

# A note on the minimum cardinality of critical sets of inertias for irreducible zero-nonzero patterns of order 4

Ber-Lin Yu and Ting-Zhu Huang

*Abstract*—If there exists a nonempty, proper subset  $\mathcal{S}$  of the set of all  $(n+1)(n+2)/2$  inertias such that  $\mathcal{S} \subseteq i(\mathcal{A})$  is sufficient for any  $n \times n$  zero-nonzero pattern  $\mathcal{A}$  to be inertially arbitrary, then  $\mathcal{S}$  is called a critical set of inertias for zero-nonzero patterns of order  $n$ . If no proper subset of  $\mathcal{S}$  is a critical set, then  $\mathcal{S}$  is called a minimal critical set of inertias. In [Kim, Olesky and Driessche, Critical sets of inertias for matrix patterns, Linear and Multilinear Algebra, 57 (3) (2009) 293-306], identifying all minimal critical sets of inertias for  $n \times n$  zero-nonzero patterns with  $n \geq 3$  and the minimum cardinality of such a set are posed as two open questions by Kim, Olesky and Driessche. In this note, the minimum cardinality of all critical sets of inertias for  $4 \times 4$  irreducible zero-nonzero patterns is identified.

*Keywords*—Zero-nonzero pattern, Inertia, Critical set of inertias, Inertially arbitrary.

## I. INTRODUCTION

A  $n \times n$  zero-nonzero pattern is a matrix  $\mathcal{A} = [\alpha_{ij}]$  with entries in  $\{*, 0\}$  where  $*$  denotes a nonzero real number. The set of all real matrices  $A = [a_{ij}]$  such that  $a_{ij} \neq 0$  if and only if  $\alpha_{ij} = *$  for all  $i$  and  $j$ . If  $A \in Q(\mathcal{A})$ , then  $A$  is a realization of  $\mathcal{A}$ . A subpattern of an  $n \times n$  zero-nonzero pattern  $\mathcal{A} = [\alpha_{ij}]$  is an  $n \times n$  zero-nonzero pattern  $\mathcal{B} = [\beta_{ij}]$  such that  $\beta_{ij} = 0$  whenever  $\alpha_{ij} = 0$ . If  $\mathcal{B}$  is a subpattern of  $\mathcal{A}$ , then  $\mathcal{A}$  is a superpattern of  $\mathcal{B}$ . A zero-nonzero pattern  $\mathcal{A}$  is reducible if there is a permutation matrix  $\mathcal{P}$  such that

$$\mathcal{P}\mathcal{A}\mathcal{P}^T = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ 0 & \mathcal{A}_{22} \end{pmatrix}$$

where  $\mathcal{A}_{11}$  and  $\mathcal{A}_{22}$  are square matrices of order at least one. A pattern is irreducible if it is not reducible.

Recall that the inertia of a matrix  $A$  is an ordered triple  $i(A) = (n_+, n_-, n_0)$  where  $n_+$  is the number of eigenvalues of  $A$  with positive real part,  $n_-$  is the number of eigenvalues of  $A$  with negative real part, and  $n_0$  is the number of eigenvalues of  $A$  with zero real part. The inertial of zero-nonzero pattern  $\mathcal{A}$  is  $i(\mathcal{A}) = \{i(A) \mid A \in Q(\mathcal{A})\}$ . An  $n \times n$  zero-nonzero pattern  $\mathcal{A}$  is an inertially arbitrary pattern (IAP) if given any ordered triple  $(n_+, n_-, n_0)$  of nonnegative integers with  $n_+ + n_- + n_0 = n$ , there exists a real matrix  $A \in Q(\mathcal{A})$  such that  $i(A) = (n_+, n_-, n_0)$ . Equivalently,  $\mathcal{A}$  is an inertially arbitrary pattern if all the  $(n+1)(n+2)/2$  ordered triples  $(n_+, n_-, n_0)$  of nonnegative integers with  $n_+ + n_- + n_0 = n$  are in  $i(\mathcal{A})$ ; see, e.g., [2-4].

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Let  $S$  be a nonempty, proper subset of the set of all  $(n+1)(n+2)/2$  inertias for any  $n \times n$  zero-nonzero pattern  $\mathcal{A}$ . If  $S \subseteq i(\mathcal{A})$  is sufficient for  $\mathcal{A}$  to be inertially arbitrary, then  $S$  is said to be a critical set of inertias for zero-nonzero patterns of order  $n$  and if no proper subset of  $S$  is a critical set of inertias,  $S$  is said to be a minimal critical set of inertias for zero-nonzero patterns of order  $n$ ; see, e.g., [3]. All minimal critical sets of inertias for irreducible zero-nonzero patterns of order 2 are identified. But as posed in [3], identifying all minimal critical sets of inertias for irreducible zero-nonzero patterns of order  $n \geq 3$  is an open question. Also open is the minimum cardinality of such a set.

In this note, we concentrate on the minimum cardinality of all critical sets of inertias for irreducible zero-nonzero patterns of order 4. It is shown that the minimum cardinality of all critical sets of inertias for  $4 \times 4$  irreducible zero-nonzero patterns is 3.

## II. PRELIMINARIES AND MAIN RESULTS

A zero-nonzero pattern  $\mathcal{A} = [\alpha_{ij}]$  has an associated digraph  $D(\mathcal{A})$  with vertex set  $\{1, 2, \dots, n\}$  and for all  $i$  and  $j$ , an arc from  $i$  to  $j$  if and only if  $\alpha_{ij} = *$ . A (directed) simple cycle of length  $k$  is a sequence of  $k$  arcs  $(i_1, i_2), (i_2, i_3), \dots, (i_k, i_1)$  such that the vertices  $i_1, \dots, i_k$  are distinct. The digraph of a matrix is defined analogously; see, e.g., [1]. A digraph is strongly connected if for each vertex  $i$  and every other vertex  $j$  ( $\neq i$ ), there is an oriented path from  $i$  to  $j$ . A zero-nonzero pattern  $\mathcal{A}$  is irreducible if and only if its digraph,  $D(\mathcal{A})$ , is strongly connected. For any digraph  $D$ , let  $G(D)$  denote the underlying multigraph of  $D$ , i.e., the multigraph obtained from  $D$  by ignoring the direction of each arc; see, e.g., [2].

The following lemma 1 was stated as Proposition 2 in [2], which is useful to determine whether a zero-nonzero pattern is inertially arbitrary or not.

**Lemma 1.** *Let  $\mathcal{A}$  be an irreducible  $n \times n$  zero-nonzero pattern and let  $A \in Q(\mathcal{A})$ . If  $T$  is a direct subgraph of  $D(\mathcal{A})$  such that  $G(T)$  is a tree, then  $\mathcal{A}$  has a realization that is diagonally similar to  $A$  such that each entry corresponding to an arc of  $T$  is 1.*

We proceed by showing the following zero-nonzero pattern is nearly inertially arbitrary.

**Theorem 1** Let

$$\mathcal{N} = \begin{pmatrix} * & * & 0 & * \\ * & * & * & 0 \\ 0 & 0 & 0 & * \\ * & 0 & * & 0 \end{pmatrix}.$$

Then the zero-nonzero pattern  $\mathcal{N}$  allows all inertias  $(n_1, n_2, n_3)$  with nonnegative integers  $n_1, n_2$  and  $n_3$  such that  $n_1 + n_2 + n_3 = 4$  except inertia  $(0, 0, 4)$ .

**Proof.** Since  $(0, 0, 4) \in i(\mathcal{N})$  if and only if  $\mathcal{N}$  allows some characteristic polynomial of the form

$$x^4 + (p + q)x^2 + pq$$

for  $p, q \geq 0$ . Suppose  $A$  is a realization of  $\mathcal{N}$ . By Lemma 1, without loss of generality, let

$$A = \begin{pmatrix} a & 1 & 0 & b \\ c & d & 1 & 0 \\ 0 & 0 & 0 & 1 \\ e & 0 & f & 0 \end{pmatrix}$$

for some nonzero real numbers  $a, b, c, d, e$  and  $f$ . Then the characteristic polynomial of  $A$  is

$$p_A(x) = x^4 - (a + d)x^3 + (ad - c - be - f)x^2 + [(a + d)f + bde]x + cf - adf - e.$$

Suppose

$$p_A(x) = x^4 + (p + q)x^2 + pq$$

Then

$$a + d = 0$$

and

$$(a + d)f + bde = 0$$

It follows that

$$bde = 0.$$

It is a contradiction. Hence,  $\mathcal{N}$  does not allow  $(0, 0, 4)$ .

Next we show that the zero-nonzero pattern  $\mathcal{N}$  allows all the remaining inertias. Note that for an arbitrary zero-nonzero pattern  $\mathcal{N}$ ,  $(n_+, n_-, n_0) \in i(\mathcal{N})$  if and only if  $(n_-, n_+, n_0) \in i(\mathcal{N})$ . So to complete the proof, it suffices to show that  $\mathcal{N}$  allows inertias  $(1, 0, 3)$ ,  $(2, 0, 2)$ ,  $(1, 1, 2)$ ,  $(3, 0, 1)$ ,  $(2, 1, 1)$ ,  $(4, 0, 0)$ ,  $(3, 1, 0)$  and  $(2, 2, 0)$ .

Consider realizations of  $\mathcal{N}$

$$\begin{pmatrix} -2 & 1 & 0 & \frac{1}{2} \\ -\frac{22}{3} & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{4}{3} & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & \frac{4}{3} \\ -\frac{1}{2} & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{3}{4} & 0 & \frac{1}{2} & 0 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1}{2} & 1 & 0 & 2 \\ \frac{1}{4} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -3 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 & -2 \\ 4 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & 2 & 0 \end{pmatrix},$$

$$\begin{pmatrix} \frac{1}{2} & 1 & 0 & \frac{2}{3} \\ -\frac{3}{4} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 0 & 3 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 0 & \frac{11}{2} \\ 4 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & 4 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 0 & -2 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 3 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 & 0 & 1 \\ \frac{1}{2} & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{3}{2} & 0 & 1 & 0 \end{pmatrix}$$

with inertias  $(1, 0, 3)$ ,  $(2, 0, 2)$ ,  $(1, 1, 2)$ ,  $(3, 0, 1)$ ,  $(2, 1, 1)$ ,  $(4, 0, 0)$ ,  $(3, 1, 0)$  and  $(2, 2, 0)$ , respectively. It follows that  $\mathcal{N}$  allows all inertias except  $(0, 0, 4)$ .

**Corollary 1.** Let  $S$  be a nonempty, proper subset of the set of all  $(n + 1)(n + 2)/2$  inertias for  $4 \times 4$  irreducible zero-nonzero patterns. If  $S$  is a critical set of inertias, then  $(0, 0, 4) \in S$ .

**Proof.** By a way of contradiction assume that  $(0, 0, 4)$  does not belong to  $S$ . Then  $S$  must contain some of the rest of inertias. By Theorem 1,  $S \subseteq i(\mathcal{N})$  and  $\mathcal{N}$  is not inertially arbitrary. It follows that  $S$  is not a critical set of inertias; a contradiction.

The following result was stated as Theorem 4 in [2].

**Lemma 2.** Let the zero-nonzero pattern of order 4

$$\mathcal{M} = \begin{pmatrix} 0 & * & 0 & 0 \\ * & 0 & * & 0 \\ 0 & 0 & * & * \\ * & 0 & 0 & * \end{pmatrix}.$$

Then  $\mathcal{M}$  allows all inertias  $(n_1, n_2, n_3)$  with nonnegative integers  $n_1, n_2$  and  $n_3$  such that  $n_1 + n_2 + n_3 = 4$  except  $(1, 0, 3)$ ,  $(0, 1, 3)$ ,  $(2, 0, 2)$  and  $(0, 2, 2)$ .

The following corollary indicates that the minimum cardinality of critical sets of inertias for irreducible  $4 \times 4$  zero-nonzero patterns is at least 2.

**Corollary 2.** There is no critical set of inertias with a single inertia for irreducible  $4 \times 4$  zero-nonzero patterns. Moreover, if  $S$  is a critical set of inertias for irreducible  $4 \times 4$  zero-nonzero patterns, then  $S$  must contain  $(0, 0, 4)$  and one of the inertias  $(1, 0, 3)$ ,  $(0, 1, 3)$ ,  $(2, 0, 2)$  and  $(0, 2, 2)$ .

**Proof.** The first part of Corollary 2 follows directly from Theorem 1 and Lemma 2. If  $S$  is a critical set of inertias, then  $(0, 0, 4) \in S$  by Corollary 1. If none of the inertias  $(1, 0, 3)$ ,  $(0, 1, 3)$ ,  $(2, 0, 2)$  and  $(0, 2, 2)$  is in  $S$ , the  $S \subseteq i(\mathcal{M})$  in Lemma 2. But it is clear that  $\mathcal{M}$  is not inertially arbitrary. It follows that  $S$  is not a critical set of inertias; a contradiction.

**Theorem 2.** Let the zero-nonzero pattern of order 4

$$\mathcal{P} = \begin{pmatrix} * & * & * & * \\ * & * & 0 & 0 \\ * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \end{pmatrix}.$$

Then  $\mathcal{P}$  allows all inertias  $(n_1, n_2, n_3)$  with nonnegative integers  $n_1, n_2$  and  $n_3$  such that  $n_1 + n_2 + n_3 = 4$  except the only inertias  $(4, 0, 0)$ ,  $(0, 4, 0)$ ,  $(3, 1, 0)$ ,  $(1, 3, 0)$  and  $(2, 2, 0)$ .

**Proof.** Since  $\mathcal{P}$  requires singularity, it follows that all of the inertias  $(4, 0, 0)$ ,  $(0, 4, 0)$ ,  $(3, 1, 0)$ ,  $(1, 3, 0)$  and  $(2, 2, 0)$  are not allowed by  $\mathcal{P}$ .

Consider realizations of  $\mathcal{P}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 1 & 1 \\ -3 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 2 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 1 & 1 & 1 \\ -3 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$

with inertias  $(0, 0, 4)$ ,  $(1, 0, 3)$ ,  $(2, 0, 2)$ ,  $(1, 1, 2)$ ,  $(3, 0, 1)$  and  $(2, 1, 1)$ , respectively. It follows that the zero-nonzero pattern  $\mathcal{P}$  allows all inertias except  $(4, 0, 0)$ ,  $(0, 4, 0)$ ,  $(3, 1, 0)$ ,  $(1, 3, 0)$  and  $(2, 2, 0)$ .

It was known that the set  $\{(0, 0, 4), (1, 0, 3), (4, 0, 0)\}$  is a minimal critical set of inertias for irreducible zero-nonzero patterns of order 4. Other minimal critical sets on inertias can be obtained by replacing  $(4, 0, 0)$  or  $(1, 0, 3)$  by its reversal; see, e.g., [3, Theorem 7]. As mentioned in Section 6 in [3], for  $n = 4$ , it is unknown that whether there are other critical sets of inertias. Also mentioned is that the minimum cardinality of all critical sets of inertias for  $4 \times 4$  irreducible zero-nonzero patterns is at most 3. The next theorem answers this problem completely.

**Theorem 3.** *The minimum cardinality of all critical sets of inertias for irreducible  $4 \times 4$  zero-nonzero patterns is 3.*

**Proof.** By a way of contradiction suppose that the minimum cardinality of all critical sets of inertias is 2. Let  $S$  be an arbitrary critical set of inertias with cardinality 2. Then, by Corollary 2,  $S$  must contain  $(0, 0, 4)$  and only one of the inertias  $(1, 0, 3)$ ,  $(0, 1, 3)$ ,  $(2, 0, 2)$  and  $(0, 2, 2)$ .

**Case 1.**  $S$  contains inertias  $(0, 0, 4)$  and  $(1, 0, 3)$  or its reversal. Then  $S$  does not contain all the inertias  $(4, 0, 0)$ ,  $(0, 4, 0)$ ,  $(3, 1, 0)$ ,  $(1, 3, 0)$  and  $(2, 2, 0)$ . By Theorem 2, we have  $S \subseteq i(\mathcal{P})$  and  $\mathcal{P}$  is not inertially arbitrary. It follows that  $S$  is not a critical set of inertias for irreducible zero-nonzero patterns of order 4, which is a contradiction.

**Case 2.** The case that  $S$  contains inertias  $(0, 0, 4)$  and  $(2, 0, 2)$  or its reversal is similar to Case 1. We omit its proof.

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