup_{t\in[0,\omega]} | y_k(t) |, \ k = 1, 2, \ldots, n$
then Y and Z are Banach spaces.

Set
$Ly = y'$ and $Py = \frac{1}{\omega}\int_0^{\omega} y(t)dt, \ y \in Y; \ Qz = \frac{1}{\omega}\int_0^{\omega} z(t)dt, \ z \in Z$
and
$Ny = (G_1^y(t), G_2^y(t), \ldots, G_n^y(t))^T, \ y \in Y.$
Obviously, $KerL = \{y \in Y, y = h, h \in R^n\}, \ ImL = \{y \in Y, \int_0^{\omega} y(s)ds = 0\}$ and
$dimKerL = n = codimImL.$
So, ImL is closed in Z and L is a Fredholm mapping of index zero. It is easy to show that P and Q are continuous projectors satisfying
$ImP = KerL, \ ImL = KerQ = Im(I - Q).$
Furthermore, through an easy computation, we can find that the inverse $K_p : ImL \to KerP \cap DomL$ of L_p has the form
$K_p(z) = \int_0^t z(s)ds - \frac{1}{\omega}\int_0^{\omega}\int_0^t z(s)dsdt.$
Thus
$QNy = (\frac{1}{\omega}\int_0^{\omega} G_1^y(t)dt, \ldots, \frac{1}{\omega}\int_0^{\omega} G_n^y(t)dt)^T, \ y \in Y$
and

$$K_p(I-Q)Ny = \begin{pmatrix} \int_0^t G_1^y(s)ds \\ \vdots \\ \int_0^t G_j^y(s)ds \\ \vdots \\ \int_0^t G_n^y(s)ds \end{pmatrix} - \begin{pmatrix} \frac{1}{\omega}\int_0^{\omega}\int_0^t G_1^y(s)dsdt \\ \vdots \\ \frac{1}{\omega}\int_0^{\omega}\int_0^t G_j^y(s)dsdt \\ \vdots \\ \frac{1}{\omega}\int_0^{\omega}\int_0^t G_n^y(s)dsdt \end{pmatrix} - \begin{pmatrix} (\frac{t}{\omega} - \frac{t}{2})\int_0^{\omega} G_1^y(s)ds \\ \vdots \\ (\frac{t}{\omega} - \frac{t}{2})\int_0^{\omega} G_j^y(s)ds \\ \vdots \\ (\frac{t}{\omega} - \frac{t}{2})\int_0^{\omega} G_n^y(s)ds \end{pmatrix}$$

Clearly, QN and $K_P(I - Q)N$ are continuous. Using the Arzela-Ascoli theorem, it is not difficult to show that $QN(\overline{\Omega}), K_P(I-Q)N(\overline{\Omega})$ are relatively compact for any open bounded set $\Omega \subset Y$. Therefore, N is L-compact on $\overline{\Omega}$ for any open bounded set $\Omega \subset Y$.

Now we reach the position to search for an appropriate open, bounded subset Ω, for the application of the continuation theorem. Corresponding to the operator equation $Ly = \lambda Ny, \lambda \in (0, 1)$, we have

$$\begin{aligned} y_i'(t) =& \lambda\{-a_i(t)y_i(t) + \prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}\sum_{j=1}^{n}[a_{ij}(t) \\ &\times f_j(\prod_{0\le t_k<t}(1-\gamma_{jk})y_j(t)) \\ &+ b_{ij}(t)f_j(\prod_{0\le t_k<t-\tau_{ij}(t)}(1-\gamma_{jk})y_j(t-\tau_{ij}(t))) \\ &+ c_{ij}(t)\int_{-\infty}^{t} k_{ij}(t-s)f_j(\prod_{0\le t_k<s}(1-\gamma_{jk})y_j(s))ds] \\ &+ \prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}I_i(t)\}, \\ & y \in Y \ \ i = 1, 2, \ldots, n. \end{aligned} \tag{4}$$

Suppose that $y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T \in Y$ is a solution of system (4) for some $\lambda \in (0, 1)$. Integrating $y_i(t)y_i'(t)$ over the interval $[0, \omega]$, we obtain

$$\begin{aligned} 0 =& \frac{1}{2}y_i^2(t) |_0^{\omega} = \int_0^{\omega} y_i(t)y_i'(t)dt \\ =& \lambda\int_0^{\omega}\{-a_i(t)y_i(t)y_i(t) + \prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}y_i(t) \\ &\times \sum_{j=1}^{n}[a_{ij}(t)f_j(\prod_{0\le t_k<t}(1-\gamma_{jk})y_j(t)) \end{aligned}$$

$$
\begin{aligned}
&+ b_{ij}(t)f_j(\prod_{0\le t_k<t-\tau_{ij}(t)}(1-\gamma_{jk})y_j(t-\tau_{ij}(t)))\\
&+ c_{ij}(t)\int_{-\infty}^{t}k_{ij}(t-s)f_j(\prod_{0\le t_k<s}(1-\gamma_{jk})y_j(s))ds]\\
&+ \prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}y_i(t)I_i(t)\}dt
\end{aligned}
$$

That is

$$
\begin{aligned}
\int_0^{\omega}a_i(t)y_i^2(t)dt &= \int_0^{\omega}\prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}\\
&\times y_i(t)\sum_{j=1}^{n}[a_{ij}(t)f_j(\prod_{0\le t_k<t}(1-\gamma_{jk})y_j(t))\\
&+ b_{ij}(t)f_j(\prod_{0\le t_k<t-\tau_{ij}(t)}(1-\gamma_{jk})y_j(t-\tau_{ij}(t)))\\
&+ c_{ij}(t)\int_{-\infty}^{t}k_{ij}(t-s)f_j(\prod_{0\le t_k<s}(1-\gamma_{jk})y_j(s))ds]\\
&+ \int_0^{\omega}(1-\gamma_{ik})^{-1}y_i(t)I_i(t)dt,\ \ i=1,2,\ldots,n
\end{aligned}
$$

Obviously

$$
\begin{aligned}
\int_{-\infty}^{t}k_{ij}(t-s)ds &= -\int_{-\infty}^{t}k_{ij}(t-s)d(t-s)\\
&= -\int_{+\infty}^{0}k_{ij}(u)du = \int_0^{+\infty}k_{ij}(u)du = 1.
\end{aligned}
$$

From conditions (H$_2$), (H$_4$) and (H$_5$), it follows that

$$
\begin{aligned}
&\underline{a}_i\int_0^{\omega}\mid y_i^2(t)\mid dt \le \int_0^{\omega}\prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}\mid y_i(t)\mid\\
&\times\sum_{j=1}^{n}[\mid a_{ij}(t)\mid\mid f_j(\prod_{0\le t_k<t}(1-\gamma_{jk})y_j(t))\mid\\
&+\mid b_{ij}(t)\mid\mid f_j(\prod_{0\le t_k<t-\tau_{ij}(t)}(1-\gamma_{jk})y_j(t-\tau_{ij}(t)))\mid\\
&+\mid c_{ij}(t)\mid\int_{-\infty}^{t}\mid k_{ij}(t-s)f_j(\prod_{0\le t_k<s}(1-\gamma_{jk})y_j(s))ds\mid]\\
&+\int_0^{\omega}(1-\gamma_{ik})^{-1}\mid y_i(t)\mid\mid I_i(t)\mid dt\\
&\le\int_0^{\omega}\sum_{j=1}^{n}(\overline{a}_{ij}+\overline{b}_{ij}+\overline{c}_{ij})M_j\prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}\mid y_i(t)\mid dt\\
&+\overline{I}_i\int_0^{\omega}\prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}\mid y_i(t)\mid dt\\
&\le(\sum_{j=1}^{n}(\overline{a}_{ij}+\overline{b}_{ij}+\overline{c}_{ij})M_j+\overline{I}_i)(\int_0^{\omega}\prod_{0\le t_k<t}(1-\gamma_{ik})^{-2}dt)^{\frac12}\\
&\times(\int_0^{\omega}\mid y_i(t)\mid^2 dt)^{\frac12}\\
&= N_i(\sum_{j=1}^{n}(\overline{a}_{ij}+\overline{b}_{ij}+\overline{c}_{ij})M_j+\overline{I}_i)(\int_0^{\omega}\mid y_i(t)\mid^2 dt)^{\frac12},\\
&i=1,2,\ldots,n.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(\textstyle\int_0^{\omega}\mid y_i^2(t)\mid dt)^{\frac12} &\le \frac{Ni}{\underline{a}_i}(\sum_{j=1}^{n}(\overline{a}_{ij}+\overline{b}_{ij}+\overline{c}_{ij})M_j+\overline{I}_i))\\
&:= S_i, i=1,2,\ldots,n \qquad (5)
\end{aligned}
$$

Let $\underline{t}_i\in[0,\omega]\neq t_k, k=1,2,\ldots,m.$ such that $\mid y_i(\underline{t}_i)\mid = inf_{t\in[0,\omega]}\mid y_i(t)\mid, i=1,2,\ldots,n$. Then, by (5), we have
$\mid y_i(\underline{t}_i)\mid\sqrt{\omega}=\mid y_i(\underline{t}_i)\mid(\int_0^{\omega}dt)^{\frac12}\le(\int_0^{\omega}\mid y_i^2(t)\mid dt)^{\frac12}\le S_i$
thus,

$$
\mid y_i(\underline{t}_i)\mid\le\frac{S_i}{\sqrt{\omega}} \qquad (6)
$$

From (6), and since $y_i(t)=y_i(\underline{t}_i)+\int_{\underline{t}_i}^{t}y_i'(s)ds$, it follows that

$$
\mid y_i(t)\mid\le\frac{S_i}{\sqrt{\omega}}+\int_0^{\omega}\mid y_i'(t)\mid dt \qquad (7)
$$

On the other hand, from (4) and conditions (H$_2$), (H$_4$), (H$_5$), (H$_7$), we have

$$
\begin{aligned}
\int_0^{\omega}\mid y_i'(t\mid dt &< \overline{a}_i\int_0^{\omega}\mid y_i(t)\mid dt+(\sum_{j=1}^{n}\mid a_{ij}(t)\mid+\mid b_{ij}(t)\mid\\
&+\mid c_{ij}(t)\mid M_j+\overline{I}_i)\int_0^{\omega}\prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}dt)\\
&\le\overline{a}_i\sqrt{\omega}(\int_0^{\omega}\mid y_i(t)\mid^2 dt)^{\frac12}+(\sum_{j=1}^{n}(\overline{a}_{ij}+\overline{b}_{ij}\\
&+\overline{c}_{ij})M_j+\overline{I}_i)\sqrt{\omega}\int_0^{\omega}\prod_{0\le t_k<t}((1-\gamma_{ik})^{-2}dt)^{\frac12}\\
&=\overline{a}_i\sqrt{\omega}(\int_0^{\omega}\mid y_i(t)\mid^2 dt)^{\frac12}+N_i\sqrt{\omega}(\sum_{j=1}^{n}(\overline{a}_{ij}+\overline{b}_{ij}\\
&+\overline{c}_{ij})M_j+\overline{I}_i)
\end{aligned}
$$

Together with (5), we get

$$
\begin{aligned}
\int_0^{\omega}\mid y_i'(t)\mid dt &< \overline{a}_i\sqrt{\omega}S_i+N_i\sqrt{\omega}(\sum_{j=1}^{n}(\overline{a}_{ij}+\overline{b}_{ij}\\
&+\overline{c}_{ij})M_j+\overline{I}_i):=D_i. \qquad (8)
\end{aligned}
$$

in view of (7) and (8), we obtain

$$
\mid y_i(t)\mid<\frac{S_i}{\sqrt{\omega}}+D_i:=R_i,\ \ i=1,2,\ldots,n. \qquad (9)
$$

Denote $A=\sum_{i=1}^{m}R_i+K$, where K is a sufficiently large positive constant, clearly, A is independent of λ . Now, take $\Omega=\{y\in Y:\parallel y(t)\parallel< A\}$. It is clear that Ω satisfies the requirement (a) in Lemma 2.2.

When $y\in\partial\Omega\cap KerL, y=(y_1,y_2\ldots,y_n)^T$ is a constant vector in R^n with $\parallel y\parallel=A$. Then $QNy=$

$(\frac{1}{\omega}\int_0^\omega G_1^y dt, \ldots, \frac{1}{\omega}\int_0^\omega G_n^y dt),\ \ y \in Y$ where

$$\begin{aligned} G_i^y =& - a_i(t)y_i(t) + \prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}\sum_{j=1}^{n}[a_{ij}(t) \\ & \times f_j(\prod_{0\le t_k<t}(1-\gamma_{jk})y_j(t)) \\ & + b_{ij}(t)f_j(\prod_{0\le t_k<t-\tau_{ij}(t)}(1-\gamma_{jk})y_j(t-\tau_{ij}(t))) \\ & + c_{ij}(t)\int_{-\infty}^{t}k_{ij}(t-s)f_j(\prod_{0\le t_k<s}(1-\gamma_{jk})y_j(s))ds] \\ & + \prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}I_i(t), i=1,2,\ldots,n \end{aligned}$$

Take $J : ImQ \to KerL, r \to r$. Then, if necessary, we can let K be greater such that $y^T JQNy < 0$. So, for any $y \in \partial\Omega \cap KerL, QNy \neq 0$. Furthermore, let $\phi(\gamma; y) = -\gamma y + (1-\gamma)JQNy$, then for any $y \in \partial\Omega \cap KerL, y^T\phi(\gamma; y) < 0$, we get

$deg\{JQN, \Omega \cap KerL, 0\} = deg\{-y, \Omega \cap KerL, 0\} \neq 0.$

So, condition (b) of Lemma 2.2 is also satisfied. We now know that Ω satisfies all the requirements in Lemma 2.2. Therefore, (3) has at least one ω-periodic solution. As a sequence system (1) has at least one ω-periodic solution. The proof is complete.

IV. Global exponential stability of the periodic solution

Suppose that $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ is a periodic solution of system (1). In this section, we will use a technique of differential inequality to study the global exponential stability of this periodic solution.

Theorem 4.1. Assume (H_1)-(H_8) hold. Moreover, suppose that matrix $F - \alpha\beta(A + B + C)L$ is an M-matrix, where $F = diag(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n), A = (\overline{a}_{ij})_{n\times n}, B = (\overline{b}_{ij})_{n\times n}, C = (\overline{c}_{ij})_{n\times n}, L = diag(L_1, L_2, \ldots, L_n), \alpha = \max\limits_{1\le i\le n}\{sup_{t\in[0,\omega]}\prod_{0\le t_k<t}(1-\gamma_{ik})\}$, $\beta = \max\limits_{1\le i\le n}\{sup_{t\in[0,\omega]}\prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}\}$. Then the ω-periodic solution of system (1) is globally exponentially stable.

Proof. According to Theorem 3.1, we know that (1) has an ω-periodic solution $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$. Suppose that $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ is an arbitrary solution of (1).

Let $y(t) = x(t) - x^*$, then (1) can be written as

$$\begin{cases} \frac{dy_i(t)}{dt} = -a_i(t)y_i(t) + \sum\limits_{j=1}^{n}[a_{ij}(t)g_j(y_j(t)) \\ \qquad +b_{ij}(t)g_j(y_j(t-\tau_{ij}(t))) \\ \qquad +c_{ij}(t)\int_{-\infty}^{0}k_{ij}(t-s)g_j(y_j(s))ds],\ \ t\neq t_k \\ \Delta y_i(t_k) = -\gamma_{ik}y_i(t_k),\ \ t\ge 0,\ \ i=1,2,\ldots,n, \\ \qquad k=1,2,\ldots, \end{cases} \tag{10}$$

where

$g_j(y_j(t)) = f_j(x_j(t)) - f_j(x_j^*),\ \ j = 1, 2, \ldots, n$

Due to the assumption of (H_8), we know that $0 \le\mid g_i(y_i)\mid\le L_i \mid y_i \mid,\ \ i = 1, 2, \ldots, n$. The initial condition of (10) is $\Psi(s) = \phi(s) - x^*,\ s \in (-\infty, 0]$.

Also according to Lemma 2.1, we consider the following nonimpulsive delay differential system:

$$\begin{aligned} \frac{du_i(t)}{dt} =& - a_i(t)u_i(t) + \prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}\sum_{j=1}^{n}[a_{ij}(t) \\ & \times g_j(\prod_{0\le t_k<t}(1-\gamma_{jk})u_j(t)) \\ & + b_{ij}(t)g_j(\prod_{0\le t_k<t-\tau_{ij}(t)}(1-\gamma_{jk})u_j(t-\tau_{ij}(t))) \\ & + c_{ij}(t)\int_{-\infty}^{t}k_{ij}(t-s)g_j(\prod_{0\le t_k<s}(1-\gamma_{jk})u_j(s))ds], \\ & i=1,2,\ldots,n. \end{aligned} \tag{11}$$

with initial conditions $u(s) = \Psi(s) = \phi(s) - x^*, s \in (-\infty, 0]$.

Let $z_i(t) =\mid u_i(t)\mid$, then the upper right derivative $D^+z_i(t)$ along the solutions of system (11) is as follows:

$$\begin{aligned} D^+z_i(t) =& D^+\mid u_i(t)\mid = u_i(t)^{'}sgn(u_i(t)) \\ \le& -\underline{a}_i\mid u_i(t)\mid + \prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}\sum_{j=1}^{n}[\mid a_{ij}(t) \\ & \times\mid L_j\mid u_j(t)\mid\prod_{0\le t_k<t}(1-\gamma_{jk}) \\ & +\mid b_{ij}(t)\mid L_j\mid\overline{u}_j(t)\mid\prod_{0\le t_k<t}(1-\gamma_{jk}) \\ & +\mid c_{ij}(t)\mid L_j\mid u_j(t)\mid\prod_{0\le t_k<t}(1-\gamma_{jk})] \\ \le& -\underline{a}_i\mid u_i(t)\mid + \prod_{0\le t_k<t}(1-\gamma_{ik})^{-1}\sum_{j=1}^{n}[\overline{a}_{ij}L_j \\ & \times\mid u_j(t)\mid\prod_{0\le t_k<t}(1-\gamma_{jk}) \\ & +\overline{b}_{ij}L_j\mid\overline{u}_j(t)\mid\prod_{0\le t_k<t}(1-\gamma_{jk}) \\ & +\overline{c}_{ij}L_j\mid u_j(t)\mid\prod_{0\le t_k<t}(1-\gamma_{jk})],\ i=1,2,\ldots,n. \end{aligned}$$

Hence

$$\begin{aligned} D^+z_i(t) \le& -\underline{a}_i\mid u_i(t)\mid +\beta\sum_{j=1}^{n}[\overline{a}_{ij}L_j\mid u_j(t)\mid\alpha \\ & +\overline{b}_{ij}L_j\mid\overline{u}_j(t)\mid\alpha + \overline{c}_{ij}L_j\mid u_j(t)\mid\alpha] \\ \le& -\underline{a}_i\mid u_i(t)\mid +\alpha\beta\sum_{j=1}^{n}(\overline{a}_{ij}+\overline{c}_{ij})L_j\mid u_j(t)\mid \\ & +\alpha\beta\sum_{j=1}^{n}\overline{b}_{ij}L_j\mid\overline{u}_j(t)\mid \\ \le& -\underline{a}_i z_i(t) + \alpha\beta\sum_{j=1}^{n}(\overline{a}_{ij}+\overline{c}_{ij})L_j z_j(t) \\ & +\alpha\beta\sum_{j=1}^{n}\overline{b}_{ij}L_j\overline{z}_j(t),\ i=1,2,\ldots,n. \end{aligned}$$

That is

$D^+z_i(t) \leq (-F + \alpha\beta(A+C)L)z(t) + \alpha\beta BL\overline{z}(t),\ t \geq 0,\ i = 1, 2, \ldots, n.$

where $F = diag(\underline{a}_1, \underline{a}_2, \ldots, \underline{a}_n), A = (\overline{a}_{ij})_{n\times n}, B = (\bar{b}_{ij})_{n\times n}, C = (\overline{c}_{ij})_{n\times n}$,
$L = diag(L_1, L_2, \ldots, L_n)$.

By initial conditions $x_i(s) = \phi_i(s) \neq 0, s \in (-\infty, 0],\ i = 1, 2, \ldots, n$, we know that $\overline{z}_i(0) > 0$, according to Lemma 2.3, if the matrix $-[-F + \alpha\beta(A+C)L + \alpha\beta BL] = F - \alpha\beta(A + B + C)L$ is an M-matrix, then there must exist constants $\mu > 0,\ r_i > 0\ (i = 1, 2, \ldots, n)$ such that

$z_i(t) =\mid u_i(t) \mid\leq r_i \sum\limits_{j=1}^{n} \overline{z}_j(0)e^{-\mu t} = r_i \sum\limits_{j=1}^{n} \mid \overline{u}_j(0) \mid e^{-\mu t},\ \ i = 1, 2, \ldots, n.$

By initial conditions, we have $\overline{u}(0) = \overline{\Psi}(0) = \overline{\phi}(0) - x^*$, then the solution of (10) satisfies

$$\begin{aligned}
\mid y_i(t) \mid &= \prod_{0\leq t_k<t} (1-\gamma_{ik}) \mid u_i(t) \mid \\
&\leq \prod_{0\leq t_k<t} (1-\gamma_{ik}) r_i \sum_{j=1}^{n} \mid \overline{u}_j(0) \mid e^{-\mu t} \\
&\leq \prod_{0\leq t_k<t} (1-\gamma_{ik}) r_i \sum_{j=1}^{n} \mid \overline{\phi}_j(0) - x_j^* \mid e^{-\mu t} \\
&\leq \alpha r_i \sum_{j=1}^{n} \mid \overline{\phi}_j(0) - x_j^* \mid e^{-\mu t} \\
&= \alpha r_i \sum_{i=1}^{n} \mid \overline{\phi}_i(0) - x_i^* \mid e^{-\mu t}, \quad i = 1, 2, \ldots, n.
\end{aligned}$$

That is

$$\begin{aligned}
\mid x_i(t) - x_i^* \mid &\leq \alpha r_i \sum_{i=1}^{n} \mid \overline{\phi}_i(0) - x_i^* \mid e^{-\mu t} \\
&= \alpha r_i [\sup_{s\in(-\infty,0]} (\sum_{i=1}^{n} \mid \phi_i(s) - x_i^* \mid)] e^{-\mu t} \\
&= \alpha r_i \parallel \phi - x^* \parallel e^{-\mu t},\ \ i = 1, 2, \ldots, n.
\end{aligned}$$

From Definition 2.2, we can see the ω -periodic solution $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ of system (1) is globally exponentially stable.

Acknowledgment

The author would like to thank the Natural Science Foundation of Anhui Province (No. KJ2011Z290).

References

[1] J. Cao, New results concerning exponential stability and periodic solutions of delayed cellular neural networks with delays, Phys Lett A 307 (2003) 136-147.
[2] Y. Li, S. Gao, Global Exponential Stability for Impulsive BAM Neural Networks with Distributed Delays on Time Scales, Neural Processing Letters 31 (1) (2010) 65-91.
[3] Z. Liu and A. Chen, Existence and global exponential stability of periodic solutions of cellular neural networks with time-vary delays, J. Math Anal. Appl. 290 (2004) 247-262.
[4] Y. Li, Existence and stability of periodic solutions for Cohen¨CGrossberg neural networks with multiple delays. Chaos, Solitons & Fractals 20 (2004) 459-466.
[5] Y. Li, P. Liu, Existence and stability of positive periodic solution for BAM neural networks with delays, Math Comput Model 40 (2004) 757-770.
[6] Y. Li, C. Liu, L. Zhu, Global exponential stability of periodic solution for shunting inhibitory CNNs with delays. Phys Lett A 337 (2005) 46-54.
[7] Y. Li, T. Zhang, Global exponential stability of fuzzy internal delayed neural networks with impulses on time scales, Int. J. Neural Syst.19 (6) (2009) 449-456.
[8] Y. Chen, Global stability of neural networks with distributed delays, Neural Networks 15 (2002) 867-871.
[9] C. Feng, R. Plamondon, On the stability analysis of delayed neural networks systems, Neural Networks 14 (2011) 1181-1188.
[10] J. Zhang, Globally exponential stability of neural networks with variables delays. IEEE Trans Circuit Syst-I 50 (2003) 288-291.
[11] Y. Li, Global exponential stability of BAM neural networks with delays and impulses, Chaos, Solitons & Fractals 24 (2005) 279-285.
[12] Y. Ren, Y. Li, Periodic solutions of recurrent neural networks with distributed delays and impulses on time scales, Int. J. Comp. Math. Sci. 5 (2011) 209-218.
[13] Z. Guan, L. James, G. Chen, On impulsive auto-associative neural networks, Neural Networks 13 (2000) 63-69.
[14] Y. Li, W. Xing, L. Lu, Existence and global exponential stability of periodic solution of a class of neural networks with impulses, Chaos, Solitons & Fractals 27 (2006) 437-445.
[15] Y. Li, W. Xing, Existence and global exponential stability of periodic solution of CNNs with impulses, Chaos, Solitons & Fractals 33 (2007) 1686-1693.
[16] R. Gains, J. Mawhin, Coincidence degree and nonlinear differential equations, Springer-Verlag, Berlin, 1977.
[17] H. Tokumarn, N. Adachi, T. Amemiya, Macroscopic stability of interconnected systems, 6th IFAC Congress, paper ID44.4, 1975, 1-7.