International Journal of Mechanical, Industrial and Aerospace Sciences
ISSN: 2517-9950
Vol:4, No:3, 2010

Vibration of Functionally Graded Cylindrical Shells Under Effect Clamped-Free Boundary Conditions Using Hamilton's Principle

M.R. Isvandzibaei, M.R. Alinaghizadeh, and A.H. Zaman

Abstract—In the present work, study of the vibration of thin cylindrical shells made of a functionally gradient material (FGM) composed of stainless steel and nickel is presented. Material properties are graded in the thickness direction of the shell according to volume fraction power law distribution. The objective is to study the natural frequencies, the influence of constituent volume fractions and the effects of boundary conditions on the natural frequencies of the FG cylindrical shell. The study is carried out using third order shear deformation shell theory. The analysis is carried out using Hamilton's principle. The governing equations of motion of FG cylindrical shells are derived based on shear deformation theory. Results are presented on the frequency characteristics, influence of constituent volume fractions and the effects of clamped-free boundary conditions

Keywords—Vibration, FGM, Cylindrical shell, Hamilton's principle, Clamped supported.

I. Introduction

Cylindrical shells have found many applications in the industry. They are often used as load bearing structures for aircrafts, ships and buildings. Understanding of vibration behavior of cylindrical shells is an important aspect for the successful applications of cylindrical shells. Researches on free vibrations of cylindrical shells have been carried out extensively [1-5]. Recently, the present authors presented studies on the influence of boundary conditions on the frequencies of a multi–layered cylindrical shell [6]. In all the above works, different thin shell theories based on Love–hypothesis were used. Vibration of cylindrical shells with ring support is considered by Loy and Lam [7]. The concept of functionally graded materials (FGMs) was first introduced in 1984 by a group of materials scientists in Japan [8-9] as a means of preparing thermal barrier materials. Since then, FGMs have attracted much interest as heat-shielding materials. FGMs are made by combining different materials using power metallurgy methods [10]. They possess variations in constituent volume fractions that lead to continuous change in the composition, microstructure, porosity, etc., resulting in gradients in the mechanical and thermal properties [11-12].

Vibration study of FGM shell structures is important. In this paper a study on the vibration of FG cylindrical shells is presented. The FGMs considered are composed of stainless steel and nickel where the volume fractions follow a power-law distribution. The study is carried out based on third order shear deformation shell theory. The analysis is carried out using Hamilton's principle. Studies are carried out for cylindrical shells with clamped-free (C-F) boundary conditions. Results are presented on the frequency characteristics, influence of constituent volume fractions and the effects of clamped-free boundary conditions.

M.R. Isvandzibaei and A.H. Zaman are with the Department of Mechanical Engineering, Islamic Azad University, Andimeshk Branch, Andimeshk, Iran e-mail: (esvandzebaei@yahoo.com).

M.R. Alinaghizadeh is with the Department of Mathematics, Islamic Azad University, Shushtar Branch, Shushtar, Iran.

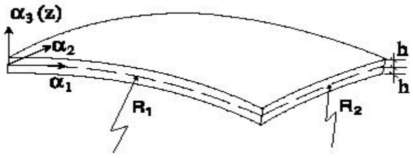

Fig. 1. Geometry of a generic shell

II. Functionally Graded Materials

For the cylindrical shell made of FGM the material properties such as the modulus of elasticity$E$, Poisson ratio$\nu$and the mass density$\rho$are assumed to be functions of the volume fraction of the constituent materials when the coordinate axis across the shell thickness is denoted by $z$and measured from the shell's middle plane. The functional relationships between$E$,ν and ρ with z for a stainless steel and nickel FGM shell are assumed as [13].

$$E = (E_1 - E_2)\left(\frac{2Z+h}{2h}\right)^N + E_2 \tag{1}$$

$$\nu = (\nu_1 - \nu_2)\left(\frac{2Z+h}{2h}\right)^N + \nu_2 \tag{2}$$

$$\rho = (\rho_1 - \rho_2)\left(\frac{2Z+h}{2h}\right)^N + \rho_2 \tag{3}$$

The strain-displacement relationships for a thin shell [14].

$$\in_{11}= \frac{1}{A_1(1+\frac{\alpha_3}{R_1})}\left[\frac{\partial U_1}{\partial \alpha_1} + \frac{U_2}{A_2}\frac{\partial A_1}{\partial \alpha_2} + U_3\frac{A_1}{R_1}\right] \tag{4}$$

$$\in_{22} \frac{1}{A_2(1+\frac{\alpha_3}{R_2})}\left[\frac{\partial U_2}{\partial \alpha_2} + \frac{U_1}{A_1}\frac{\partial A_2}{\partial \alpha_1} + U_3\frac{A_2}{R_2}\right] \tag{5}$$

$$\in_{33}= \frac{\partial U_3}{\partial \alpha_3} \tag{6}$$

$$\in_{12}= \frac{A_1(1+\frac{\alpha_3}{R_1})}{A_2(1+\frac{\alpha_3}{R_2})}\frac{\partial}{\partial\alpha_2}(\frac{U_1}{A_1(1+\frac{\alpha_3}{R_1})}) + \frac{A_2(1+\frac{\alpha_3}{R_2})}{A_1(1+\frac{\alpha_3}{R_1})}\frac{\partial}{\partial\alpha_1}(\frac{U_2}{A_2(1+\frac{\alpha_3}{R_2})}) \quad (7)$$

$$\in_{13}= A_1(1+\frac{\alpha_3}{R_1})\frac{\partial}{\partial\alpha_3}(\frac{U_1}{A_1(1+\frac{\alpha_3}{R_1})}) + \frac{1}{A_1(1+\frac{\alpha_3}{R_1})}\frac{\partial U_3}{\partial\alpha_1} \quad (8)$$

$$\in_{23}= A_2(1+\frac{\alpha_3}{R_2})\frac{\partial}{\partial\alpha_3}(\frac{U_2}{A_2(1+\frac{\alpha_3}{R_2})}) + \frac{1}{A_2(1+\frac{\alpha_3}{R_2})}\frac{\partial U_3}{\partial\alpha_2} \quad (9)$$

$$A_1 = \left|\frac{\partial\bar{r}}{\partial\alpha_1}\right|, A_{ij} \quad (10)$$

where A_1 and A_2 are the fundamental form parameters or Lame parameters,U_1, U_2 and U_3 are the displacement at any point (α_1,α_2,α_3),R_1 and $R_2$are the radius of curvature related to α_1,α_2 andα_3 respectively. The third- order theory of Reddy used in the present study is based on the following displacement field:

$$\begin{cases} U_1 = u_1(\alpha_1,\alpha_2) + \alpha_3.\phi_1(\alpha_1,\alpha_2) + \alpha_3^2.\psi_1(\alpha_1,\alpha_2) \\ \quad +\alpha_3^3.\beta_1(\alpha_1,\alpha_2) \\ U_2 = u_2(\alpha_1,\alpha_2) + \alpha_3.\phi_2(\alpha_1,\alpha_2) + \alpha_3^2.\psi_2(\alpha_1,\alpha_2) \\ \quad +\alpha_3^3.\beta_2(\alpha_1,\alpha_2) \\ U_3 = u_3(\alpha_1,\alpha_2) \end{cases} \quad (11)$$

These equations can be reduced by satisfying the stress-free conditions on the top and bottom faces of the laminates, which are equivalent to $\in_{13}$=$\in_{23}$= 0at $Z = \pm\frac{h}{2}$ Thus,

$$\begin{cases} U_1 = u_1(\alpha_1,\alpha_2) + \alpha_3.\phi_1(\alpha_1,\alpha_2) \\ \quad -C_1.\alpha_3^3(-\frac{u_1}{R_1} + \phi_1 + \frac{\partial u_3}{A_1\partial\alpha_1}) \\ U_2 = u_2(\alpha_1,\alpha_2) + \alpha_3.\phi_2(\alpha_1,\alpha_2) \\ \quad -C_1.\alpha_3^3(-\frac{u_2}{R_2} + \phi_2 + \frac{\partial u_3}{A_2\partial\alpha_2}) \\ U_3 = u_3(\alpha_1,\alpha_2) \end{cases} \quad (12)$$

where $C_1 = \frac{4}{3h^2}$. Substituting Eq. 12 into nonlinear strain-displacement relation 4 - 9, it can be obtained for the third-order theory of Reddy

$$\begin{Bmatrix} \in_{11} \\ \in_{22} \\ \in_{12} \end{Bmatrix} = \begin{Bmatrix} \in_{11}^0 \\ \in_{22}^0 \\ \in_{12}^0 \end{Bmatrix} + \alpha_3 \begin{Bmatrix} k_{11} \\ k_{22} \\ k_{12} \end{Bmatrix} + \alpha_3^3 \begin{Bmatrix} k'_{11} \\ k'_{22} \\ k'_{12} \end{Bmatrix} \quad (13)$$

$$\begin{Bmatrix} \in_{13} \\ \in_{23} \end{Bmatrix} = \begin{Bmatrix} \gamma_{13}^0 \\ \gamma_{23}^0 \end{Bmatrix} + \alpha_3^2 \begin{Bmatrix} \gamma_{13}^2 \\ \gamma_{23}^2 \end{Bmatrix} + \alpha_3^3 \begin{Bmatrix} \gamma_{13}^3 \\ \gamma_{23}^3 \end{Bmatrix} \quad (14)$$

where

$$\begin{Bmatrix} \in_{11}^0 \\ \in_{22}^0 \\ \in_{12}^0 \end{Bmatrix} = \begin{Bmatrix} (\frac{1}{A_1}\frac{\partial u_1}{\partial\alpha_1} + \frac{u_2}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2} + \frac{u_3}{R_1}) \\ (\frac{1}{A_2}\frac{\partial u_2}{\partial\alpha_2} + \frac{u_1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1} + \frac{u_3}{R_2}) \\ \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}(\frac{u_2}{A_2}) + \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}(\frac{u_1}{A_1}) \end{Bmatrix} \quad (15)$$

$$\begin{Bmatrix} k_{11} \\ k_{22} \\ k_{12} \end{Bmatrix} = \begin{Bmatrix} (\frac{1}{A_1}\frac{\partial\phi_1}{\partial\alpha_1} + \frac{\phi_2}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}) \\ (\frac{1}{A_2}\frac{\partial\phi_2}{\partial\alpha_2} + \frac{\phi_1}{A_1A_2}\frac{\partial A_2}{\partial\alpha_1}) \\ (\frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}(\frac{\phi_2}{A_2}) + \frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}(\frac{\phi_1}{A_1})) \end{Bmatrix} \quad (16)$$

$$\begin{Bmatrix} k'_{11} \\ k'_{22} \\ k'_{12} \end{Bmatrix} = -C_1 \begin{Bmatrix} (\frac{1}{A_1}(-\frac{\partial u_1}{R_1\partial\alpha_1} + \frac{\partial\phi_1}{\partial\alpha_1} + \frac{\partial^2 u_3}{A_1\partial\alpha_1^2} \\ \quad -\frac{\partial A_1}{A_1^2\partial\alpha_1}\frac{\partial u_3}{\partial\alpha_1}) \\ \quad +\frac{\partial A_1}{\partial\alpha_2}\frac{1}{A_1A_2}(-\frac{u_2}{R_2} \\ \quad +\phi_2 + \frac{\partial u_3}{A_2\partial\alpha_2})) \\ (\frac{1}{A_2}(-\frac{\partial u_2}{R_2\partial\alpha_2} + \frac{\partial\phi_2}{\partial\alpha_2} + \frac{\partial^2 u_3}{A_2\partial\alpha_2^2} \\ \quad -\frac{\partial A_2}{A_2^2\partial\alpha_2}\frac{\partial u_3}{\partial\alpha_2}) + \frac{\partial A_2}{\partial\alpha_1}\frac{1}{A_1A_2} \\ \quad (-\frac{u_1}{R_1} + \phi_1 + \frac{\partial u_3}{A_1\partial\alpha_1})) \\ (\frac{A_2}{A_1}(-\frac{\partial}{R_2\partial\alpha_1}(\frac{u_2}{A_2}) + \frac{\partial}{\partial\alpha_1}(\frac{\phi_2}{A_2}) \\ \quad +\frac{1}{A_2^2}\frac{\partial^2 u_3}{\partial\alpha_1\partial\alpha_2} - \frac{1}{A_2^4}\frac{\partial A_2^2}{\partial\alpha_1}\frac{\partial u_3}{\partial\alpha_2})+ \\ +\frac{A_1}{A_2}(-\frac{\partial}{R_1\partial\alpha_2}(\frac{u_1}{A_1}) + \frac{\partial}{\partial\alpha_2}(\frac{\phi_1}{A_1}) \\ \quad +\frac{1}{A_1^2}\frac{\partial^2 u_3}{\partial\alpha_1\partial\alpha_2} - \frac{1}{A_1^4}\frac{\partial A_1^2}{\partial\alpha_2}\frac{\partial u_3}{\partial\alpha_1})) \end{Bmatrix} \quad (17)$$

$$\begin{Bmatrix} \gamma_{13}^0 \\ \gamma_{23}^0 \end{Bmatrix} = \begin{Bmatrix} (\phi_1 - \frac{u_1}{R_1} + \frac{1}{A_1}\frac{\partial u_3}{\partial\alpha_1}) \\ (\phi_2 - \frac{u_2}{R_2} + \frac{1}{A_2}\frac{\partial u_3}{\partial\alpha_2}) \end{Bmatrix} \quad (18)$$

$$\begin{Bmatrix} \gamma_{13}^2 \\ \gamma_{23}^2 \end{Bmatrix} = 3C_1 \begin{Bmatrix} (-\frac{u_1}{R_1} + \phi_1 + \frac{\partial u_3}{A_1\partial\alpha_1}) \\ (-\frac{u_2}{R_2} + \phi_2 + \frac{\partial u_3}{A_2\partial\alpha_2} \end{Bmatrix} \quad (19)$$

$$\begin{Bmatrix} \gamma_{13}^3 \\ \gamma_{23}^3 \end{Bmatrix} = C_1 \begin{Bmatrix} \frac{(-\frac{u_1}{R_1}+\phi_1+\frac{\partial u_3}{A_1\partial\alpha_1})}{R_1} \\ \frac{(-\frac{u_2}{R_2}+\phi_2+\frac{\partial u_3}{A_2\partial\alpha_2})}{R_2} \end{Bmatrix} \quad (20)$$

where(ε^0 , γ^0) are the membranes strains and (k , k' , γ^2 , γ^3) are the bending strains, known as the curvatures.

III. FORMULATION

Consider a cylindrical shell as shown in Fig. 2, where$R$ is the radius, L the length and h the thickness of the shell. The reference surface is chosen to be the middle surface of the cylindrical shell where an orthogonal coordinate system x, θ, z is fixed. The displacements of the shell with reference this coordinate system are denoted byU_1,U_2 and U_3 in thex, θ and z directions, respectively.

For a thin cylindrical shell, the stress -strain relationship are defined as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \in_{11} \\ \in_{22} \\ \in_{23} \\ \in_{13} \\ \in_{12} \end{Bmatrix} \tag{21}$$

For a isotropic cylindrical shell the reduced stiffness $Q_{ij}(i$, j=1, 2 and 6) are defined as

$$Q_{11} = Q_{22} = \frac{E}{1-\nu^2}\ , Q_{12} = \frac{\nu\ E}{1-\nu^2} \tag{22}$$

$$Q_{44} = Q_{55} = Q_{66} = \frac{E}{2(1+\nu)} \tag{23}$$

where E is the Young's modulus and ν is Poisson's ratio. Defining

$$\{A_{ij}\ , B_{ij}\ ,\ D_{ij}\ ,\ E_{ij}\ ,\ F_{ij}\ ,\ G_{ij}\ ,\ H_{ij}\} = \int_{-h/2}^{h/2} Q_{ij} \left\{ 1\ ,\ \alpha_3\ ,\ \alpha_3^2,\ \alpha_3^3,\ \alpha_3^4,\ \alpha_3^5,\ \alpha_3^6 \right\}\ d\alpha_3 \tag{24}$$

where Q_{ij} are functions of z for functionally gradient materials. Here A_{ij} denote the extensional stiffness, D_{ij} the bending stiffness,B_{ij} the bending-extensional coupling stiffness and $E_{ij}, F_{ij}, G_{ij}, H_{ij}$ are the extensional, bending, coupling, and higher-order stiffness.

For a thin cylindrical shell the force and moment results are defined as

$$\begin{Bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} d\alpha_3 \tag{25}$$

$$\begin{Bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \alpha_3^3 d\alpha_3 \tag{26}$$

$$\begin{Bmatrix} P_{11} \\ P_{22} \\ P_{12} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \alpha_3^3 d\alpha_3 \tag{27}$$

$$\begin{Bmatrix} P_{13} \\ \\ P_{23} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{13} \\ \\ \sigma_{23} \end{Bmatrix} \alpha_3^3 d\alpha_3 \tag{28}$$

$$\begin{Bmatrix} Q_{13} \\ Q_{23} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix} d\alpha_3 \tag{29}$$

$$\begin{Bmatrix} R_{13} \\ R_{23} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} \sigma_{13} \\ \sigma_{23} \end{Bmatrix} \alpha_3^2 d\alpha_3 \tag{30}$$

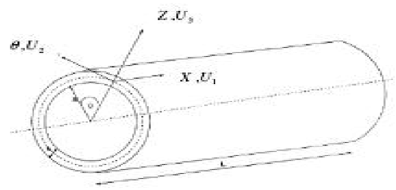

Fig. 2. Geometry of a cylindrical shell

IV. The equations of motion for vibration of a generic shell

The equations of motion for vibration of a generic shell can be derived by using Hamilton's principle which is described by

$$\delta \int_{t_1}^{t_2} (\Pi - K) dt = 0, \Pi = U - V \tag{31}$$

where $K, \Pi, U$and V are the total kinetic, potential, strain and loading energies, $t_1$and $t_2$are arbitrary time. The kinetic, strain and loading energies of a cylindrical shell can be written as:

$$K = \frac{1}{2} \int_{\alpha_1} \int_{\alpha_2} \int_{\alpha_3} \rho(\dot{U}_1^2 + \dot{U}_2^2 + \dot{U}_3^2) dV \tag{32}$$

$$U = \int_{\alpha_1} \int_{\alpha_2} \int_{\alpha_3} (\sigma_{11} \in_{11} + \sigma_{22} \in_{22} + \sigma_{12} \in_{12} \tag{33}$$

$$+ \sigma_{13} \in_{13} + \sigma_{23} \in_{23}) dV \tag{34}$$

$$V = \int_{\alpha_1} \int_{\alpha_2} (q_1 \delta U_1 + q_2 \delta U_2 + q_3 \delta U_3) A_1 A_2 d\alpha_1 d\alpha_2 \tag{35}$$

The infinitesimal volume is given by

$$dV = A_1 A_2 d\alpha_1 d\alpha_2 d\alpha_3 \tag{36}$$

with the use of Eqs. (11)-(20) and substituting into Eq. (28), we get the equations of motions a generic shell.

$$\begin{aligned} &-\frac{\partial(N_{11}A_2)}{\partial\alpha_1} + N_{22}\frac{\partial A_2}{\partial\alpha_1} - \frac{\partial(N_{12}A_1^2)}{A_1\partial\alpha_2} - \frac{Q_{13}}{R_1}A_1A_2 \\ &-\frac{\partial}{\partial\alpha_1}(\frac{P_{11}C_1A_2}{R_1}) + \frac{P_{22}C_1}{R_1}\frac{\partial A_2}{\partial\alpha_1} - \frac{\partial}{\partial\alpha_2}(\frac{P_{12}C_1A_1^2}{R_1})\frac{1}{A_1} \\ &+\frac{3C_1R_{13}}{R_1}A_1A_2 - \frac{C_1P_{13}A_1A_2}{R_1^2} \\ &= -(\ddot{u}_1 I_\circ + \ddot{\phi}_1 I_1 + [-C_1\ (-\frac{\ddot{u}_1}{R_1} + \ddot{\phi} + \frac{\partial\ddot{u}_3}{A_1\partial\alpha_1}) + \frac{C_1\ddot{u}_1}{R_1}\ ]\, I_3 \\ &+\frac{C_1\ddot{\phi}_1}{R_1}I_4 - \frac{C_1^2}{R_1}(-\frac{\ddot{u}_1}{R_1} + \ddot{\phi}_1 + \frac{\partial\ddot{u}_3}{A_1\partial\alpha_1})I_6) \end{aligned} \tag{37}$$

$$
\begin{aligned}
&\tfrac{\partial(N_{22}A_1)}{\partial\alpha_2} - N_{11}\tfrac{\partial A_1}{\partial\alpha_2} + \tfrac{\partial(N_{12}A_2^2)}{A_2\partial\alpha_1} + \tfrac{Q_{23}}{R_2}A_1A_2 \\
&+\tfrac{\partial}{\partial\alpha_2}(\tfrac{P_{22}C_1A_1}{R_2}) - \tfrac{P_{11}C_1}{R_2}\tfrac{\partial A_1}{\partial\alpha_2} + \tfrac{\partial}{\partial\alpha_1}(\tfrac{P_{12}C_1A_2^2}{R_2})\tfrac{1}{A_2} \\
&-\tfrac{3C_1R_{23}}{R_2}A_1A_2 + \tfrac{C_1P_{23}A_1A_2}{R_2^2} = (\ddot{u}_2I_\circ + \ddot{\phi}_2I_1 + \tfrac{C_1\ddot{\phi}_2}{R_2}I_2 + \\
&\left[-c_1(-\tfrac{\ddot{u}_2}{R_2} + \ddot{\phi}_2 + \tfrac{\partial\ddot{u}_3}{A_2\partial\alpha_2}) + \tfrac{C_1\ddot{u}_2}{R_2}\right]I_3 \\
&-\tfrac{C_1^2}{R_2}(-\tfrac{\ddot{u}_2}{R_2} + \ddot{\phi}_2 + \tfrac{\partial u_3}{A_2\partial\alpha_2})I_6)
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
&(-\tfrac{\partial^2(P_{11}C_1A_2/A_1)}{\partial\alpha_1^2} + N_{11}\tfrac{A_1A_2}{R_1} + \tfrac{\partial}{\partial\alpha_2}(\tfrac{C_1P_{11}}{A_2}\tfrac{\partial A_1}{\partial\alpha_2}) \\
&+N_{22}\tfrac{A_1A_2}{R_2} - \tfrac{\partial^2(P_{22}A_1C_1/A_2)}{\partial\alpha_2^2} + \tfrac{\partial}{\partial\alpha_1}(\tfrac{P_{22}C_1}{A_1}\tfrac{\partial A_2}{\partial\alpha_1}) \\
&-\tfrac{\partial^2(P_{12}C_1)}{\partial\alpha_1\partial\alpha_2} - \tfrac{\partial}{\partial\alpha_2}(\tfrac{P_{12}C_1}{A_2^2}\tfrac{\partial A_2^2}{\partial\alpha_1}) - \tfrac{\partial^2(P_{12}C_1)}{\partial\alpha_1\partial\alpha_2} \\
&-\tfrac{\partial}{\partial\alpha_1}(\tfrac{P_{12}C_1}{A_1^2}\tfrac{\partial A_1^2}{\partial\alpha_2}) - \tfrac{\partial(Q_{13}A_2)}{\partial\alpha_1} + \tfrac{\partial(3C_1R_{13}A_2)}{\partial\alpha_1} \\
&-\tfrac{\partial}{\partial\alpha_1}(\tfrac{P_{13}C_1A_2}{R_1}) - \tfrac{\partial(Q_{23}A_1)}{\partial\alpha_2} + \tfrac{\partial(3C_1R_{23}A_1)}{\partial\alpha_2} \\
&-\tfrac{\partial}{\partial\alpha_2}(\tfrac{C_1P_{23}A_1}{R_2}) - \tfrac{\partial}{\partial\alpha_1}(\tfrac{P_{11}C_1A_2}{A_1^2}\tfrac{\partial A_1}{\partial\alpha_1}) \\
&-\tfrac{\partial}{\partial\alpha_2}(P_{22}C_1\tfrac{A_1}{A_2^2}\tfrac{\partial A_2}{\partial\alpha_2})) \\
&= -\{\ddot{u}_3\ I_\circ + C_1\left[\tfrac{\partial}{\partial\alpha_1}(\tfrac{u_1}{A_1}) + \tfrac{\partial}{\partial\alpha_2}(\tfrac{u_2}{A_2})\right]I_3 \\
&+C_1\left[\tfrac{\partial}{\partial\alpha_1}(\tfrac{\ddot{\phi}_1}{A_1}) + \tfrac{\partial}{\partial\alpha_2}(\tfrac{\ddot{\phi}_2}{A_2})\right]I_4 \\
&-C_1^2I_6((-\tfrac{\partial}{R_2\partial\alpha_2}(\tfrac{\ddot{u}_2}{A_2}) + \tfrac{\partial}{\partial\alpha_2}(\tfrac{\ddot{\phi}_2}{A_2}) + \tfrac{1}{A_2}\tfrac{\partial^2\ddot{u}_3}{\partial\alpha_2^2} - \tfrac{\partial A_2}{A_1^2\partial\alpha_2}\tfrac{\partial\ddot{u}_3}{\partial\alpha_2}) \\
&+(-\tfrac{\partial}{R_1\partial\alpha_1}(\tfrac{\ddot{u}_1}{A_1}) + \tfrac{\partial}{\partial\alpha_1}(\tfrac{\ddot{\phi}_1}{A_1}) + \tfrac{1}{A_1}\tfrac{\partial^2\ddot{u}_3}{\partial\alpha_1^2} - \tfrac{\partial A_1}{A_1^2\partial\alpha_1}\tfrac{\partial\ddot{u}_3}{\partial\alpha_1}))
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
&-\tfrac{\partial(M_{11}A_2)}{\partial\alpha_1} + \tfrac{\partial(C_1P_{11}A_2)}{\partial\alpha_1} + M_{22}\tfrac{\partial A_2}{\partial\alpha_1} \\
&-C_1P_{22}\tfrac{\partial A_2}{\partial\alpha_1} - \tfrac{\partial(M_{12}A_1^2)}{A_1\partial\alpha_2} + \tfrac{\partial(P_{12}C_1A_1^2)}{A_1\partial\alpha_2} - 3C_1R_{13}A_1A_2 \\
&+A_1A_2Q_{13} + \tfrac{C_1P_{13}}{R_1}A_1A_2 \\
&= -\Big[\ddot{u}_1I_1 + \ddot{\phi}_1I_2 - C_1\ddot{u}_1I_3 + (-2C_1\ddot{\phi}_1 + C_1\tfrac{\ddot{u}_1}{R_1} \\
&-\tfrac{C_1}{A_1}\tfrac{\partial\ddot{u}_3}{\partial\alpha_1})I_4 + C_1^2(-\tfrac{\ddot{u}_1}{R_1} + \ddot{\phi}_1 + \tfrac{\partial\ddot{u}_3}{A_1\partial\alpha_1})I\ _6]
\end{aligned}
\tag{40}
$$

$$
\begin{aligned}
&-\tfrac{\partial(M_{22}A_1)}{\partial\alpha_2} + \tfrac{\partial(C_1A_1P_{22})}{\partial\alpha_2} + M_{11}\tfrac{\partial A_1}{\partial\alpha_2} - C_1P_{11}\tfrac{\partial A_1}{\partial\alpha_2} \\
&-\tfrac{\partial(M_{12}A_2^2)}{A_2\partial\alpha_1} + \tfrac{\partial(P_{12}C_1A_2^2)}{A_2\partial\alpha_1} - 3C_1R_{23}A_1A_2 + A_1A_2Q_{23} \\
&+\tfrac{C_1P_{23}}{R_2}A_1A_2 = -[\ddot{u}_2\ I_1 + \ddot{\phi}_2I_2 - C_1\ddot{u}_2I_3 \\
&+(-2C_1\ddot{\phi}_2 + C_1\tfrac{\ddot{u}_2}{R_2} - \tfrac{C_1}{A_2}\tfrac{\partial\ddot{u}_3}{\partial\alpha_2})I_4 \\
&+C_1^2(\tfrac{\ddot{u}_2}{R_2} + \ddot{\phi}_2 + \tfrac{\partial\ddot{u}_3}{A_2\partial\alpha_2})I\ _6].
\end{aligned}
\tag{41}
$$

For Eqs. (33)-(37) are defining as

$$
I_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho\alpha_3^i d\alpha_3 \tag{42}
$$

V. Equations of Motion for Vibration of Cylindrical Shell

The curvilinear coordinates and fundamental from parameters for a cylindrical shell are:

$$
R_2 = a, \frac{1}{R} = 0, A_2 = a, A_1 = 0, \alpha_3 = \alpha_3, \alpha_2 = \theta, \alpha_1 = x \tag{43}
$$

Substituting relationship (39) into Eqs. (33)-(37) the equations of motions for vibration of cylindrical shell with the third-order theory of Reddy are converted to

$$
\begin{aligned}
a\frac{\partial N_{11}}{\partial x} + \frac{\partial N_{12}}{\partial\theta} &= I_0\ddot{u}_1 + (I_1 - C_1I_3)\ddot{\phi}_1 \\
&\quad -C_1I_3\frac{\partial\ddot{u}_3}{\partial x}
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
&\frac{\partial N_{22}}{\partial\theta} + C_1\frac{\partial P_{12}}{\partial x} + Q_{23} - 3C_1R_{23} + C_1P_{23} \\
&\qquad = (I_0 + 2\frac{C_1}{a}I_3 + \frac{C_1^2}{a^2}I_6)\ddot{u}_2 \\
&\qquad +(I_1 - C_1I_3 + \frac{C_1}{a}I_4 - \frac{C_1^2}{a}I_6)\ddot{\phi}_2 \\
&\qquad -(\frac{C_1}{a}I_3 - \frac{C_1^2}{a^2}I_6)\frac{\partial\ddot{u}_3}{\partial\theta}
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
&-C_1a\frac{\partial^2P_{11}}{\partial x^2} + N_{22} - \frac{C_1}{a}\frac{\partial^2P_{22}}{\partial\theta^2} - 2C_1\frac{\partial^2P_{12}}{\partial x\partial\theta} \\
&\qquad -a\frac{\partial Q_{13}}{\partial x} \\
&+3C_1a\frac{\partial R_{13}}{\partial x} - \frac{\partial Q_{23}}{\partial\theta} + 3C_1\frac{\partial R_{23}}{\partial\theta} - \frac{C_1}{a}\frac{\partial P_{23}}{\partial\theta} \\
&\qquad = -C_1I_3\frac{\partial u_1}{\partial x} - \frac{C_1}{a}I_3\frac{\partial u_2}{\partial\theta} \\
&+(-C_1I_4 + C_1^2I_6)\frac{\partial\ddot{\phi}_1}{\partial x} + (-\frac{C_1}{a}I_4 + \frac{C_1^2}{a}I_6)\frac{\partial\ddot{\phi}_2}{\partial\theta} \\
&-\frac{C_1^2}{a^2}I_6\frac{\partial\ddot{u}_2}{\partial\theta} + C_1^2I_6\frac{\partial^2\ddot{u}_3}{\partial x^2} + \frac{C_1^2}{a}I_6\frac{\partial^2\ddot{u}_3}{\partial\theta^2} - \ddot{u}_3I_0
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
&-a\frac{\partial M_{11}}{\partial x} + C_1a\frac{\partial P_{11}}{\partial x} - \frac{\partial M_{12}}{\partial\theta} + C_1\frac{\partial P_{12}}{\partial\theta} \\
&\quad -3C_1R_{13}a + aQ_{13} = -I_1\ddot{u}_1 + C_1I_3\ddot{u}_1 \\
&+(-I_2 + 2C_1I_4 - C_1^2I_6)\ddot{\phi}_1 + (C_1I_4 - C_1^2I_6)\frac{\partial\ddot{u}_3}{\partial x}
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
&-\frac{\partial M_{22}}{\partial\theta} - C_1\frac{\partial P_{22}}{\partial\theta} - a\frac{\partial M_{12}}{\partial x} + C_1a\frac{\partial P_{12}}{\partial x} \\
&-3C_1R_{23}a + aQ_{23} + C_1R_{23} = (-I_1C_1I_3 - \frac{C_1}{a}I_4)\ddot{u}_2 \\
&\qquad +(-I_2 + 2C_1I_4)\ddot{\phi}_2 - \frac{C_1}{a}I_4\frac{\partial\ddot{u}_3}{\partial\theta}
\end{aligned}
\tag{48}
$$

The displacement fields for a FG cylindrical shell and the displacement fields which satisfy these boundary conditions can be written as

$$
\begin{aligned}
u_1 &= \bar{A}\,\tfrac{\partial\phi(x)}{\partial x}\ \cos(n\theta)\cos(\omega t) \\
u_2 &= \bar{B}\,\phi(x)\ \sin(n\theta)\cos(\omega t) \\
u_3 &= \bar{C}\phi(x)\cos(n\theta)\ \cos(\omega t) \\
\phi_1 &= \bar{D}\tfrac{\partial\phi(x)}{\partial x}\cos(n\theta)\cos(\omega t) \\
\phi_2 &= \bar{E}\phi(x)\ \sin(n\theta)\cos(\omega t)
\end{aligned}
\tag{49}
$$

where, $\bar{A}$,$\bar{B}$,$\bar{C}$,$\bar{D}$ and $\bar{E}$ are the constants denoting the amplitudes of the vibrations in the x, θ and z directions,ϕ_1 and ϕ_2 are the displacement fields for higher order deformation theories for a cylindrical shell, $\phi(x)$ is the axial function that

TABLE I
PROPERTIES OF MATERIALS

Coefficients	Stainless Steel E	ν	ρ
P_0	201.04×109	0.3262	8166
P_{-1}	0	0	0
P_1	3.079×10-4	-2.002×10-4	0
P_2	-6.534×10-7	3.797×10-7	0
P_3	0	0	0
	2.07788× 1011	0.317756	8166

Coefficients	Nickel E	ν	ρ
P_0	223.95×109	0.3100	8900
P_{-1}	0	0	0
P_1	-2.794×10-4	0	0
P_2	-3.998×10-9	0	0
P_3	0	0	0
	2.05098×1011	0.3100	8900

satisfies the geometric boundary conditions. The axial function $\phi\,(x)$is chosen as the beam function as

$$\phi(x) = \gamma_1 \cosh(\frac{\lambda_m x}{L}) + \gamma_2 \cos(\frac{\lambda_m x}{L})$$
$$-\zeta_m(\gamma_3 \sinh(\frac{\lambda_m x}{L}) + \gamma_4 \sin(\frac{\lambda_m x}{L})) \quad (50)$$

The geometric boundary conditions for free and clamped boundary conditions can be expressed mathematically in terms of $\phi\,(x)$as:

Free boundary condition

$$\phi''(0) = \phi'''(L) = 0 \quad (51)$$

Clamped boundary condition

$$\phi(0) = \phi'(L) = 0 \quad (52)$$

Substituting Eq. (45) into Eqs. (40) - (44) for third order theory we can be expressed

$$\det\,(C_{ij} - M_{ij}\,\omega^2) = 0 \quad (53)$$

Expanding this determinant, a polynomial in even powers ofω is obtained

$$\beta_\circ\omega^{10} + \beta_1\omega^8 + \beta_2\omega^6 + \beta_3\omega^4 + \beta_4\omega^2 + \beta_5 = \circ \quad (54)$$

where $\beta_i(i = 0, 1, 2, 3, 4, 5)$ are some constants. Eq. (50) is solved five positive and five negative roots are obtained. The five positive roots obtained are the natural angular frequencies of the cylindrical shell based third-order theory. The smallest of the five roots is the natural angular frequency studied in the present study. The FGM cylindrical shell is composed of Nickel at its inner surface and Stainless steel at its outer surface. The material properties for stainless steel and nickel, calculated at $T = 300K$, are presented in table 1 where $P_\circ, P_{-1}, P_1, P_2$ and P_3 are the coefficients of temperature $T(K)$ expressed in Kelvin and are unique to the constituent materials. The material properties $P$of FGMs are a function of the material properties and volume fractions of the constituent material.

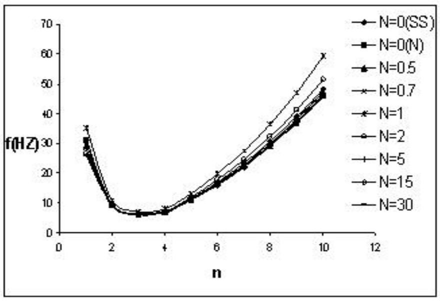

Fig. 3. Natural frequencies FG cylindrical shell associated with various power law exponent for C-F boundary condition.

VI. RESULTS AND DISCUSSION

For simplicity, we actually vary the value of power law exponent whenever we need to change the volume fraction. Varying the value of power law exponent N of the FG cylindrical shell, natural frequencies are computed for clamped-free boundary conditions. Results are also computed for pure stainless steel and pure nickel shells. All these results are plotted in Fig. 3.

VII. CONCLUSIONS

A study on the free vibration of functionally graded (FG) cylindrical shell composed of stainless steel and nickel has been presented. Material properties are graded in the thickness direction of the shell according to volume fraction power law distribution. The study is carried out using third order shear deformation shell theory. The analysis is carried out using Hamilton's principle. Studies are carried out for cylindrical shells with clamped-free (C-F) boundary conditions. The study showed that in this boundary conditions the frequency first decreases and then increases as the circumferential wave number n increases. The results showed that one could easily vary the natural frequency of the FG cylindrical shell by varying the volume fraction.

REFERENCES

[1] Arnold, R.N., Warburton, G.B., 1948. Flexural vibrations of the walls of thin cylindrical shells. Proceedings of the Royal Society of London A; 197:238-256.
[2] Ludwig, A., Krieg, R., 1981.An analysis quasi-exact method for calculating eigen vibrations of thin circular shells. J. Sound vibration; 74,155-174.
[3] Chung, H., 1981. Free vibration analysis of circular cylindrical shells. J. Sound vibration; 74, 331-359.
[4] Soedel, W., 1980.A new frequency formula for closed circular cylindrical shells for a large variety of boundary conditions. J. Sound vibration; 70,309-317.
[5] Forsberg, K., 1964. Influence of boundary conditions on modal characteristics of cylindrical shells. AIAA J; 2, 182- 189.
[6] Lam, K.L., Loy, C.T., 1995. Effects of boundary conditions on frequencies characteristics for a multi- layered cylindrical shell. J. Sound vibration; 188, 363-384.
[7] Loy, C.T., Lam, K.Y., 1996.Vibration of cylindrical shells with ring support. I.Joumal of Impact Engineering; 1996; 35:455.
[8] Koizumi, M., 1993. The concept of FGM Ceramic Transactions, Functionally Gradient Materials.

[9] Makino A, Araki N, Kitajima H, Ohashi K. Transient temperature response of functionally gradient material subjected to partial, stepwise heating. Transactions of the Japan Society of Mechanical Engineers, Part B 1994; 60:4200-6(1994).
[10] Anon, 1996.FGM components: PM meets the challenge. Metal powder Report. 51:28-32.
[11] Zhang, X.D., Liu, D.Q., Ge, C.C., 1994. Thermal stress analysis of axial symmetry functionally gradient materials under steady temperature field. Journal of Functional Materials; 25:452-5.
[12] Wetherhold, R.C., Seelman, S., Wang, J.Z., 1996. Use of functionally graded materials to eliminate or control thermal deformation. Composites Science and Technology; 56:1099-104.
[13] Najafizadeh, M.M., Hedayati, B. Refined Theory for Thermoelastic Stability of Functionally Graded Circular Plates. Journal of thermal stresses; 27:857-880.
[14] Soedel, W., 1981. Vibration of shells and plates. MARCEL DEKKER, INC, New York.