

Viscous potential flow analysis of electrohydrodynamic capillary instability through porous media

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Abstract—The effect of porous medium on the capillary instability of a cylindrical interface in the presence of axial electric field has been investigated using viscous potential flow theory. In viscous potential flow, the viscous term in Navier-Stokes equation vanishes as vorticity is zero but viscosity is not zero. Viscosity enters through normal stress balance in the viscous potential flow theory and tangential stresses are not considered. A dispersion relation that accounts for the growth of axisymmetric waves is derived and stability is discussed theoretically as well as numerically. Stability criterion is given by critical value of applied electric field as well as critical wave number. Various graphs have been drawn to show the effect of various physical parameters such as electric field, viscosity ratio, permittivity ratio on the stability of the system. It has been observed that the axial electric field and porous medium both have stabilizing effect on the stability of the system.

Keywords—Capillary instability, Viscous potential flow, Porous media, Axial electric field.

I. INTRODUCTION

Capillary instability arises when a fluid cylinder in an infinite fluid collapses under the action of capillary forces due to surface tension [9], [12]. The capillary instability occurs in various situations such as film boiling, Breaking of liquid jet and in many Chemical and Metallurgical processes.

The instability of long cylindrical column under the action of capillary force was first studied by Rayleigh [5] following earlier work by Plateau [10]. His study was based on potential flow of an inviscid fluid neglecting the effect of outside fluid. Rayleigh [6] again established the result for viscous effects neglecting the surrounding fluid. Weber [3] made another extension to Rayleigh's theory by considering an effect of viscosity and that of surrounding air on the stability of columnar jet. Tomotika [13] extended the work to include the effect of surrounding fluid. Lee and Flumerfelt [15] studied the same problem without making the approximation used by Tomotika [13] and for some limiting cases investigated the effects of viscosity ratio for various values of Ohnesorge number and a fixed value of density ratio.

Viscous potential flow theory has played an important role in studying various stability problems. In viscous potential flow, we consider irrotational flow, so the viscous term i.e. $\mu \nabla^2 \mathbf{u}$ in the Navier-Stokes equation is identically zero when the vorticity is zero but the viscous stresses are not zero, where μ denotes viscosity and \mathbf{u} denotes velocity of fluid

flow. Tangential stresses are not considered in viscous potential theory and viscosity enters through normal stress balance [4]. Funada and Joseph [14] studied the viscous potential flow analysis of capillary instability and observed that viscous potential flow is better approximation of the exact solution than the inviscid model.

As the electric field plays an important role in many practical problems of chemical engineering and other related fields, there is increasing interests in the study of electrohydrodynamic instability. Elhefnawy and Moatimid [2] have studied the effect of an axial electric field on the stability of cylindrical flows in the presence of mass and heat transfer and absence of gravity. They observed that the electric field has strong stabilizing influence for all short and long wavelengths. Elcoot [1] has studied the nonlinear analysis of capillary instability of viscous fluids in the presence of axial electric field.

Recently, Asthana and Agrawal [11] have studied the viscous potential flow analysis of electrohydrodynamic Kelvin-Helmholtz instability at the plane interface and concluded that the tangential electric field has stabilizing effect on the critical value of relative velocity while relative velocity has destabilizing effect on the critical value of electric field. Awasthi and Agrawal [7] has studied the viscous contribution to the pressure for the potential flow analysis of capillary instability with axial electric field and observed that the axial electric field has stabilizing effect on the stability of the system. Awasthi and Asthana [8] have studied the effect of porous medium on the capillary instability when there is heat and mass transfer across the interface and observed that porous medium have stabilizing effect.

In the present article, viscous potential flow analysis of capillary instability through porous media in the presence of an axial electric field has been carried out for axisymmetric disturbances. Both the fluids are taken as incompressible, viscous and dielectric with different kinematic viscosities and permittivities, respectively, which have not been considered earlier. The effect of gravity and free surface charges at the interface is neglected. A dispersion relation is derived and stability is discussed theoretically as well as numerically. A critical value of the electric field as well as the critical wave number is obtained. The effect of electric field and ratio of permittivity of fluids on stability of the system is also studied and shown graphically.

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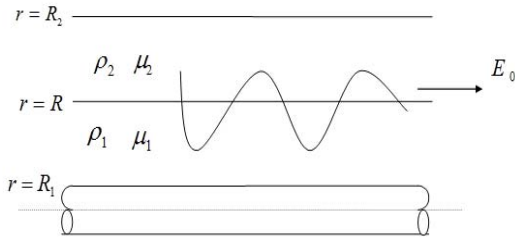


Fig. 1. Equilibrium configuration of the system.

II. PROBLEM FORMULATION

A system of two incompressible and viscous fluids, separated by a cylindrical interface, is considered in an annular porous medium with constant porosity ϵ and constant permeability k_1 as shown in Figure 1. A cylindrical system of coordinates (r, θ, z) is assumed so that in the equilibrium state z -axis is the axis of symmetry of the system. The undisturbed cylindrical interface is taken at radius R . In the formulation the subscript 1 and 2 denote variables associated with the fluid inside and outside the interface, respectively. In the undisturbed state, viscous fluid of thickness h_1 , density ρ_1 , viscosity μ_1 and permittivity β_1 occupies the inner region $r_1 < r < R$ and viscous fluid of thickness h_2 , density ρ_2 , viscosity μ_2 and permittivity β_2 occupies the outer region $R < r < r_2$. The bounding surfaces $r = r_1$ and $r = r_2$ are considered to be rigid. The only resistance term taken is $-\frac{\mu}{k_1}v$ where μ denotes the fluid viscosity, k_1 represents the medium permeability and v is the Darcian velocity. Both the fluids are assumed to be incompressible and irrotational. Assuming that Darcy's law is valid, we obtain the following momentum and continuity equations:

$$\rho \frac{dv}{dt} = -\nabla p - \frac{\mu}{k_1}v \quad (1)$$

$$\nabla \cdot v = 0 \quad (2)$$

Here p represents pressure, μ denotes to the fluid viscosity, k_1 is the medium permeability and v is the Darcian (filter) velocity. Taking an average of the fluid velocity over a volume V we get the intrinsic average velocity \bar{v} , which is related to by the Dupuit-Forchheimer relationship $\bar{v} = \epsilon v$, where ϵ represents the porosity of the medium which is defined as the fraction of the total volume of the medium that is occupied by void space.

Small axisymmetric disturbances are superimposed on the basic rest state. After disturbance, the interface is given by

$$F(r, z, t) = r - R - \eta(z, t) = 0 \quad (3)$$

where η is the perturbation in the radius of the interface from the equilibrium value R , and for which the outward unit

normal vector is given by

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \left\{ 1 + \left(\frac{\partial \eta}{\partial z} \right)^2 \right\}^{-1/2} \left(\mathbf{e}_r - \frac{\partial \eta}{\partial z} \mathbf{e}_z \right) \quad (4)$$

where \mathbf{e}_r and \mathbf{e}_z are unit vectors along the r and z directions, respectively.

The velocity is expressed as the gradient of the potential function and the potential functions satisfy the Laplace equation as a consequence of the incompressibility constraint. i.e.

$$\nabla^2 \phi_j = 0 \quad \text{for } (j = 1, 2) \quad (5)$$

where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$.

The two fluids are subjected to an external electric field E_0 , acting along z -axis i.e. $\mathbf{E} = E_0 \mathbf{e}_z$.

It is assumed that the quasi-static approximation is valid for the problem, hence the electric field can be derived from electric scalar potential function $\psi(r, z, t)$ such that

$$\mathbf{E}_j = E_0 \mathbf{e}_z - \nabla \psi_j, \quad (j = 1, 2) \quad (6)$$

Gauss's law requires that the electric potentials also satisfy Laplace's equation i.e.

$$\nabla^2 \psi_j = 0, \quad (j = 1, 2) \quad (7)$$

The boundary conditions at the rigid cylindrical surfaces $r = r_1$ and $r = r_2$ are given by

$$\frac{\partial \phi_j}{\partial r} = 0 \quad \text{at } r = r_j \quad (j = 1, 2) \quad (8)$$

$$\frac{\partial \psi_j}{\partial z} = 0 \quad \text{at } r = r_j \quad (j = 1, 2) \quad (9)$$

The tangential component of the electric field must be continuous across the interface i.e.

$$\|E_t\| = 0 \quad (10)$$

where $E_t (= |\mathbf{n} \times \mathbf{E}|)$ is the tangential component of the electric field. where $\|X\|$ represents the difference in a quantity across the interface, it is defined as $\|X\| = X^2 - X^1$.

There is discontinuity in the normal current across the interface; charge accumulation within a material element is balanced by conduction from bulk fluid on either side of the surface. The boundary condition, corresponding to normal component of the electric field, at the interface is given by

$$\|E_n\| = 0 \quad (11)$$

where $E_n (= \mathbf{n} \cdot \mathbf{E})$ is the normal component of the electric field.

The interfacial condition for conservation of momentum is given by;

$$\rho_1 (\nabla \phi_1 \cdot \nabla F) \left(\frac{\partial F}{\partial t} + \nabla \phi_1 \cdot \nabla F \right) = \rho_2 (\nabla \phi_2 \cdot \nabla F) \left(\frac{\partial F}{\partial t} + \nabla \phi_2 \cdot \nabla F \right) + (p_2 - p_1 - 2\mu_2 \frac{\partial v_2}{\partial r} + 2\mu_1 \frac{\partial v_1}{\partial r} + \sigma \nabla \cdot \mathbf{n}) |\nabla F|^2 \quad (12)$$

where p represents the pressure and T denotes the surface tension. Surface tension has been assumed to be a constant, neglecting its dependence on temperature. Pressure can be obtained using Bernoulli's equation.

III. LINEAR EQUATIONS AND SOLUTION

It has been observed that the asymmetric disturbances are always stable for capillary instability. A long cylinder of liquid is unstable to the axisymmetric disturbances with wavelengths greater than $2\pi R$, where R is the radius of the cylinder. Hence, we considered only axisymmetric disturbances in this analysis. Now, axisymmetric disturbances are imposed on the equations (10), (11) and (12) and retaining the linear terms we can get the following equations.

$$\left[\left[\frac{\partial \phi}{\partial z} \right] \right] = 0 \quad (13)$$

$$\left[\left[\beta \left(\frac{\partial \phi}{\partial r} + E_0 \frac{\partial \eta}{\partial z} \right) \right] \right] = 0 \quad (14)$$

$$\left[\left[\rho \left(\frac{1}{\varepsilon} \frac{\partial \phi}{\partial t} + \frac{1}{\rho k_1} \phi \right) + \frac{2\mu}{\varepsilon} \frac{\partial^2 \phi}{\partial r^2} + \beta E_0 \frac{\partial \psi}{\partial z} \right] \right] = \sigma \left(\frac{\partial^2 \eta}{\partial z^2} + \frac{\eta}{R^2} \right) \quad (15)$$

Now we use normal mode technique to find the solution of the governing equations. Let the interface elevation be given by

$$\eta = C \exp(ikz - i\omega t) + c.c. \quad (16)$$

where C is constant, i is the imaginary unit, k is the real wave number, ω is the growth rate and $c.c.$ refers the complex conjugate of the preceding term.

On solving equations (4) and (6) with the help of boundary conditions, we get

$$\phi_1 = -\frac{i\omega C}{k} A_0(kr) \exp(ikz - i\omega t) + c.c. \quad (17)$$

$$\phi_2 = -\frac{i\omega C}{k} B_0(kr) \exp(ikz - i\omega t) + c.c. \quad (18)$$

$$\psi_1 = \frac{iE_0 k (\beta^{(2)} - \beta^{(1)}) g_2(k)}{\beta^{(1)} g_2(k) G_1(k) - \beta^{(2)} g_1(k) G_2(k)}$$

$$[I_0(kr) K_0(kr_1) - I_0(kr_1) K_0(kr)] C \exp(ikz - i\omega t) + c.c. \quad (19)$$

$$\psi_2 = \frac{iE_0 k (\beta^{(2)} - \beta^{(1)}) g_1(k)}{\beta^{(1)} g_2(k) G_1(k) - \beta^{(2)} g_1(k) G_2(k)}$$

$$[I_0(kr) K_0(kr_2) - I_0(kr_2) K_0(kr)] C \exp(ikz - i\omega t) + c.c. \quad (20)$$

where

$$A_0(kr) = \frac{I_0(kr) K_1(kr_1) + I_1(kr_1) K_0(kr)}{I_1(kR) K_1(kr_1) - I_1(kr_1) K_1(kR)}$$

$$B_0(kr) = \frac{I_0(kr) K_1(kr_2) + I_1(kr_2) K_0(kr)}{I_1(kR) K_1(kr_2) - I_1(kr_2) K_1(kR)}$$

$$g_j(k) = I_0(kr_j) K_0(kR) - K_0(kr_j) I_0(kR)$$

$$G_j(k) = I_1(kR) K_0(kr_j) + K_1(kR) I_0(kr_j), \quad j = 1, 2$$

and symbols $I_0(kR)$ and $K_0(kR)$ are zero order modified Bessel functions of first and second kind, respectively.

IV. DISPERSION RELATION

Substituting the values of η , ϕ_1 , ϕ_2 , ψ_1 and ψ_2 in equation (15) we get the dispersion relation

$$D(\omega, k) = a_2 \omega^2 + i a_1 \omega - a_0 = 0 \quad (21)$$

where

$$a_2 = \frac{\rho_1}{\varepsilon} A_0(kR) - \frac{\rho_2}{\varepsilon} B_0(kR)$$

$$a_1 = \left[\frac{\mu_1}{k_1} A_0(kR) - \frac{\mu_2}{k_1} B_0(kR) \right] + 2k^2 \left[\frac{\mu_1}{\varepsilon} A_t(kR) - \frac{\mu_2}{\varepsilon} B_t(kR) \right]$$

$$a_0 = -\frac{\sigma k}{R^2} (1 - k^2 R^2) - \frac{k^2 E_0^2 g_1(k) g_2(k) (\beta_2 - \beta_1)^2}{\beta_1 g_2(k) G_1(k) - \beta_2 g_1(k) G_2(k)}$$

$$A_t(kR) = A_0(kR) - \frac{1}{kR}, \quad B_t(kR) = B_0(kR) - \frac{1}{kR}$$

Let $\omega = \omega_R + i\omega_I$ and equating the real and imaginary parts of equation (21), we have

$$a_2 (\omega_R^2 - \omega_I^2) - a_1 \omega_I - a_0 = 0 \quad (22)$$

and

$$2a_0 \omega_R \omega_I + a_1 \omega_R = 0 \quad (23)$$

so

$$\omega_R = 0$$

Putting this value in equation (23), we get the dispersion relation for the growth rate ω_0 :

$$a_2 \omega_I^2 + a_1 \omega_I + a_0 = 0 \quad (24)$$

Neutral curves are obtained by putting $\omega_I(k) = 0$. Equation (38) reduces to $a_0 = 0$ which in turn implies that

$$\frac{\sigma k}{R^2} (k_c^2 R^2 - 1) = \frac{k_c^2 (E_0)_c^2 g_1(k_c) g_2(k_c) (\varepsilon_1 - \varepsilon_2) (\sigma_1 - \sigma_2)}{\sigma_1 g_2(k_c) G_1(k_c) - \sigma_2 g_1(k_c) G_2(k_c)} \quad (25)$$

For values $E_0 \geq (E_0)_c$ the system is linearly stable and for $E_0 < (E_0)_c$ the system is unstable.

V. DIMENSIONLESS FORM OF DISPERSION RELATION

Introducing dimensionless groups

$$\hat{r} = \frac{r}{H}, \quad \hat{z} = \frac{z}{H}, \quad \hat{\eta} = \frac{\eta}{H}, \quad \hat{t} = \frac{t}{\tau}, \quad \hat{\omega}_I = \omega_I \tau$$

$$\hat{k} = kH, \quad \hat{h} = \frac{h_1}{H}, \quad \hat{R} = \hat{r}_1 + \hat{h}, \quad \hat{E}^2 = \frac{\beta_1 E_0^2 H}{\sigma},$$

where the length scale H and time scale τ are defined as

$$H = r_2 - r_1, \quad \tau = \sqrt{\frac{\rho_1 H^3}{\sigma}}$$

Also

$$\rho = \frac{\rho_2}{\rho_1}, \quad \mu = \frac{\mu_2}{\mu_1}, \quad \sigma = \frac{\sigma_2}{\sigma_1}, \quad \varepsilon = \frac{\varepsilon_2}{\varepsilon_1}, \quad Oh = \frac{\sqrt{\rho_1 \sigma H}}{\mu_1},$$

where Oh is the Ohnesorge number.

Eliminating the ' \wedge ' on the dimensionless variables, the dimensionless form of (24) is

$$b_2 \omega_I^2 + b_1 \omega_I + b_0 = 0 \quad (26)$$

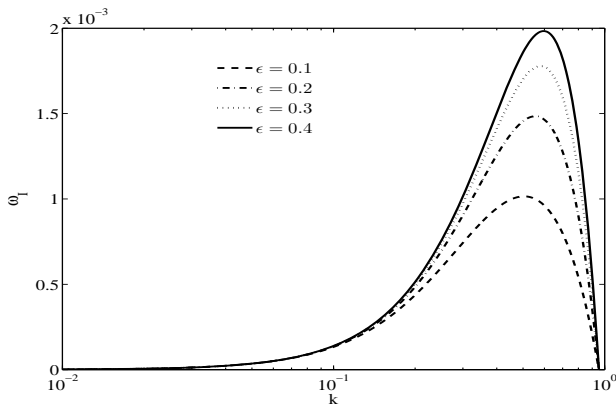


Fig. 2. The growth rate ω_0 vs k for different value of ϵ at $\hat{h} = 0.05, \hat{p}_1 = 1/0.004$.

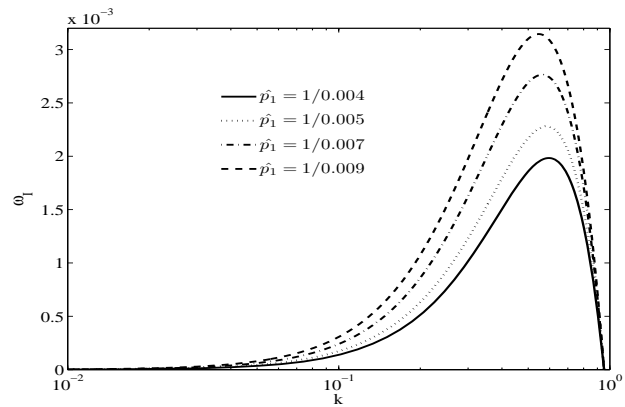


Fig. 3. The growth rate ω_0 vs k for different value of \hat{p}_1 at $\hat{h} = 0.05, \epsilon = 0.5$.

where

$$b_2 = \left(\frac{1}{\epsilon} A_0(kr) - \frac{\rho}{\epsilon} B_0(kr) \right)$$

$$b_1 = \frac{1}{oh} \left(p_1 (A_0(kr) - \mu B_0(kr)) + \frac{2k^2}{\epsilon} (A_t(kr) - \mu B_t(kr)) \right)$$

$$b_0 = \frac{E^2 k^2 (\beta - 1)^2 g_1(k) g_2(k)}{g_2(k) G_1(k) - \beta g_1(k) G_2(k)} + k \left(k^2 - \frac{1}{R^2} \right).$$

From equation (25) the expression for neutral curves becomes

$$\frac{1}{R^2} (k_c^2 R^2 - 1) = \frac{k_c E_c^2 g_1(k_c) g_2(k_c) (1 - \epsilon) (1 - \sigma)}{g_2(k_c) G_1(k_c) - \sigma g_1(k_c) G_2(k_c)} \quad (27)$$

VI. RESULTS AND DISCUSSIONS

The dispersion relation for the linear analysis of viscous capillary instability is quadratic in growth rate and instability occurs due to the positive values of the disturbance growth rate (i.e. $\omega_I > 0$). If ω_I is negative, the perturbation decays with time while if $\omega_I > 0$, the system is unstable as the perturbation grows exponentially with time. The case $\omega_I = 0$ is the marginal stability case. The diameters of the inner and outer cylinders are 1cm and 2 cm, respectively. In the following the effect of various physical parameters on the onset of stability are interpreted through various Figures.

In Figure 2, the effect of porosity of the medium ϵ has been studied when the inside fluid thickness $\hat{h} = 0.05$. It has been observed that as the porosity of the medium increases, the growth rates also increases. This shows that the porosity of the medium destabilizes the system. The variation of non-dimensional number \hat{p}_1 has been shown in Figure 3. As \hat{p}_1 decreases, the growth rate increases and concludes that \hat{p}_1 has stabilizing effect. As the medium permeability is inversely proportional to \hat{p}_1 hence, medium permeability has destabilizing influence.

In figure 4, the growth rate for different values of electric field have been plotted for $\epsilon = 0.5, \hat{p}_1 = 1/0.004$. Here inner fluid thickness is taken as $h = 0.05$. It is observed that as E increases, growth rate decreases i.e. axial electric field has stabilizing effect on the capillary instability. Awasthi and

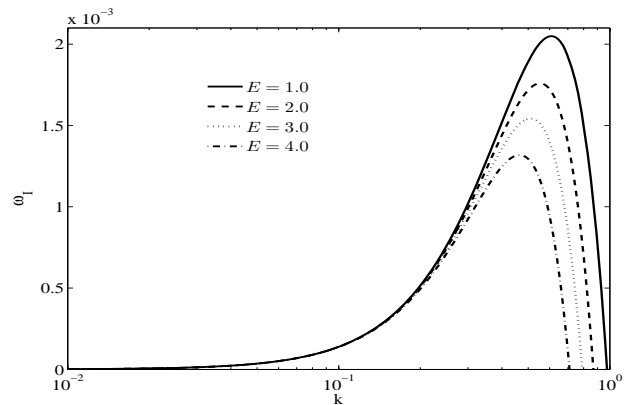


Fig. 4. The growth rate ω_0 vs k for different value of E at $\hat{h} = 0.05, \epsilon = 0.5, \hat{p}_1 = 1/0.004$.

Agrawal [2011] have studied the effect of irrotational shearing stresses on the viscous potential flow analysis of capillary instability in the presence of axial electric field. They have also observed that the axial electric field has stabilizing effect. Therefore, the medium does not affect the behavior of axial electric field on the capillary instability. Figure 5 shows the growth rate curves for different values of β for $E = 4$ and $\epsilon = 0.5, \hat{p}_1 = 1/0.004$. As β increases, growth rate first increases and then decreases therefore, it is concluded that β has dual effect on the stability analysis.

In Figure 7, growth rate curves for different values of inside fluid thickness \hat{h} have been plotted for $\epsilon = 0.5, \hat{p}_1 = 1/0.004$ and observed that inside fluid thickness has destabilizing effect. From Figure 6, we have observed that the wavelength of unstable waves is decreased on increasing the inside fluid thickness. On the other hand from Figure 4, it has observed that the wavelength of unstable waves is increased on increasing the value of electric field.

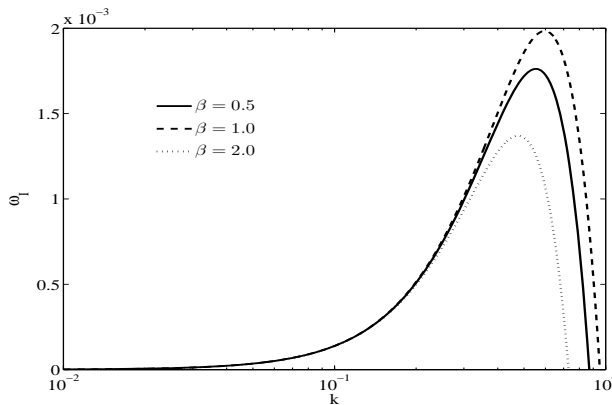


Fig. 5. The growth rate ω_0 vs k for different value of β at $\hat{h} = 0.05$, $\epsilon = 0.5$, $p_1 = 1/0.004$.

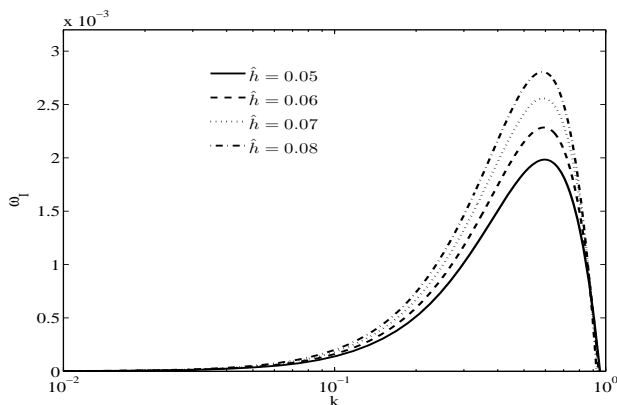


Fig. 6. The growth rate ω_0 vs k for different value of \hat{h} at $\epsilon = 0.5$, $p_1 = 1/0.004$.

VII. CONCLUSIONS

Viscous potential flow analysis of capillary instability through porous media in the presence of an axial electric field has been carried out. The dispersion relation is obtained which is a quadratic equation in growth rate. The stability condition is obtained by applying Routh-Hurwitz criterion for stability. A critical value of electric field as well as critical wave number is obtained. The system is unstable when the electric field is less than the critical value of electric field, otherwise it is stable. It is observed that the axial electric field increases the stability of the system in the presence of porous material. It is found that the ratio of permittivity plays dual role on the stability analysis. Porosity and permeability both have destabilizing effect on the stability of the system.

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