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Comparing interval estimators for reliability in a dependent set-up

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Abstract—In this paper some procedures for building confidence intervals for the reliability in stress-strength models are discussed and empirically compared. The particular case of a bivariate normal set-up is considered. The confidence intervals suggested are obtained employing approximations or asymptotic properties of maximum likelihood estimators. The coverage and the precision of these intervals are empirically checked through a simulation study. An application to real paired data is also provided.

Keywords—approximate estimators, asymptotic theory, confidence interval, Monte Carlo simulations, stress-strength, variance estimation.

I. Introduction

THE aim of this paper is to propose confidence intervals for the probability $P(X > Y)$, where X and Y are two normal random variables with joint distribution $\phi(x, y)$ and when sample values x and y are jointly observed.

The recent years has seen a lot of publications on this subject, perhaps because of its practical applications encapsulated by the term "stress-strength". When speaking of stress-strength models, we usually mean the assessment of "reliability" of a "component" in terms of random variables Y representing "stress" experienced by the component and X representing the "strength" of the component available to overcome the stress. According to this simplified scenario, if the stress exceeds the strength ($Y > X$) the component fails; and vice versa. Reliability is defined as the probability of not failing: $R = P(X > Y)$. The reliability problem arises in the fields of aeronautical, civil, mechanical and electronic engineering; but it can be also applied to other contexts.

The germ of this idea was introduced by Birnbaum [1] and developed by Birnbaum and McCarty [2]. From then onwards a lot of research has been done, devoted to probabilistic problems associated with evaluation of R and construction of efficient estimators of this parameter, based on sample values with various assumptions on the distributions of X and Y. Most of it presuppose that both random variables have distribution belonging to the same family, and more significantly assume independence between them. There have, however, been studies in which X and Y admit certain specified form of dependence: see for example [5], [10], [16] for the bivariate normal case, and [7], [11]–[13] for other parametric families; see [8] for a more complete review.

In this paper the estimation of reliability when X and Y are two non-independent normal variables, with unknown means and variances and unknown correlation coefficient, will be considered. Based upon maximum likelihood (ML) estimators, approximate interval estimators appeared in literature will be considered and refined and their performance in terms of coverage and precision will be empirically investigated and compared through an extensive simulation study.

The paper is structured as follows: in Section II some methods for interval estimation of reliability for the normal dependent setup are presented; in Section III the simulation design is described, the results are shown and discussed; in Section IV the interval estimators are applied to a real dataset.

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II. Available methods

As said in the introduction, parametric estimation of reliability under the assumption of independence between X and Y has been studied by many authors. Yet, the case of a dependent set-up for the two variables is also interesting, since in many real situations stress and strength are someway correlated; as pointed out in [10], the use of a joint distribution for stress and strength is justified by the practice of using stronger components in worse environments which cause greater stresses (which statistically means positive correlation between stress and strength).

In the general case, if we denote with $f_{xy}(x, y; \boldsymbol{\eta})$ the joint probability density function of (X, Y), where $\boldsymbol{\eta}$ is a vector of parameters, the reliability is given by

$$R = \int_{-\infty}^{\infty} \int_{-\infty}^{x} f_{xy}(x, y; \boldsymbol{\eta}) dy dx. \tag{1}$$

The assumption of a bivariate normal distribution for the couple (X, Y) has been more widely considered in literature. Let $(X, Y) \sim N(\boldsymbol{\mu}, \Sigma)$, where $\boldsymbol{\mu} = (\mu_x, \mu_y)$ is the vector of means and

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix},$$

the covariance matrix. The expression of R for this set-up is easily derived (see Owen *et al.* [15] or Church and Harris [3]):

$$R = \Phi\left[\frac{\mu_x - \mu_y}{\sqrt{\sigma_x^2 + \sigma_y^2 - 2\rho\sigma_x\sigma_y}}\right] = \Phi\left[\frac{\mu_x - \mu_y}{\sigma_d}\right], \tag{2}$$

with $\sigma_d^2 = \sigma_x^2 + \sigma_y^2 - 2\rho\sigma_x\sigma_y$. When n random sample paired observations (x_i, y_i), $i = 1, 2, \ldots, n$, are available from (X, Y) and all the parameters are unknown, Mukherjee and Sharan [10] propose the ML estimator for R given by:

$$\hat{R} = \Phi\left(\frac{\bar{y} - \bar{x}}{\hat{\sigma}_x^2 + \hat{\sigma}_y^2 - 2\hat{\rho}\hat{\sigma}_x\hat{\sigma}_y}\right), \tag{3}$$

where $\hat{\rho}$ is the sample correlation coefficient. They calculate its asymptotic variance as well:

$$V(\hat{R}) = \frac{1}{2\pi n} e^{-(\mu_x-\mu_y)^2/\sigma_d^2} \left[1 + \frac{(\mu_x-\mu_y)^2}{2\sigma_d^2}\right]. \quad (4)$$

With regard to interval estimation of R, Nandi and Aich [14] considered the problem of obtaining confidence bounds for R when the r.v. X and Y follows a bivariate normal distribution, and it is assumed their variances are equal but unknown, and the correlation coefficient between them is known.

Gupta and Subramanian [9] overcome the problem of ρ estimation by supposing for the marginals of the bivariate normal an equal coefficient of variation ν. Parameters are estimated using the method of maximum likelihood, and an asymptotic confidence interval for R, based upon approximate variance estimators, is derived.

Govindarazulu [5] gives explicit expression for the minimum sample size n for specified confidence length and confidence level when X and Y are bivariate normal with all the parameters unknown. Implicitly he proposes a confidence interval (henceforth CI) of R, based on the point estimate $\hat{R}_G = \Phi\left[(\bar{x}-\bar{y})/s_d\right]$. To get an approximate $(1-\alpha)$ CI, the equation to be solved is

$$\begin{aligned} P(R_1 < \hat{R} < R_2) &= P\left(\Phi^{-1}(R_1) < \frac{\bar{X}-\bar{Y}}{S_d} < \Phi^{-1}(R_2)\right) \\ &= 1-\alpha. \end{aligned} \quad (5)$$

Since $\sqrt{n}\cdot\dfrac{\bar{X}-\bar{Y}}{S_d}$ has a non-central T distribution with $f = n-1$ degrees of freedom and non-centrality parameter $\lambda = n^{1/2}\Phi^{-1}(R)$, Equation 5 reduces to

$$\begin{aligned} &P(R_1 < \hat{R} < R_2) = \\ &\quad P\left(n^{1/2}\Phi^{-1}(R_1) < (Z+\lambda)/W < n^{1/2}\Phi^{-1}(R_2)\right) \\ &\qquad = 1-\alpha, \end{aligned} \quad (6)$$

where Z is the normal standard variable, and W is the square root of a chi-square random variable divided by its degrees of freedom. Since $E(W) \approx 1$ and $V(W) = 1/(2f)$, we can write $\lambda/W \approx \lambda$, and then

$$\begin{aligned} &P(R_1 < \hat{R} < R_2) \\ &\quad = P\left(n^{1/2}\Phi^{-1}(R_1) < T + \lambda < n^{1/2}\Phi^{-1}(R_2)\right) \\ &\qquad = P\left(n^{1/2}(\Phi^{-1}(R_1) - \Phi^{-1}(R)) < T \right. \\ &\qquad\quad \left. < n^{1/2}(\Phi^{-1}(R_2) - \Phi^{-1}(R))\right) = 1-\alpha \end{aligned} \quad (7)$$

where T is a random variable with a central Student's T distribution. Substituting R with its estimate $\hat{R}$, we get the equation

$$n^{1/2}\Phi^{-1}(R_1) - n^{1/2}\Phi^{-1}(\hat{R}) = t_{\alpha/2} \quad (8)$$

and an analogous one for R_2, then the approximate $(1-\alpha)$ CI is then given by:

$$\left(\Phi\left[\Phi^{-1}(\hat{R}) + \frac{t_{\alpha/2,n-1}}{\sqrt{n}}\right], \Phi\left[\Phi^{-1}(\hat{R}) + \frac{t_{1-\alpha/2,n-1}}{\sqrt{n}}\right]\right). \quad (9)$$

Approximate CIs based upon asymptotic properties of the ML estimator can be easily obtained. The asymptotic variance of the estimator $\hat{R}$ by Mukherjee and Sharan (Equation 4) can be estimated by:

$$v(\hat{R}) = \frac{1}{2\pi n} e^{-(\bar{x}-\bar{y})^2/\hat{\sigma}_d^2} \left[1 + \frac{(\bar{x}-\bar{y})^2}{2\hat{\sigma}_d^2}\right], \quad (10)$$

with $\hat{\sigma}_d^2 = s_d^2 \cdot \dfrac{n-1}{n}$, obtained by substituting in the expression of $V(\hat{R})$ all the unknown quantities with their ML estimates. Then an approximate CI can be computed as

$$\left(\hat{R} + z_{\alpha/2}\sqrt{v(\hat{R})}, \hat{R} + z_{1-\alpha/2}\sqrt{v(\hat{R})}\right). \quad (11)$$

Of course, one has to pay attention when the lower bound of this interval falls below 0, or its upper bound exceeds 1: this may happen when the sample estimate of R is very low (very high) and the sample size is small. Then, the previous CI can be adjusted in the following way:

$$\left(\max\left\{0, \hat{R} + z_{\alpha/2}\sqrt{v(\hat{R})}\right\}, \min\left\{1, \hat{R} + z_{1-\alpha/2}\sqrt{v(\hat{R})}\right\}\right). \quad (12)$$

Another way to get an approximate CI is to consider the argument of the Φ function in the expression of R in Equation 2, $d = (\mu_x - \mu_y)/\sigma_d$, and obtain a CI for it, based upon its ML estimator, $\hat{d}$; then transform its lower and upper bounds through the Φ function. This way of proceeding is suggested by the fact that $\hat{d}$ is more likely to approximately follow a normal distribution than $\hat{R}$: as pointed out in [6], it seems more reasonable to base inference on a normal approximation to $\hat{d}$ than on the normal approximation to $\hat{R}$, since d is unbounded, whereas R is bounded in $(-1;+1)$. The asymptotic variance of $\hat{d}$ can be derived following the same steps in [10], and can be estimated by:

$$v(\hat{d}) = \frac{1}{n}\left(1 + (\bar{x}-\bar{y})^2/(2\hat{\sigma}_d^2)\right) \quad (13)$$

and then an approximate CI for d is given by

$$(d_L, d_U) = \left(\hat{d} + z_{\alpha/2}\sqrt{v(\hat{d})}, \hat{d} + z_{1-\alpha/2}\sqrt{v(\hat{d})}\right) \quad (14)$$

and the corresponding CI for $\hat{R}$ by

$$(\Phi(d_L), \Phi(d_U)) \quad (15)$$

This way, we also avoid the concern about the feasibility of lower and upper bounds.

III. Simulation study

The simulation study that has been performed aims at empirically checking the statistical properties of the proposed estimators, specifically coverage and average length at a nominal level (95%).

The CIs empirically investigated by simulation are:

- Approximate CI based on the asymptotic variance of ML point estimator variance (ANR, Eq. 12)
- Approximate CI based on asymptotic variance of $\hat{d}$ in ML point estimator (AND, Eq. 15)
- Govindarazulu approximate CI (APP , Eq. 9)

TABLE I
PARAMETERS FOR THE DEPENDENT SET-UP

par.	values							
μ_x	0.5	0.5	1	1	0.5	0.5	1	1
μ_y	0	0	0	0	0	0	0	0
σ_x	2	1	2	1	2	1	2	1
σ_y	1	1	1	1	1	1	1	1
ρ	0.5	0.5	0.5	0.5	0.8	0.8	0.8	0.8
R	0.614	0.692	0.718	0.841	0.646	0.786	0.772	0.943
#	1	2	3	4	5	6	7	8

An array of eight different scenarios has been considered, each corresponding to a different combination of distribution parameters (and thus different reliabilities), coded with a progressive number; they are reported in Table I. Without any loss in generality, we set $\mu_y = 0$, $\sigma_y = 1$ and varied the parameters μ_x, σ_x and ρ. ρ here takes two values, 0.5 and 0.8. The five parameters has been jointly set in order to assure a high value (> 0.5) for the reliability, since in real practice there is concern for high reliability for the study component. Nevertheless, the analyzed scenarios cover a large range of reliability, since R goes from .614 to .943. Different sample sizes ($n = 10, 20, 30, 50$) are used in order to test the reliance of the estimators also for small samples. The number of Monte Carlo runs has been fixed at $1,000$.

The simulation study for the comparison of interval estimators works as follows:

1) set the parameters $\boldsymbol{\eta} = (\mu_x, \mu_y, \sigma_x, \sigma_y, \rho)$ for the bivariate r.v. (X, Y) (Table I);
2) compute R (Equation 2);
3) draw a sample (x, y) of size n from (X, Y);
4) estimate R and a CI for R, using each of the listed estimators;
5) check out if this CI contains R; compute its length;
6) repeat 3-5 $nSim$ $(10,000)$ times and compute the overall CI coverage (rate of the CI's containing R) and average length (computed over the $nSim$ MC runs) for each interval estimator.

The results of the simulation study are reported in Tables II and III.

From the results, an empirically best performer does not stand out; yet, overall, AND gives more stable results (the coverage is never smaller than 0.918, while for ANR the minimum coverage rate is 0.766 and for APP 0.820) and is preferable when expecting high reliability values; under scenarios 3, 4, 6, 7, 8 it is the best performer for each sample size. As n increases, the coverage rates of the three estimators get closer, and approach the nominal level. With regard to the average length, the three intervals estimators show very similar values, apart from scenario #8, where AND average length is sensibly greater than the other two estimators, but is compensated by a better coverage.

Figure 1 graphically compares the three methods from a coverage perspective (scenarios $1 \div 4$ are here examined).

TABLE II
SIMULATION RESULTS: COVERAGE

$n = 10$	scenario							
method	1	2	3	4	5	6	7	8
ANR	0.891	0.869	0.885	0.834	0.893	0.848	0.876	0.766
AND	0.930	0.918	0.939	0.928	0.932	0.922	0.939	0.932
APP	0.947	0.931	0.933	0.885	0.942	0.908	0.925	0.820
$n = 20$	scenario							
method	1	2	3	4	5	6	7	8
ANR	0.919	0.908	0.903	0.883	0.918	0.906	0.892	0.845
AND	0.935	0.931	0.941	0.937	0.937	0.933	0.945	0.935
APP	0.943	0.920	0.922	0.892	0.939	0.915	0.914	0.820
$n = 30$	scenario							
method	1	2	3	4	5	6	7	8
ANR	0.941	0.936	0.940	0.914	0.944	0.926	0.933	0.874
AND	0.955	0.949	0.958	0.957	0.956	0.961	0.959	0.954
APP	0.960	0.939	0.945	0.906	0.952	0.922	0.931	0.831
$n = 50$	scenario							
method	1	2	3	4	5	6	7	8
ANR	0.939	0.947	0.941	0.942	0.937	0.947	0.933	0.917
AND	0.941	0.953	0.944	0.960	0.945	0.952	0.947	0.952
APP	0.945	0.945	0.933	0.917	0.939	0.927	0.918	0.824

TABLE III
SIMULATION RESULTS: AVERAGE LENGTH

$n = 10$	scenario							
method	1	2	3	4	5	6	7	8
ANR	0.460	0.427	0.414	0.300	0.450	0.357	0.375	0.147
AND	0.439	0.417	0.408	0.334	0.432	0.371	0.383	0.221
APP	0.482	0.445	0.432	0.322	0.470	0.376	0.393	0.176
$n = 20$	scenario							
method	1	2	3	4	5	6	7	8
ANR	0.335	0.317	0.308	0.241	0.329	0.279	0.284	0.125
AND	0.326	0.310	0.302	0.246	0.320	0.277	0.281	0.154
APP	0.336	0.310	0.297	0.217	0.327	0.260	0.266	0.112
$n = 30$	scenario							
method	1	2	3	4	5	6	7	8
ANR	0.276	0.262	0.256	0.203	0.272	0.232	0.238	0.109
AND	0.271	0.258	0.252	0.204	0.267	0.230	0.236	0.123
APP	0.275	0.253	0.244	0.175	0.268	0.210	0.219	0.086
$n = 50$	scenario							
method	1	2	3	4	5	6	7	8
ANR	0.215	0.205	0.200	0.162	0.212	0.183	0.187	0.092
AND	0.213	0.203	0.198	0.162	0.209	0.182	0.186	0.097
APP	0.213	0.197	0.189	0.137	0.207	0.164	0.169	0.067

IV. EXAMPLE OF APPLICATION

The example is drawn from [4]. Trace metals in drinking water affect the flavor, and unusually high concentration can pose a health hazard. A study selected six river locations and determined the zinc concentration (mg/L) for both surface water and bottom water at each location. The six pairs of observations are displayed in Table IV. The objective is constructing a confidence interval for the probability that true average concentration in bottom water exceeds that of the surface water, i.e. $P(X > Y)$.

TABLE IV
SAMPLE DATA FOR THE EXAMPLE OF APPLICATION

X	Y
0.430	0.415
0.266	0.238
0.567	0.390
0.531	0.410
0.707	0.605
0.716	0.609

For these data, $\bar{x} = 0.5361$, $\bar{y} = 0.4445$, $\hat{\sigma}_x^2 = 0.02446$, $\hat{\sigma}_y^2 = 0.01675$, $\rho = 0.94216$ and then $\hat{R} = 0.951$. The confidence intervals, constructed according to the three methods described in Section 2 supposing a bivariate normal

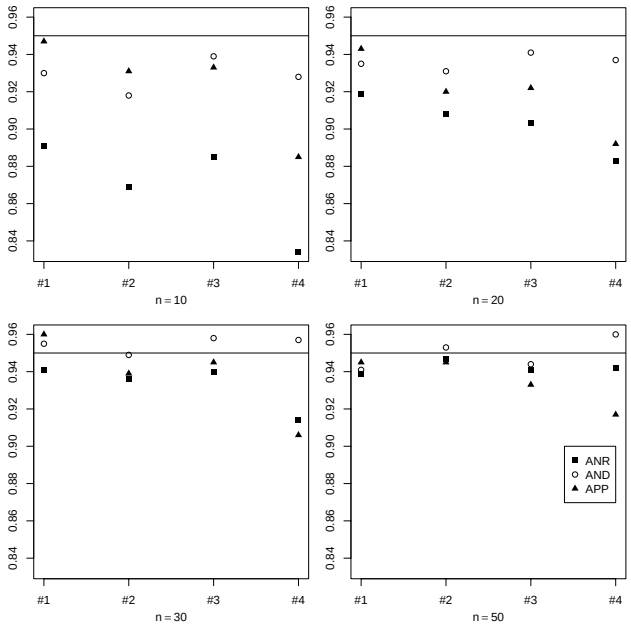

Fig. 1. Monte Carlo coverage for the three methods (first four scenarios).

distribution underlying the data, are reported in Table V, for different nominal levels (90%, 95%, 99%).

TABLE V
CONFIDENCE INTERVALS FOR THE EXAMPLE

method	90%		95%		99%	
ANR	0.846	1.000	0.826	1.000	0.787	1.000
AND	0.733	0.996	0.664	0.998	0.514	0.999
APP	0.754	0.990	0.678	0.995	0.446	0.999

It can be noted that ANR always provides 1 as an upper bound, and this occurred because of the high value of the reliability estimate; whereas AND and APP provide similar results and a larger confidence interval.

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