

Two New Collineations of some Moufang-Klingenberg Planes

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Abstract—In this paper we are interested in Moufang-Klingenberg planes $M(\mathcal{A})$ defined over a local alternative ring \mathcal{A} of dual numbers. We introduce two new collineations of $M(\mathcal{A})$.

Keywords—Moufang-Klingenberg planes, local alternative ring, projective collineation.

I. INTRODUCTION

The number of collineations of any projective plane is huge. For example; the Fano plane has 168 collineations, the non-Desarguesian projective Veblen-Wedderburn plane of order 9 (which is denoted by $\pi_N(9)$) has 311,040 collineations [8, p. 366]. It is easy to see that the composite of any two collineations is a collineation, as the inverses of any collineation. Function composition is always associative; thus the collineations of any projective or affine plane form a group. For more detailed information about these groups, the reader is referred to the books of [5], [8].

In this paper we deal with the class (which we will denote by $M(\mathcal{A})$) of Moufang-Klingenberg (MK) planes coordinatized by a local alternative ring $\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$ (an alternative field \mathbf{A} , $\varepsilon \notin \mathbf{A}$ and $\varepsilon^2 = 0$) introduced by Blunck in [3]. We will introduce two collineations of $M(\mathcal{A})$, different from the collineations given in [4].

The paper is organized as follows. Section 2 includes some basic definitions and results from the literature. In Section 3 we will give two transformations of $M(\mathcal{A})$ and show that the transformations are collineations $M(\mathcal{A})$.

II. PRELIMINARIES

Let $M = (\mathbf{P}, \mathbf{L}, \in, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \in)$ (points, lines, incidence) and an equivalence relation ' \sim ' (neighbour relation) on \mathbf{P} and on \mathbf{L} , respectively. Then M is called a *projective Klingenberg plane* (PK-plane), if it satisfies the following axioms:

(PK1) If P, Q are non-neighbour points, then there is a unique line PQ through P and Q .

(PK2) If g, h are non-neighbour lines, then there is a unique point $g \cap h$ on both g and h .

(PK3) There is a projective plane $M^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$ and an incidence structure epimorphism $\Psi : M \rightarrow M^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim Q, \quad \Psi(g) = \Psi(h) \Leftrightarrow g \sim h$$

hold for all $P, Q \in \mathbf{P}$, $g, h \in \mathbf{L}$.

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A point $P \in \mathbf{P}$ is called *near* a line $g \in \mathbf{L}$ iff there exists a line $h \sim g$ such that $P \in h$.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a *collineation* of M .

A *Moufang-Klingenberg plane* (MK-plane) is a PK-plane M that generalizes a Moufang plane, and for which M^* is a Moufang plane (for the exact definition see [1]).

An *alternative ring (field)* \mathbf{R} is a not necessarily associative ring (field) that satisfies the alternative laws

$$a(ab) = a^2b, \quad (ba)a = ba^2, \quad \forall a, b \in \mathbf{R}.$$

An alternative ring \mathbf{R} with identity element 1 is called *local* if the set \mathbf{I} of its non-unit elements is an ideal.

We are now ready to give consecutively two important lemmas related to alternative rings.

Lemma 2.1: The subring generated by any two elements of an alternative ring is associative (cf. [7, Theorem 3.1]).

Lemma 2.2: The identities

$$\begin{aligned} x(y(xz)) &= (xyx)z \\ ((yx)z)x &= y(xzx) \\ (xy)(zx) &= x(yz)x \end{aligned}$$

which are known as Moufang identities are satisfied in every alternative ring (cf. [6, p. 160]).

We summarize some basic concepts about the coordinatization of MK-planes from [1].

Let \mathbf{R} be a local alternative ring. Then $M(\mathbf{R}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$ is the incidence structure with neighbour relation defined as follows:

$$\begin{aligned} \mathbf{P} &= \{(x, y, 1) : x, y \in \mathbf{R}\} \\ &\cup \{(1, y, z) : y \in \mathbf{R}, z \in \mathbf{I}\} \\ &\cup \{(w, 1, z) : w, z \in \mathbf{I}\}, \\ \mathbf{L} &= \{[m, 1, p] : m, p \in \mathbf{R}\} \\ &\cup \{[1, n, p] : p \in \mathbf{R}, n \in \mathbf{I}\} \\ &\cup \{[q, n, 1] : q, n \in \mathbf{I}\}, \\ [m, 1, p] &= \{(x, xm + p, 1) : x \in \mathbf{R}\} \\ &\cup \{(1, zp + m, z) : z \in \mathbf{I}\}, \\ [1, n, p] &= \{(yn + p, y, 1) : y \in \mathbf{R}\} \\ &\cup \{(zp + n, 1, z) : z \in \mathbf{I}\}, \\ [q, n, 1] &= \{(1, y, yn + q) : y \in \mathbf{R}\} \\ &\cup \{(w, 1, wq + n) : w \in \mathbf{I}\} \end{aligned}$$

and

$$\begin{aligned} P = (x_1, x_2, x_3) \sim (y_1, y_2, y_3) = Q &\Leftrightarrow \\ x_i - y_i \in \mathbf{I} \quad (i = 1, 2, 3), \forall P, Q \in \mathbf{P}; & \\ g = [x_1, x_2, x_3] \sim [y_1, y_2, y_3] = h &\Leftrightarrow \\ x_i - y_i \in \mathbf{I} \quad (i = 1, 2, 3), \forall g, h \in \mathbf{L}. & \end{aligned}$$

Now it is time to give the following theorem from [1].

Theorem 2.1: $\mathbf{M}(\mathbf{R})$ is an MK-plane, and each MK-plane is isomorphic to some $\mathbf{M}(\mathbf{R})$.

Let \mathbf{A} be an alternative field and $\varepsilon \notin \mathbf{A}$. Consider $\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$ with componentwise addition and multiplication as follows:

$$(a_1 + a_2\varepsilon)(b_1 + b_2\varepsilon) = a_1b_1 + (a_1b_2 + a_2b_1)\varepsilon,$$

where $a_i, b_i \in \mathbf{A}$, $i = 1, 2$. Then \mathcal{A} is a local alternative ring with ideal $\mathbf{I} = \mathbf{A}\varepsilon$ of non-units. The set of formal inverses of the non-units of \mathcal{A} is denoted as \mathbf{I}^{-1} . Calculations with the elements of \mathbf{I}^{-1} are defined as follows [2]:

$$\begin{aligned} (a\varepsilon)^{-1} + t &: = (a\varepsilon)^{-1} := t + (a\varepsilon)^{-1}, \\ q(a\varepsilon)^{-1} &: = (aq^{-1}\varepsilon)^{-1}, \\ (a\varepsilon)^{-1}q &: = (q^{-1}a\varepsilon)^{-1}, \\ ((a\varepsilon)^{-1})^{-1} &: = a\varepsilon, \end{aligned}$$

where $(a\varepsilon)^{-1} \in \mathbf{I}^{-1}$, $t \in \mathcal{A}$, $q \in \mathcal{A} \setminus \mathbf{I}$. (Other terms are not defined.). For more information about \mathcal{A} and its relation to MK-planes, the reader is referred to the papers of Blunck [2], [3]. In [3], the centre $\mathbf{Z}(\mathcal{A})$ is defined to be the (commutative, associative) subring of \mathcal{A} which is commuting and associating with all elements of \mathcal{A} . It is $\mathbf{Z}(\mathcal{A}) := \mathbf{Z}(\varepsilon) = \mathbf{Z} + \mathbf{Z}\varepsilon$ where $\mathbf{Z} = \{z \in \mathbf{A} : za = az, \forall a \in \mathbf{A}\}$ is the centre of \mathbf{A} . If \mathbf{A} is not associative, then \mathbf{A} is a Cayley division algebra over its centre \mathbf{Z} . Throughout this paper we assume $\text{char} \mathbf{A} \neq 2$ and we restrict ourselves to the MK-planes $\mathbf{M}(\mathcal{A})$. In the next section, we will introduce two collineations of $\mathbf{M}(\mathcal{A})$.

III. TWO COLLINEATIONS OF $\mathbf{M}(\mathcal{A})$

In this section we will give two transformations. We will show that these are collineations of $\mathbf{M}(\mathcal{A})$.

Now we start with giving the transformations, where $w, z, q, n \in \mathbf{A}$: For any $s \notin \mathbf{I}$, the map J_s transforms points and lines as follows:

$$\begin{aligned} (x, y, 1) &\rightarrow (ys^{-1}, xs, 1), \\ (1, y, z\varepsilon) &\rightarrow (1, sy^{-1}s, s(y^{-1}z)) \quad \text{if } y \notin \mathbf{I}, \\ (1, y, z\varepsilon) &\rightarrow (s^{-1}ys^{-1}, 1, s^{-1}z) \quad \text{if } y \in \mathbf{I}, \\ (w\varepsilon, 1, z\varepsilon) &\rightarrow (1, sws, sz) \end{aligned}$$

and

$$\begin{aligned} [m, 1, k] &\rightarrow [sm^{-1}s, 1, -(km^{-1})s] \quad \text{if } m \notin \mathbf{I}, \\ [m, 1, k] &\rightarrow [1, s^{-1}ms^{-1}, ks^{-1}] \quad \text{if } m \in \mathbf{I}, \\ [1, n\varepsilon, p] &\rightarrow [sns, 1, ps], \\ [q\varepsilon, n\varepsilon, 1] &\rightarrow [sn, s^{-1}q, 1]. \end{aligned}$$

For any $s \notin \mathbf{I}$, the map H_s transforms points and lines as follows:

$$\begin{aligned} (x, y, 1) &\rightarrow (s((y+s)^{-1}x), (s(y+s)^{-1})y, 1) \\ &\text{if } y+s \notin \mathbf{I}, \\ (x, y, 1) &\rightarrow (1, x^{-1}y, (x^{-1}(y+s))s^{-1}) \\ &\text{if } y+s \in \mathbf{I} \wedge x \notin \mathbf{I}, \\ (x, y, 1) &\rightarrow (y^{-1}x, 1, y^{-1}((y+s)s^{-1})) \\ &\text{if } y+s \in \mathbf{I} \wedge x \in \mathbf{I}, \\ (1, y, z\varepsilon) &\rightarrow (s(y+zs)^{-1}, (s(y+zs)^{-1})y, 1) \\ &\text{if } y \notin \mathbf{I}, \\ (1, y, z\varepsilon) &\rightarrow (1, y, z+ys^{-1}) \\ &\text{if } y \in \mathbf{I}, \\ (w\varepsilon, 1, z\varepsilon) &\rightarrow ((s(1+zs)^{-1})w, s(1+zs)^{-1}, 1) \end{aligned}$$

and

$$\begin{aligned} [m, 1, k] &\rightarrow \left[\begin{array}{c} m - (ms^{-1})((s(s+k)^{-1})k), \\ 1, (s(s+k)^{-1})k \end{array} \right] \\ &\text{if } s+k \notin \mathbf{I}, \\ [m, 1, k] &\rightarrow [1, s^{-1}((s+k)m^{-1}), -km^{-1}] \\ &\text{if } s+k \in \mathbf{I} \wedge m \notin \mathbf{I}, \\ [m, 1, k] &\rightarrow [-mk^{-1}, k^{-1}((s+k)s^{-1}), 1] \\ &\text{if } s+k \in \mathbf{I} \wedge m \in \mathbf{I}, \\ [1, n\varepsilon, p] &\rightarrow [(sn-p)^{-1}s, 1, -p((sn-p)^{-1}s)] \\ &\text{if } p \notin \mathbf{I}, \\ [1, n\varepsilon, p] &\rightarrow [1, n-s^{-1}p, p] \\ &\text{if } p \in \mathbf{I}, \\ [q\varepsilon, n\varepsilon, 1] &\rightarrow [-q(s(1+ns)^{-1}), 1, s(1+ns)^{-1}]. \end{aligned}$$

Now we are ready to give the main result of the paper.

Theorem 3.1: The transformations J_s and H_s , defined above, are collineations of $\mathbf{M}(\mathcal{A})$.

Proof: The proof can be done by direct computation with using Moufang identities and properties of the local alternative rings (cf [1]). We will only show that J_s preserves the incidence relation (i.e. $P \in l \Leftrightarrow J_s(P) \in J_s(l)$) and the neighbour relation (i.e. $P \sim Q \Leftrightarrow J_s(P) \sim J_s(Q)$ and $g \sim h \Leftrightarrow J_s(g) \sim J_s(h)$).

Case 1. Let $P = (x, y, 1)$. Then $J_s(P) = (ys^{-1}, xs, 1)$.

1.1. Let $l = [m, 1, k]$.

1.1.1. If $m \in \mathbf{I}$, then since $J_s(P) = (ys^{-1}, xs, 1)$ and $J_s(l) = [1, s^{-1}ms^{-1}, ks^{-1}]$, we have $J_s(P) \in J_s(l) \Leftrightarrow ys^{-1} = (xs)(s^{-1}ms^{-1}) + ks^{-1}$. By Lemma 2.1 and 2.2, we get $J_s(P) \in J_s(l) \Leftrightarrow ys^{-1} = ((xss^{-1})m)s^{-1} + ks^{-1} = (xm)s^{-1} + ks^{-1}$ and multiplying by s on the right, we find $y = xm + k \Leftrightarrow P \in l$.

1.1.2. If $m \notin \mathbf{I}$, then since $J_s(P) = (ys^{-1}, xs, 1)$ and $J_s(l) = [sm^{-1}s, 1, -(km^{-1})s]$, we have $J_s(P) \in J_s(l) \Leftrightarrow xs = (ys^{-1})(sm^{-1}s) - (km^{-1})s$. Again by Lemma 2.1 and 2.2, we get $J_s(P) \in J_s(l) \Leftrightarrow xs = ((ys^{-1}s)m^{-1})s +$

$-(km^{-1})s = (ym^{-1})s - (km^{-1})s$ and multiplying by s^{-1} on the right, we obtain $J_s(P) \in J_s(l) \Leftrightarrow x = ym^{-1} - km^{-1}$. Finally, by multiplying both sides on the right by m , $y = xm + k \Leftrightarrow P \in l$.

1.2. Let $l = [1, n\varepsilon, p]$ where $n \in \mathbf{A}$. Then since $J_s(P) = (ys^{-1}, xs, 1)$ and $J_s(l) = [sns, 1, ps]$, we have $J_s(P) \in J_s(l) \Leftrightarrow xs = (ys^{-1})(sns) + ps \Leftrightarrow xs = ((ys^{-1}s)n)s + ps \Leftrightarrow x = yn + p \Leftrightarrow P \in l$.

1.3. Let $l = [q\varepsilon, n\varepsilon, 1]$ where $q, n \in \mathbf{A}$. In this case $P \notin l$. Since $J_s(P) = (ys^{-1}, xs, 1)$ and $J_s(l) = [sn, s^{-1}q, 1]$ then $J_s(P) \notin J_s(l)$. So, $P \notin l \Leftrightarrow J_s(P) \notin J_s(l)$.

Case 2. Let $P = (1, y, z\varepsilon)$ where $z \in \mathbf{A}$.

2.1. Let $l = [m, 1, k]$.

2.1.1. If $m \in \mathbf{I}$ and $y \in \mathbf{I}$ then since $J_s(P) = (s^{-1}ys^{-1}, 1, s^{-1}z)$ and $J_s(l) = [1, s^{-1}ms^{-1}, ks^{-1}]$. In this case, we have $J_s(P) \in J_s(l) \Leftrightarrow s^{-1}ys^{-1} = s^{-1}ms^{-1} + (s^{-1}z)(ks^{-1})$. By Lemma 2.1, we obtain $J_s(P) \in J_s(l) \Leftrightarrow s^{-1}ys^{-1} = s^{-1}ms^{-1} + s^{-1}(zk)s^{-1}$. By multiplying both sides on the right and left by s we find $J_s(P) \in J_s(l) \Leftrightarrow y = m + zk$ and so $J_s(P) \in J_s(l) \Leftrightarrow P \in l$.

2.1.2. If $m \in \mathbf{I}$ and $y \notin \mathbf{I}$ then $y = m + zk \in \mathbf{I}$, which is a contradiction. That is, $P \notin l$. Also as a direct consequence of the coordinatization of $\mathbf{M}(\mathcal{A})$, $J_s(P) \notin J_s(l)$.

2.1.3. If $m \notin \mathbf{I}$ and $y \in \mathbf{I}$ then $y - zk = m \in \mathbf{I}$, which is a contradiction. That is, $P \notin l$. Also as a direct consequence of the coordinatization of $\mathbf{M}(\mathcal{A})$, $J_s(P) \notin J_s(l)$.

2.1.4. If $m \notin \mathbf{I}$ and $y \notin \mathbf{I}$ then since $J_s(P) = (1, sy^{-1}s, s(y^{-1}z))$ and $J_s(l) = [sm^{-1}s, 1, -(km^{-1})s]$. In this case, we have $P \in l \Rightarrow y = m + zk$. where $y^{-1} = m^{-1} - m^{-1}(zk)m^{-1}$. By Lemma 2.2, we get $y^{-1} = m^{-1} - (m^{-1}z)(km^{-1})$. Note that $m^{-1}z = y^{-1}z$ where $z \in \mathbf{I}$. So, $y^{-1} = m^{-1} - (y^{-1}z)(km^{-1})$. By multiplying both sides on the right and left by s , we find $sy^{-1}s = sm^{-1}s - s((y^{-1}z)((km^{-1}))s)$. By Lemma 2.2, we obtain $sy^{-1}s = sm^{-1}s - (s(y^{-1}z))((km^{-1})s)$ which means that $J_s(P) \in J_s(l)$. Conversely, let $J_s(P) \in J_s(l) \Rightarrow sy^{-1}s = sm^{-1}s - (s(y^{-1}z))((km^{-1})s)$. Since $m \notin \mathbf{I}$ and $y \notin \mathbf{I}$, there exists $m_1, m_2, y_1, y_2 \in \mathbf{A}$ such that $m_1 \neq 0 \neq y_1$ and $m = m_1 + m_2\varepsilon, y = y_1 + y_2\varepsilon$. Then using the inverses $m^{-1} = m_1^{-1} - m_1^{-1}m_2m_1^{-1}\varepsilon$ and $y^{-1} = y_1^{-1} - y_1^{-1}y_2y_1^{-1}\varepsilon$ of m and y , respectively;

$$(1, sy^{-1}s, s(y^{-1}z)) \in [sm^{-1}s, 1, -(km^{-1})s] \\ \Leftrightarrow \begin{cases} y_1^{-1} = m_1^{-1} \Leftrightarrow y_1 = m_1 \\ y_1^{-1}y_2y_1^{-1} = m_1^{-1}m_2m_1^{-1} \\ + (y_1^{-1}z)(km_1^{-1}) \end{cases}$$

(in which k has the form $k_1 + k_2\varepsilon$ where $k_1, k_2 \in \mathbf{A}$) and so the solution of this equation system is

$$y_1^{-1}y_2y_1^{-1} = y_1^{-1}m_2y_1^{-1} + (y_1^{-1}z)(k_1y_1^{-1}).$$

Since all terms of this equation are elements of Cayley division ring \mathbf{A} , Moufang identities are valid. Therefore,

$$y_1^{-1}y_2y_1^{-1} = y_1^{-1}m_2y_1^{-1} + y_1^{-1}(zk_1)y_1^{-1} \\ = y_1^{-1}(m_2 + zk_1)y_1^{-1} \\ = y_1^{-1}((m_2 + zk_1)y_1^{-1})$$

is obtained. Then we have

$$y_1^{-1}y_2y_1^{-1} = y_1^{-1}((m_2 + zk_1)y_1^{-1}) \Leftrightarrow y_2 = m_2 + zk_1$$

from Lemma 2.1. Finally we have $y_1 = m_1$ and $y_2 = m_2 + zk_1$ which means that $P \in l$.

2.2. Let $l = [1, n\varepsilon, p]$ where $n \in \mathbf{A}$. In this case, $J_s(l) = [sns, 1, ps]$ and $P \notin l$.

2.2.1. If $y \in \mathbf{I}$, then $J_s(P) = (s^{-1}ys^{-1}, 1, s^{-1}z)$. Also as a direct consequence of the coordinatization of $\mathbf{M}(\mathcal{A})$, $J_s(P) \notin J_s(l)$.

2.2.2. If $y \notin \mathbf{I}$, then $J_s(P) = (1, sy^{-1}s, s(y^{-1}z))$. In this case we have $J_s(P) \in J_s(l) \Leftrightarrow sy^{-1}s = sns + (s(y^{-1}z))(ps)$. By Lemma 2.2, $J_s(P) \in J_s(l) \Leftrightarrow sy^{-1}s = sns + s((y^{-1}z)p)s$. By multiplying both sides on the right and left by s^{-1} , we find $J_s(P) \in J_s(l) \Leftrightarrow y^{-1} = n + (y^{-1}z)p \in \mathbf{I}$ which contradicts with our hypothesis $y \notin \mathbf{I}$. That is, $J_s(P) \notin J_s(l)$.

2.3. Let $l = [q\varepsilon, n\varepsilon, 1]$ where $q, n \in \mathbf{A}$. In this case, $J_s(l) = [sn, s^{-1}q, 1]$.

2.3.1. If $y \in \mathbf{I}$, then $J_s(P) = (s^{-1}ys^{-1}, 1, s^{-1}z)$. So we have $J_s(P) \in J_s(l) \Leftrightarrow s^{-1}z = (s^{-1}ys^{-1})(sn) + s^{-1}q$. By Lemma 2.1 and 2.2, we get $J_s(P) \in J_s(l) \Leftrightarrow s^{-1}z = s^{-1}(y(s^{-1}sn)) + s^{-1}q$. By multiplying both sides on the left by s , we find $J_s(P) \in J_s(l) \Leftrightarrow z = yn + q$. So, $J_s(P) \in J_s(l) \Leftrightarrow P \in l$.

2.3.2. If $y \notin \mathbf{I}$, then $J_s(P) = (1, sy^{-1}s, s(y^{-1}z))$. In his case, we have $J_s(P) \in J_s(l) \Leftrightarrow s(y^{-1}z) = sn + (sy^{-1}s)(s^{-1}q)$. By Lemma 2.1 and 2.2, we get $J_s(P) \in J_s(l) \Leftrightarrow s(y^{-1}z) = sn + s(y^{-1}(ss^{-1}q))$. By multiplying both sides on the left by s , we find $J_s(P) \in J_s(l) \Leftrightarrow y^{-1}z = n + y^{-1}q$. By multiplying both sides on the left by y , we obtain $J_s(P) \in J_s(l) \Leftrightarrow z = yn + q$. So, we get $J_s(P) \in J_s(l) \Leftrightarrow P \in l$.

Case 3. Let $P = (w\varepsilon, 1, z\varepsilon)$ where $w, z \in \mathbf{A}$. Then $J_s(P) = (1, sws, sz)$.

3.1. Let $l = [m, 1, k]$. Then from the coordinatization of $\mathbf{M}(\mathcal{A})$ we obviously have $P \notin l$.

3.1.1. If $m \in \mathbf{I}$, then $J_s(l) = [1, s^{-1}ms^{-1}, ks^{-1}]$. Also as a direct consequence of the coordinatization of $\mathbf{M}(\mathcal{A})$, $J_s(P) \notin J_s(l)$.

3.1.2. If $m \notin \mathbf{I}$, then $J_s(l) = [sm^{-1}s, 1, -(km^{-1})s]$. In this case, we have $J_s(P) \in J_s(l) \Leftrightarrow sws = sm^{-1}s - (sz)((km^{-1})s)$. By Lemma 2.2, we get $J_s(P) \in J_s(l) \Leftrightarrow sws = sm^{-1}s - s(z(km^{-1}))s$. By multiplying both sides on the right and left by s^{-1} , we find $J_s(P) \in J_s(l) \Leftrightarrow w = m^{-1} - z(km^{-1})$. So, we obtain $J_s(P) \in J_s(l) \Leftrightarrow m^{-1} = w + z(km^{-1}) \in \mathbf{I}$ which contradicts with our hypothesis $m \notin \mathbf{I}$. That is, $J_s(P) \notin J_s(l)$.

3.2. Let $l = [1, n\varepsilon, p]$ where $n \in \mathbf{A}$. Then $J_s(P) = [sns, 1, ps]$. In this case, $J_s(P) \in J_s(l) \Leftrightarrow sws = sns + (sz)(ps)$. By Lemma 2.2, we have $J_s(P) \in J_s(l) \Leftrightarrow sws = sns + s(zp)s$. By multiplying both sides on the right and left by s^{-1} , we obtain $J_s(P) \in J_s(l) \Leftrightarrow w = n + zp$ which means that $P \in l$.

3.3. Let $l = [q\varepsilon, n\varepsilon, 1]$ where $q, n \in \mathbf{A}$. Then $J_s(l) = [sn, s^{-1}q, 1]$. In this case we have $J_s(P) \in J_s(l) \Leftrightarrow sz = sn + (sws)(s^{-1}q) = sn + s(wq)$ by Lemma 2.1 and 2.2. By multiplying both sides on the left by s^{-1} , we find $J_s(P) \in J_s(l) \Leftrightarrow z = n + wq$ which means that $P \in l$.

Now, we will show that J_s preserves the neighbour relation for the point and the lines by using properties of ideals. The case in which the most complicated computations arise is when $m, u \notin \mathbf{I}$ for the lines $[m, 1, k]$ and $[u, 1, v]$. Therefore we give the proof for only this case. Then $J_s([m, 1, k]) = [sm^{-1}s, 1, -(km^{-1})s]$ and $J_s([u, 1, v]) = [su^{-1}s, 1, -(vu^{-1})s]$ and also

$$\begin{aligned} [sm^{-1}s, 1, -(km^{-1})s] &\sim [su^{-1}s, 1, -(vu^{-1})s] \\ \Leftrightarrow m^{-1} - u^{-1} \in \mathbf{I} \wedge vu^{-1} - km^{-1} \in \mathbf{I} \\ \Leftrightarrow m_1^{-1} - u_1^{-1} = 0, v_1u_1^{-1} - k_1m_1^{-1} = 0 \\ \Leftrightarrow m_1 = u_1, v_1 = k_1 \\ \Leftrightarrow m_1 - u_1 = 0, v_1 - k_1 = 0 \\ \Leftrightarrow m - u \in \mathbf{I} \wedge v - k \in \mathbf{I} \text{ (or } k - v \in \mathbf{I}) \\ \Leftrightarrow [m, 1, k] \sim [u, 1, v]. \end{aligned}$$

■

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