

# Adomian Decomposition Method Associated with Boole's Integration Rule for Goursat Problem

Mohd Agos Salim Nasir, Ros Fadilah Deraman, and Siti Salmah Yasiran

**Abstract**—The Goursat partial differential equation arises in linear and non linear partial differential equations with mixed derivatives. This equation is a second order hyperbolic partial differential equation which occurs in various fields of study such as in engineering, physics, and applied mathematics. There are many approaches that have been suggested to approximate the solution of the Goursat partial differential equation. However, all of the suggested methods traditionally focused on numerical differentiation approaches including forward and central differences in deriving the scheme. An innovation has been done in deriving the Goursat partial differential equation scheme which involves numerical integration techniques. In this paper we have developed a new scheme to solve the Goursat partial differential equation based on the Adomian decomposition (ADM) and associated with Boole's integration rule to approximate the integration terms. The new scheme can easily be applied to many linear and non linear Goursat partial differential equations and is capable to reduce the size of computational work. The accuracy of the results reveals the advantage of this new scheme over existing numerical method.

**Keywords**—Goursat problem, partial differential equation, Adomian decomposition method, Boole's integration rule.

## I. INTRODUCTION

**M**OST of the phenomena that arise in several sciences and engineering fields can be described by partial differential equations. The Goursat problem is a partial differential equation hyperbolic type that arises in several areas of application. In physics for example, supersonic flows [8], reacting gas flow [3] and sonic barrier [7]. In engineering such as, trajectory generation for the  $N$ -Trailer [9], steering of mobile robot [6], isotropic plate [5] and micro differential operator [11]. Usually, these models relate with space and time derivatives and need to be solved in order to gain a better insight into the underlying physical problems. Many of these equations uses analytical methods and hence it cannot be utilized. Thus, numerical approximations need to be used.

In the last decades, several numerical techniques have been proposed to handle the Goursat partial differential equation, among them are Runge-Kutta, finite difference, finite elements, variational iteration method and two dimensional differential transform (see [12]).

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However it is known that many of the techniques only focus on numerical differentiation method in deriving the scheme. In this paper, we develop a new scheme by using numerical integration method for solving Goursat partial differential equation.

Numerical integration is used to describe the numerical solution of differential equation and arises in the numerous applications such as in reformulation of mathematical problems, convert mathematical problems to ordinary or partial differential equation into algebraic equation, calculate the integral transform. It can also be used in fundamental computation technique and applied statistical computation [4]. The Boole's integration rule is based on the evaluating integral at equal subinterval and efficient in solving several numerical problems.

ADM was introduced and developed by [1] and have been proved to be reliable, accurate and effective in the both of analytical solution and numerical approximation to the Goursat partial differential equation. This method can easily handle a wide class of linear or non linear, ordinary or partial differential equation, and integral equation.

Based on the high performance of ADM, we develop a new scheme by using ADM and associated with Boole's integration rule to solve (linear, derivative linear and non linear) Goursat partial differential equations. The accuracy of the proposed scheme is compared with the existing scheme in the literature.

## II. THE GOURSAT PROBLEM AND ADM

The Goursat problem arises in partial differential equation with mixed derivatives. The standard form of Goursat problem [13] is

$$\begin{aligned} u_{xy} &= f(x, y, u, u_x, u_y), \\ u(x, 0) &= g(x), \quad u(0, y) = h(y), \\ g(0) &= h(0) = u(0, 0), \\ 0 \leq x &\leq a, \quad 0 \leq y \leq b. \end{aligned} \quad (1)$$

The established finite difference scheme is based on arithmetic mean averaging of functional values and is given by [13]

$$\frac{u_{i+1,j+1} + u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = \frac{1}{4}(f_{i+1,j+1} + f_{i,j} + f_{i+1,j} + f_{i,j+1}), \quad (2)$$

where  $h$  denotes the grid size.

If the ADM is used, we obtain [14]:

In operators form, the left hand side of general Goursat problem (1) becomes:

$$L_x L_y u(x, y) = f(x, y, u, u_x, u_y), \quad (3)$$

where

$$L_x = \frac{\partial}{\partial x}, \quad L_y = \frac{\partial}{\partial y}. \quad (4)$$

The inverse operators  $L_x^{-1}$  and  $L_y^{-1}$  can be defined as

$$L_x^{-1}(\cdot) = \int_0^x (\cdot) dx, \quad L_y^{-1}(\cdot) = \int_0^y (\cdot) dy. \quad (5)$$

The Goursat problem (1) involves two distinct differential operators  $L_x$  and  $L_y$ , then the two inverse integral operator  $L_x^{-1}$  and  $L_y^{-1}$  will be used. Applying  $L_y^{-1}$  to both sides of (3) gives

$$L_x [L_y^{-1} L_y u(x, y)] = L_y^{-1} f(x, y, u, u_x, u_y), \quad (6)$$

Then, equation (6) becomes

$$L_x [u(x, y) - u(x, 0)] = L_y^{-1} f(x, y, u, u_x, u_y), \quad (7)$$

where

$$L_y^{-1} L_y u(x, y) = u(x, y) - u(x, 0), \quad (8)$$

or equivalently

$$L_x u(x, y) = L_x u(x, 0) + L_y^{-1} f(x, y, u, u_x, u_y). \quad (9)$$

Then, operating with  $L_x^{-1}$  to both sides of (9) will yield:

$$L_x^{-1} L_x u(x, y) = L_x^{-1} L_x u(x, 0) + L_x^{-1} L_y^{-1} f(x, y, u, u_x, u_y). \quad (10)$$

Now we substitute

$$L_x^{-1} L_x u(x, y) = u(x, y) - u(0, y), \quad \text{and}$$

$$L_x^{-1} L_x u(x, 0) = u(x, 0) - u(0, 0). \quad (11)$$

into (10), then

$$u(x, y) - u(0, y) = u(x, 0) - u(0, 0) + L_x^{-1} L_y^{-1} f(x, y, u, u_x, u_y). \quad (12)$$

The ADM derivation of Goursat partial differential equation (1) given as follows

$$u(x, y) = u(x, 0) + u(0, y) - u(0, 0) + L_x^{-1} L_y^{-1} f(x, y, u, u_x, u_y). \quad (13)$$

The equation (13) can be rewrite as

$$u(x_0 + h, y_0 + h) = u(x_0 + h, y_0) + u(x_0, y_0 + h) - u(x_0, y_0) + \int_{x_0}^{x_0+h} \int_{y_0}^{y_0+h} f(x, y, u, u_x, u_y) dy dx. \quad (14)$$

### III. ADM ASSOCIATED WITH BOOLE'S INTEGRATION RULE FOR THE GOURSAT PROBLEM

By indexing the independent variables, (14) becomes

$$u_{i+n, j+n} = u_{i+n, j} + u_{i, j+n} - u_{i, j} + \int_i^{i+n} \int_j^{j+n} f_{i, j} dj di, \quad (15)$$

where  $n$  is a number of subinterval in the numerical integration formula.

The Boole's integration rule with four segments is as follows [2]

$$\int_a^b f(x) dx = \frac{2}{45} h \left[ 7f(a) + 32f(a+h) + 12f(a+2h) + 32f(a+3h) + 7f(b) \right]. \quad (16)$$

By letting  $n = 4$  into (15), we obtain

$$u_{i+4, j+4} = u_{i+4, j} + u_{i, j+4} - u_{i, j} + \int_i^{i+4} \int_j^{j+4} f_{i, j} dj di. \quad (17)$$

Utilize the integration (16) to approximate the double integral in scheme (17) to obtain:

$$\int_i^{i+4} \left[ \int_j^{j+4} f_{i,j} dj \right] di = \int_i^{i+4} \frac{2h}{45} [7f_{i,j} + 32f_{i,j+1} + 12f_{i,j+2} + 32f_{i,j+3} + 7f_{i,j+4}] di$$

$$= \frac{2h}{45} \left[ \frac{14k}{45} (7f_{i,j} + 32f_{i+1,j} + 12f_{i+2,j} + 32f_{i+3,j} + 7f_{i+4,j}) + \frac{64k}{45} (7f_{i,j+1} + 32f_{i+1,j+1} + 12f_{i+2,j+1} + 32f_{i+3,j+1} + 7f_{i+4,j+1}) + \frac{24k}{45} (7f_{i,j+2} + 32f_{i+1,j+2} + 12f_{i+2,j+2} + 32f_{i+3,j+2} + 7f_{i+4,j+2}) + \frac{64k}{45} (7f_{i,j+3} + 32f_{i+1,j+3} + 12f_{i+2,j+3} + 32f_{i+3,j+3} + 7f_{i+4,j+3}) + \frac{14k}{45} (7f_{i,j+4} + 32f_{i+1,j+4} + 12f_{i+2,j+4} + 32f_{i+3,j+4} + 7f_{i+4,j+4}) \right]$$

$$= \frac{28h^2}{2025} (7f_{i,j} + 32f_{i+1,j} + 12f_{i+2,j} + 32f_{i+3,j} + 7f_{i+4,j} + 7f_{i,j+4}) + \frac{128h^2}{2025} (7f_{i,j+1} + 32f_{i+1,j+1} + 12f_{i+2,j+1} + 32f_{i+3,j+1} + 7f_{i+4,j+1}) + \frac{48h^2}{2025} (7f_{i,j+2} + 32f_{i+1,j+2} + 12f_{i+2,j+2} + 32f_{i+3,j+2} + 7f_{i+4,j+2}),$$

(18)

where  $k = h$ .

Substitute approximation (18) into scheme (17). Then, the new scheme can be written as:

$$u_{i+4,j+4} = u_{i+4,j} + u_{i,j+4} - u_{i,j} + \frac{28h^2}{2025} \left( \begin{aligned} &7f_{i,j} + 32f_{i+1,j} + 12f_{i+2,j} + \\ &32f_{i+3,j} + 7f_{i+4,j} + 7f_{i,j+4} \\ &+ 32f_{i+1,j+4} + 12f_{i+2,j+4} + \\ &32f_{i+3,j+4} + 7f_{i+4,j+4} \end{aligned} \right)$$

$$+ \frac{128h^2}{2025} \left( \begin{aligned} &7f_{i,j+1} + 32f_{i+1,j+1} + 12f_{i+2,j+1} + 32f_{i+3,j+1} \\ &+ 7f_{i+4,j+1} + 7f_{i,j+3} + 32f_{i+1,j+3} + 12f_{i+2,j+3} \\ &+ 32f_{i+3,j+3} + 7f_{i+4,j+3} \end{aligned} \right)$$

$$+ \frac{48h^2}{2025} \left( \begin{aligned} &7f_{i,j+2} + 32f_{i+1,j+2} + 12f_{i+2,j+2} \\ &+ 32f_{i+3,j+2} + 7f_{i+4,j+2} \end{aligned} \right).$$

(19)

#### IV. NUMERICAL EXPERIMENTS

We consider the following three Goursat problems

Problem 1: Linear (homogeneous).

$$\begin{aligned} u_{xy} &= u, \\ u(x,0) &= e^x, \\ u(0,y) &= e^y, \\ 0 \leq x \leq 1.2, \quad 0 \leq y \leq 1.2. \end{aligned}$$

(20)

The analytical solution is  $u(x, y) = e^{x+y}$ .

Problem 2: Linear (derivative).

$$\begin{aligned} u_{xy} &= -1 + y + u_x, \\ u(x,0) &= -1 + e^x, \\ u(0,y) &= -1 + e^y, \\ 0 \leq x \leq 2.4, \quad 0 \leq y \leq 2.4. \end{aligned}$$

(21)

The analytical solution is  $u(x, y) = -1 - xy + e^{x+y}$ .

Problem 3: Non-linear.

$$\begin{aligned} u_{xy} &= e^{2u}, \\ u(x,0) &= \frac{x}{2} - \ln(1 + e^x), \\ u(0,y) &= \frac{y}{2} - \ln(1 + e^y), \\ 0 \leq x \leq 3.2, \quad 0 \leq y \leq 3.2. \end{aligned}$$

(22)

The analytical solution is:

$$u(x, y) = \frac{x+y}{2} - \ln(e^x + e^y).$$

These problems were considered as [14], [10] and [13] respectively. We developed MATLAB program for the application of schemes (2) and (19) to problems (20), (21) and (22). The graphs and results presented below are relative errors and average relative errors at several selected grid points respectively.

Results of Problem 1:

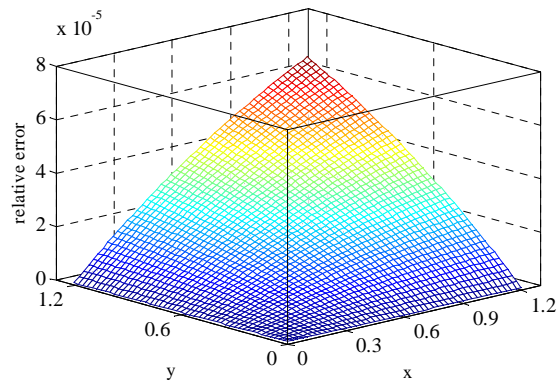


Fig. 1 Graph of relative errors for scheme (2) at  $h = 0.025$

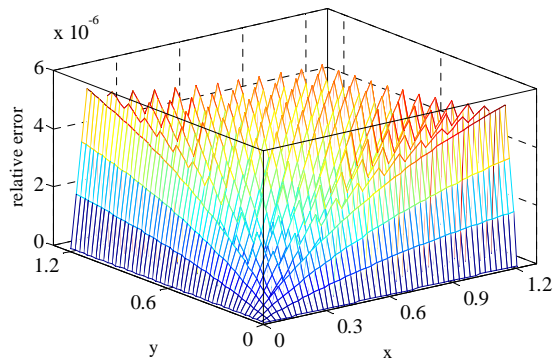


Fig. 2 Graph of relative errors for scheme (19) at  $h = 0.025$

TABLE I  
RESULT OF AVERAGE RELATIVE ERRORS

Grid size (h)	Scheme (2)	Scheme (19)
0.020	$1.3346202 \times 10^{-5}$	$1.1528706 \times 10^{-6}$
0.012	$4.7492517 \times 10^{-6}$	$2.4810242 \times 10^{-7}$
0.010	$3.2885079 \times 10^{-6}$	$1.4344073 \times 10^{-7}$
0.006	$1.1769762 \times 10^{-6}$	$3.0922875 \times 10^{-8}$

Results of Problem 2:

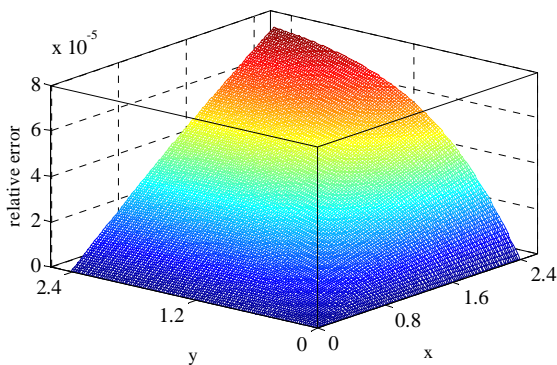


Fig. 3 Graph of relative errors for scheme (2) at h = 0.020

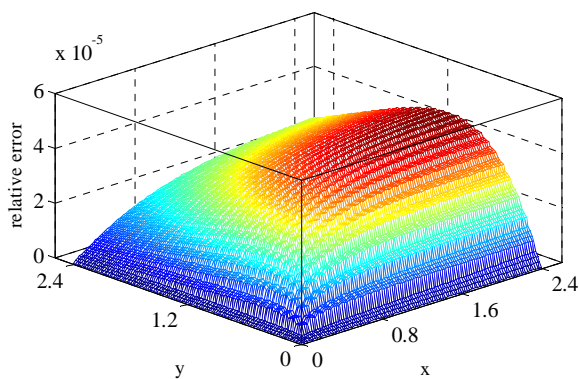


Fig. 4 Graph of relative errors for scheme (19) at h = 0.020

TABLE II  
RESULT OF AVERAGE RELATIVE ERRORS

Grid size (h)	Scheme (2)	Scheme (19)
0.040	$1.2239002 \times 10^{-4}$	$5.6937761 \times 10^{-5}$
0.030	$6.8389269 \times 10^{-5}$	$4.0549595 \times 10^{-5}$
0.025	$4.6943413 \times 10^{-5}$	$3.2972447 \times 10^{-5}$
0.024	$4.3595431 \times 10^{-5}$	$3.1840536 \times 10^{-5}$

Results of Problem 3:

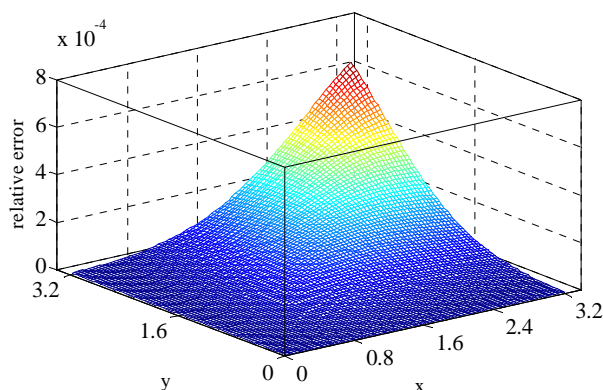


Fig. 5 Graph of relative errors for scheme (2) at h = 0.040

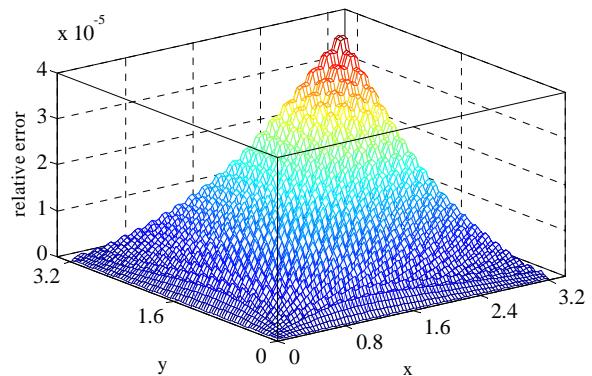


Fig. 6 Graph of relative errors for scheme (19) at h = 0.040

TABLE III  
RESULT OF AVERAGE RELATIVE ERRORS

Grid size (h)	Scheme (2)	Scheme (19)
0.025	$3.7624478 \times 10^{-5}$	$1.7847299 \times 10^{-6}$
0.020	$2.4003895 \times 10^{-5}$	$9.1161289 \times 10^{-7}$
0.016	$1.5323754 \times 10^{-5}$	$4.6584645 \times 10^{-7}$
0.010	$5.9631796 \times 10^{-6}$	$1.1339828 \times 10^{-7}$

As can be seen from the graphs and results of the average relative errors, for all Goursat problems with the grid sizes investigated, the new scheme (19) is more accurate than the standard scheme (2).

V. CONCLUSION

In this paper, we have developed a new scheme based on ADM associated with a well known Boole's integration rule for Goursat partial differential equations (linear, derivative linear and non linear). Our new scheme preserves the linearity of the Goursat problems (20) and (21). The numerical results we obtained confirm the superiority of the new scheme (19) over the established scheme (2).

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