Decomposition of Homeomorphism on Topological Spaces

Ahmet Z. Ozcelik, Serkan Narli

***Abstract*—**In this study, two new classes of generalized homeomorphisms are introduced and shown that one of these classes has a group structure. Moreover, some properties of these two homeomorphisms are obtained.

***Keywords*—**Generalized closed set, homeomorphism, gsg-homeomorphism, sgs-homeomorphism.

I. Introduction

LEVINE [9] has generalized the concept of closed sets to generalized closed sets. Bhattacharyya and Lahiri [2] have generalized the concept of closed sets to semi-generalized closed sets with the help of semi-open sets and obtained various topological properties. Arya and Nour [1] have defined generalized semi-open sets with the help of semi-openness and used them to obtain some characterizations of s-normal spaces. Devi, Balachandran and Maki [8] defined two classes of maps called semi-generalized homeomorphisms and generalized semi-homeomorphisms and also defined two classes of maps called sgc-homeomorphisms and gsc-homeomorphism. In this paper, we introduce two classes of maps called sgs-homeomorphisms and gsg-homeomorphisms and study their properties.

Throughout the present paper, (X, τ) *and* (Y,δ) denote topological spaces on which no separation axioms are assumed unless explicity stated. Let A be a subset of X. We denote the interior of A (respectively the closure of A) with respect to τ by Int(A) (respectively Cl(A))

II. Preliminaries

Since we shall use the following definitions and some properties, we recall them in this section.

a. A subset B of a topological space (X, τ) is said to be semi-closed if there exists a closed set F such that $\mathrm{Int}(F)\subset B\subset F$. A subset B of (X, τ) is called a semi-open set if its complement $X\backslash B$ is semi-closed in (X, τ). Every closed (respectively open) set is semi-closed (respectively semi-open) [3,5].

b. A mapping $f : (X, \tau) \rightarrow (Y, \delta)$ is said to be semi-closed if the image f(F) of each closed set F in (X, τ) is semi-closed in (Y, δ). Every closed mapping is semi-closed [10].

c. Let (X, τ) be a topological space and A be a subset of X. Then, the semiinterior and semiclosure of A are defined by:

$\mathrm{sInt}(A)= \cup\{G_i : G_i \text{ is a semi-open in X and } G_i\subset A\}$

$\mathrm{sCl}(A)= \cap\{K_i : K_i \text{ is a semi-closed in X and } A\subset K_i\}$

d. A subset B of a topological space (X, τ) is said to be semi-generalized closed (written in short as sg-closed) if $\mathrm{sCl}(B)\subset O$ whenever $B\subset O$ and O is semi-open [2]. The complement of a semi-generalized closed set is called a semi-generalized open. Every semi-closed set is sg-closed. The concepts of g-closed sets[7] and sg-closed sets are, in general, independent. The family of all sg-closed sets of any topological space (X, τ) is denoted by $sgc(X, \tau)$.

e. A subset B of a topological space (X, τ) is said to be generalized semi-open (written in short as gs-open) if $F\subset \mathrm{sInt}(B)$ whenever $F\subset B$ and F is closed. B is generalized semi-closed (written in short as gs-closed) if and only if $X\backslash B$ is gs-open. Every closed set (semi-closed set, g-closed set and sg-closed set) is gs-closed. The family of all gs-closed sets of any topological space (X, τ) is denoted by $gsc(X, \tau)$ [1].

f. A map $f : (X, \tau)\rightarrow (Y, \delta)$ is called a semi-generalized continuous map (written in short as sg-continuous mapping) if $f^{-1}(V)$ is sg-closed in (X, τ) for every closed set V of (Y, δ) [5].

g. A map $f : (X, \tau) \rightarrow (Y, \delta)$ is called a generalized semi-continuous map (written in short as gs-continuous mapping) if $f^{-1}(V)$ is gs-closed in (X, τ) for every closed set V of (Y, δ)[8].

h. A map $f : (X, \tau)\rightarrow(Y, \delta)$ is called a semi-generalized closed map (respectively semi-generalized open map) if f(V) is semi-generalized closed (respectively semi-generalized open) in (Y, δ) for every closed set (respectively open set) V of (X, τ). Every semi-closed map is a semi-generalized closed map. A semi-generalized closed map (respectively semi-generalized open map) is written shortly as sg-closed map

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(respectively sg open map) [7].

k. A map f : (X, τ) → *(Y, δ)* is called a generalized semi-closed map (respectively generalized semi-open map) if for each closed set (respectively open set) V of (X, τ), f(V) is gs-closed (respectively gs-open) in *(Y, δ)*. Every semi-closed map, every sg-closed map is a generalized semi-closed map. A generalized semi-closed map (respectively generalized semi-open map) is written shortly as gs-closed map (respectively gs open map) [7].

l. A map f : (X, τ)→*(Y, δ)* is said to be a semi-homeomorphism(B) (simply s.h. (B)) if f is continuous, f is semi-open (i.e. f(U) is semi-open for every open set U of (X, τ)) and f is bijective [4].

m. A map f : (X, τ)→*(Y, δ)* is said to be a semi-homeomorphism (C.H) (simply s.h.(C.H)) if f is irresolute (i.e. $f^{-1}(V)$ is semi-open for every semi-open set V of *(Y, δ)*), f is pre-semi-open (i.e. f(U) is semi-open for every semi-open set U of (X, τ)) and f is bijective [6].

n. A map f : (X, τ) → *(Y, δ)* is called a sg-irresolute map if $f^{-1}(V)$ is sg-closed in (X, τ) for every sg-closed set V of *(Y, δ)* [11].

o. A map f : (X, τ) → *(Y, δ)* is called a gs-irresolute map if $f^{-1}(V)$ is gs-closed in (X, τ) for every gs-closed set V of *(Y, δ)* [8].

p. A bijection f : (X, τ) → *(Y, δ)* is called a semi-generalized homeomorphism (abbreviated sg-homeomorphism) if f is both sg-continuous and sg-open [8].

r. A bijection f : (X, τ) → *(Y, δ)* is said to be a sgc-homeomorphism if f is sg-irresolute and its inverse f^{-1} is also sg-irresolute [8].

s. A bijection f : (X, τ) → *(Y, δ)* is called a generalized semi-homeomorphism (abbreviated gs-homeomorphism) if f is both gs-continuous and gs-open [8]

t. A bijection f : (X, τ) → *(Y, δ)* is said to be a gsc-homeomorphism if f is gs-irresolute and its inverse f^{-1} is also gs-irresolute [8].

u. A space (X, τ) is called a $T_{1/2}$ space if every g-closed set is closed, that is if and only if every gs-closed set is semi-closed [7,9].

v. A space (X, τ) is called a T_b space if every gs-closed set is closed [7].

III. GSG-HOMEOMORPHISM

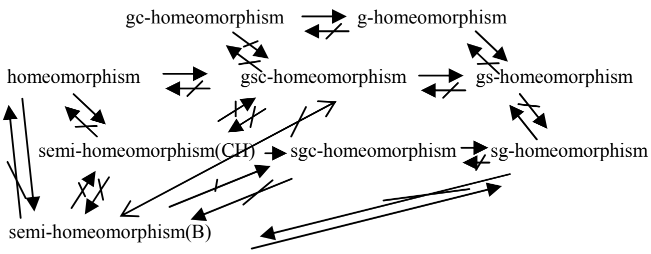
In this section, the relations between semi-homeomorphisms (B) and gsc-homeomorphisms are investigated and the diagram of implications is given. Also the gsg-homeomorphism is defined and some of its properties are obtained.

Remark 3.1. The following two examples show that the concepts of semi-homeomorphism (B) and gsc-homeomorphisms are independent of each other.

Example 3.2.
Let X = {a, b, c }, τ={ ∅, {a, b} , {b, c} , {b}, X },
δ = { ∅ , {b}, X}.
The identity map I_x : (X, τ) → *(X, δ)* is not gsc-homeomorphism. However I_x is a s.h. (B).

Example 3.3.
Let X = {a, b, c }, the topology τ on X be discrete and the topology *δ* on X be indiscrete.
The identity map I_x : (X, τ) → *(X, δ)* is not sh(B). However I_x is a gsc-homeomorphism.

Proposition 3.4. From remark 3.1 and remark 4.21 of R.Devi, K. Balachandran and H.Maki [8], we have the following diagram of implications.



Definition 3.5. A map f: (X, τ) → *(Y, δ)* is called a gsg-irresolute map if the set $f^{-1}(A)$ is sg-closed in (X, τ) for every gs-closed set A of *(Y, δ)*.

Definition 3.6. A bijection f: (X, τ) → *(Y, δ)* is called a gsg-homeomorphism if the function f and the inverse function f^{-1} are both gsg-irresolute maps. If there exists a gsg-homeomorphism from X to Y, then the spaces (X, τ) and *(Y, δ)* are said to be gsg-homeomorphic. The family of all gsg-homeomorphism of any topological space (X, τ) is denoted by gsgh(X, τ).

Remark 3.7. The following two examples show that the concepts of homeomorphism and gsg-homeomorphism are independent of each other.

Example 3.8.
Let X = {a, b, c}, τ = {∅, {b}, X}.
The identity map I_X : (X, τ) → (X, τ) is a homeomorphism but is not a gsg-homeomorphism on X.

Example 3.9.
Let X be any set which contains at least two elements; τ and δ be discrete and indiscrete topologies on X, respectively. The identity map $I_X : (X, \tau) \to (X, \delta)$ is a gsg-homeomorphism but is not a homeomorphism.

Remark 3.10. Every gsg-homeomorphism implies both a gsc-homeomorphism and a sgc- homeomorphism.

However the converse is not true as shown by the following example.

Example 3.11.
Let X = {a, b, c}, τ= {∅, {b}, X}. Then
sgc(X, τ) = { Ø, {a}, {c}, {a, c}, X} and
gsc(X, τ) = { Ø, {a}, {c}, {a, c}, {a, b}, {b, c}, X }.
The identity map $I_X : (X, \tau) \to (X, \tau)$ is both gsc-homeomorphism and sgc-homeomorphism. Since the set {b,c} is gs-closed but the set $I_X^{-1}(\{b,c\})= \{b,c\}$ is not sg-closed, then the identity map I_X is not a gsg-homeomorphism on X.

Proposition 3.12. Every gsg-homeomorphism implies both a gs-homeomorphism and a sg-homeomorphism. However its converse is not true.

Definition 3.13. Let (X, τ) and (Y, δ) be any topological spaces. If the following properties are satisfied
a) sgc(X, τ) = gsc(X, τ) and sgc(Y, δ) = gsc(Y, δ)
b) there exists a bijective map
ϕ : gsc (X, τ) $\to$ gsc(Y, δ) such that
$\forall$ A$\in$ gsc (X, τ) #(ϕ(A)) = #(A) (#(A) is cardinality of A).
then the spaces (X, τ) *and* (Y, δ) are called S-related

Theorem 3.14. The space (X, τ) and (Y, δ) are gsg-homeomorphic if and only if these spaces are S-related.

Proof. It follows from definition of gsg-homeomorphism and definitions 2.3 , 2.4

Theorem 3.15.
a) Every gsc(sgc)-homeomorphism from $T_{½}$ space onto itself is a gsg-homeomorphism.
b) Every gs(sg)-homeomorphism from T_b space onto itself is a gsg-homeomorphism.

Proof. Since for any $T_{½}$ space (X, τ) the family of sg-closed sets is equal to the family of gs-closed sets, any gsc(sgc)-homeomorphism from X to X is a gsg-homeomorphism.

In any T_b space (X, τ) every gs-closed subset is a closed subset so (b) is obvious.

Result 3.16. Let (X, τ) and *(Y, δ)* be any topological spaces. If there exists any gsg-homeomorphism from X to Y, then every gsc(sgc)-homeomorphism from X to Y is a sgc(gsc)-homeomorphism.

Proof. It is obtained by theorem 3.14

Theorem 3.17. For a topological space (X, τ) the following implications hold:
a) gsgh(X, τ) $\subset$ gsch(X, τ) $\subset$ gsh(X, τ) and
gsgh(X, τ) $\subset$ sgch(X, τ) $\subset$ sgh(X, τ)
b) If gsgh(X, τ) is nonempty then gsgh(X, τ) is a group and
sgch(X, τ) = gsch(X, τ) = gsgh(X, τ)

Proof. It follows from R. Devi, H. Maki [4], remark 3.10 and result 3.16.

Theorem 3.18. If *f:* (X, τ) → *(Y, δ)* is a gsg-homeomorphism, then it induces an isomorphism from the group gsgh(X, τ) onto gsgh(Y, δ).

Proof. The homomorphism $f_* : gsgh(X, \tau) \to gsgh(Y, \delta)$ is induced from f by $f_*(h)=fohof^{-1}$ for every h$\in$*gsgh*(X, τ). Then it easily follows that f_* is an isomorphism

IV. SGS-HOMEOMORPHISM

Definition 4.1. A map *f:* (X, τ) →*(Y, δ)* is called a sgs-irresolute map if the set $f^{-1}(A)$ is gs-closed in (X, τ) for every sg-closed set A of *(Y, δ).*

Definition 4.2. A bijection *f:* (X, τ) → *(Y, δ)* is called a sgs-homeomorphism if the function f and its inverse function f^{-1} are both sgs-irresolute maps. If there exists a sgs-homeomorphism from X to Y, then the space (X, τ) and *(Y, δ)* are said to be sgs-homeomorphic spaces.

Remark 4.3. Every sgc-homeomorphism and gsc-homeomorphism implies a sgs-homeomorphism.

Example 4.4.
Let X = Y = {a, b, c} and
τ = {{a}, {b}, {a, b}, {b, c}, X, Ø}, δ = {Ø, {b}, {a, b}, Y}.
Since sgc (X, τ) = gsc (X, τ) = þ (X)\ {{b}, {a, b}} (þ (X) is power set of X) and
sgc(Y, δ) ={{c},{a},{a, c},Ø,X},gsc(Y, δ)=þ(Y)\{{b},{a, b}},
then the identitiy map $I_X : (X, \tau) \to (Y, \delta)$ is a sgs-homeomorphism but is not a sgc-homeomorphism.

Example 4.5.
Let X = Y = {a, b, c} and
τ = {∅, {a}, X}, δ = {∅, {b}, {a, b}, Y}. Since
sgc(X, τ) ={{b}, {c}, {b, c}, X, Ø }, gsc(X, τ) ={ {b}, {c}, {b, c}, {a, c}, {a, b}, X, Ø} and
sgc(Y, δ) = { {c}, {a}, {a, c}, Y, Ø }, gsc(Y, δ) = { {a}, {c}, {a, c}, {b, c}, Y, Ø } then the mapping
f : (X, τ) → *(Y, δ),*defined by *f(a) = b, f (b) = a, f(c)= c*
is a sgs-homeomorphism but is not a gsc-homeomorphism.

Result 4.6. Every homeomorphism is a sgs-homemorphism but the converse is not true.

Remark 4.7. Every sgs-homeomorphism is a gs-homeomorphism and the converse is not true as seen from the following example:

Example 4.8.
Let X = Y = {a, b, c}, τ = {∅, {a, b}, X}, δ = {∅, {b}, {a, b}, Y} since sgc(X, τ) = gsc(X, τ) = {{c}, {a, c}, {b, c}, X, Ø }, sgc(Y, δ) = {{c}, {a}, {a, c}, Y, Ø} and gsc(Y, δ) = {{a}, {c}, {b, c}, {a, c}, Y, Ø }
Then, the identity mapping I: (X, τ) → (Y, δ) is a gs-homeomorphism but it is not sgs-homeomorphism.

Example 4.9.
The map I : (X, τ) → (Y, δ) is given by Example 4.8 is a sg-homeomorphism but is not a sgs-homeomorphism.

Result 4.10.
a) From the example 4.9 we can see that any sg-homeomorphism is not a sgs-homeomorphism.
b) Every gsg-homeomorphism is a sgs-homeomorphism and the converse is not true as seen from the following example.

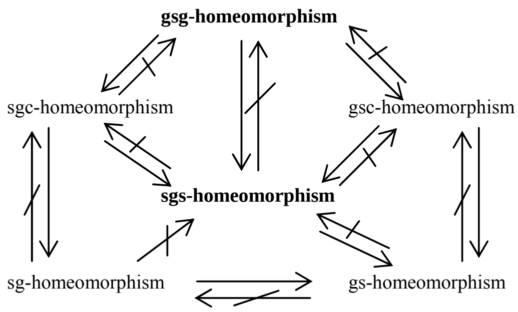
Example 4.12.
Let X = Y = {a, b, c} and
τ = {∅,{a}, {a, b}, X}, δ = {∅,{a}, {b}, {a, b},{b, c}, Y}. Then the mapping
f : (X, τ) → *(Y, δ)* defined by *f(a) = b, f (b) = a and f(c) = c* is a sgs-homeomorphism. However f is not a gsg-homeomorphism.

Theorem 4.13.
a) Every sgs-homeomorphism from a $T_{1/2}$ space onto itself is a gsg-homeomorphism. This implies that sgs-homeomorphism is both a sgc-homeomorphism and gsc-homeomorphism.
b) Every sgs-homeomorphism from a T_b space onto itself is a homeomorphism. This implies that sgs-homeomorphism is a gs-homeomorphism, a sg-homeomorphism, a sgc-homeomorphism, a gsc-homeomorphism and a gsg-homeomorphism.
c) Every sgs-homeomorphism from a $T_{1/2}$ space onto itself is a sh (CH).

Proof.
a) In a $T_{1/2}$ space, every gs-closed set is a semi-closed set.
b) In a T_b space, every gs-closed set is a closed set.
c) Follows from the definition of $T_{1/2}$ space.

V. CONCLUSION

In this paper, we introduce two classes of maps called sgs-homeomorphisms and gsg-homeomorphisms and study their properties. From all of the above statements, we have the following diagram:



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