

The Structure of Weakly Left C-wrpp Semigroups

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Abstract—In this paper, the class of weakly left C-wrpp semigroups which includes the class of weakly left C-rpp semigroups as a subclass is introduced. To particularly show that the spined product of a left C-wrpp semigroup and a right normal band which is a weakly left C-wrpp semifroup by virtue of left C-full Ehreman cyber groups recently obtained by authors Li-Shum, results obtained by Tang and Du-Shum are extended and strengthened.

Keywords—Left C-semigroup, left C-wrpp semigroup, left quasi-normal band, weakly left C-wrpp semigroup

I. INTRODUCTION

THROUGHOUT this paper, we adopt the notation and terminologies given by Howei[1] and Li-Shum[2].

By modifying Green's relations on rpp semigroups, Tang [3] has introduced a new set of Green's relations on a semigroup S and by using these new Green's relations, he was able to give a description for a wider class of C-rpp semigroups, namely, the class of C-wrpp semigroups. Tang[3] considered a Green-like right congruence relation L^{**} on a semigroup S for $a, b \in S$ $aL^{**}b$ if and only $axR ay \Leftrightarrow bxR by$ for all $x, y \in S^1$. Moreover, Tang pointed out in [3] that a semigroup S is a wrpp semigroup if and only if each L^{**} -class of S contains at least one idempotent.

Recall that a wrpp semigroup S is a C-wrpp semigroup if the idempotents of S are central. It is well known that a semigroup S is a C-wrpp semigroup if and only if S is a strong semilattice of left-R cancellative monoids (see[3]). Because a Clifford semigroup can be expressed as a strong semilattice of groups and a C-rpp semigroup can be expressed as a strong semilattice of left cancellative monoids (see[4]), we see immediately that the concept of C-wrpp semigroups is a common generalization of Clifford semigroups and C-rpp semigroups.

For wrpp semigroups, Du-Shum [5] first introduced the concept of left C-wrpp semigroups, that is, a left C-wrpp semigroup whose satisfy the following conditions: (i) for all $e \in E(L_a^{**})$, $a = ae$ where $E(L_a^{**})$ is the set of idempotents in L_a^{**} ; (ii) for all $a \in S$, there exists a unique idempotent a^+

satisfying $aL^{**}a^+$ and $a = a^+a$; (iii) for all $a \in S$, $aS \subseteq L^{**}(a)$, where $L^{**}(a)$ is the smallest left $**$ -ideal of S generated by a . For left C-wrpp semigroups, Du-Shum[5] gave a method of construction.

Guo [6] has investigated weakly left C-semigroups, and he pointed out that a semigroup S is a weakly left C-semigroup if and only if S is a completely regular semigroup with idempotents set $E(S)$ forming a left quasi-normal band.

In this paper, we first define the concept of weakly left C-wrpp semigroups. A structure theorem for weakly C-wrpp semigroups is obtained, and we prove this theorem in view of the structure of left C-full Ehreman cyber groups recently obtained by Li and Shum[2].

II. PRELIMINARIES

We first recall that some known results used in the sequel. The following results due to [2] and [7].

Let S be a semigroup, and U a subset of the set $E(S)$ which is the set of all idempotents of S . For all $a \in S$, let $U_a^l = \{u \in U \mid ua = a\}$, $U_a^r = \{u \in U \mid au = a\}$, $U_a = U_a^l \cap U_a^r = \{u \in U \mid ua = a = au\}$. According to Lawson[8] and He[7], we have the following relations on S :

$$\tilde{L}^U = \{(a, b) \in S \times S \mid U_a^r = U_b^r\}, \tilde{R}^U = \{(a, b) \in S \times S \mid U_a^l = U_b^l\}, \\ \tilde{H}^U = \tilde{L}^U \cap \tilde{R}^U, \tilde{Q}^U = \{(a, b) \in S \times S \mid U_a = U_b\}.$$

It is easy to verify that above relations are equivalent relations. For all $a \in S$, a \tilde{L}^U -class, a \tilde{R}^U -class, a \tilde{H}^U -class and a \tilde{Q}^U -class of S containing a , denoted by $\tilde{L}_a^U, \tilde{R}_a^U, \tilde{H}_a^U$ and \tilde{Q}_a^U , respectively. For the sake of convenience, we denote the semigroup S with a projective set U which is a subset of all idempotents $E(S)$ by $S(U)$.

Consider the special semigroup $S(U)$ with $U = E(S)$. Then the equivalent relations on $S = S(U)$, say $\tilde{L}^{E(S)}, \tilde{R}^{E(S)}, \tilde{H}^{E(S)}$ and $\tilde{Q}^{E(S)}$, respectively. For brevity, we write $\tilde{L}, \tilde{R}, \tilde{H}$ and \tilde{Q} , respectively.

Definition 2.1 A semigroup $S(U)$ is called a U -semi-lpp semigroup if each \tilde{R}^U -class of S contains at least one element in U , that is, $\tilde{R}_a^U \cap U \neq \emptyset$ for all $a \in S$. A semigroup $S(U)$ is called a U -semi-rpp semigroup if each \tilde{L}^U -class of S contains at least one element in U , that is, $\tilde{L}_a^U \cap U \neq \emptyset$ for all

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$a \in S$. A semigroup $S(U)$ is called a U -semiabundant semigroup if both each \tilde{L}^U -class and each \tilde{R}^U -class of S contain at least one element in U , that is, $\tilde{L}_a^U \cap U \neq \emptyset$ and $\tilde{R}_a^U \cap U \neq \emptyset$ for all $a \in S$. A semigroup $S(U)$ is called U -abundant semigroup if each \tilde{Q}^U -class of S contains at least one element in U , that is, $\tilde{Q}_a^U \cap U \neq \emptyset$. Denoted the unique element in U by a_e° if $|\tilde{Q}_a^U \cap U|=1$. In special case $U = E(S)$, a $E(S)$ -abundant semigroup is called a full abundant semigroup, in this case, a_e° is usually written as a° .

Lawson[8] point out that \tilde{L}^U is not necessarily a right congruence and \tilde{R}^U is not necessarily a left congruence on $S(U)$. We have

Definition 2.2 A semigroup $S(U)$ is called satisfying U -right(left) congruence condition if $\tilde{L}^U \in RC(S)$ ($\tilde{R}^U \in LC(S)$), a semigroup $S(U)$ is called satisfying U -congruence condition if $\tilde{L}^U \in RC(S)$ and $\tilde{R}^U \in LC(S)$, where $RC(S)$ is a lattice that all right congruences form, and $LC(S)$ is a lattice that all left congruences form.

Definition 2.3 A U -semiabundant semigroup $S(U)$ is called an Ehresmann semigroup if $S(U)$ satisfies U -congruence condition and U is a subsemilattice of U . In particular, an Ehresmann semigroup $S(U)$ is called a C-Ehresmann semigroup if U lies in center of $S(U)$.

Definition 2.4 A U -abundant semigroup $S(U)$ is called an orthodox U -abundant semigroup if U is a subsemigroup of $S(U)$ and $(ab)_U^\circ D(U) a_U^\circ b_U^\circ$ for all $a, b \in S$, where $D=L \vee R$ is a usual Green's D relation.

Definition 2.5 An orthodox U -abundant semigroup is called a left C-Ehresmann semigroup if $uS \subseteq Su$ holds for all $u \in U$.

Definition 2.6 An orthodox U -abundant semigroup is called a left C-Ehresmann cyber group if the identity $uxuy = uxy$ holds for all $u \in U, x, y \in S$.

When $U = E(S)$, we call an orthodox $E(S)$ -abundant semigroup(left C-Ehresmann semigroup, left C-Ehresmann cyber group) an orthodox full abundant semigroup(left C-full Ehresmann semigroup, left C-full Ehresmann cyber group).

Recall that the direct product $I \times T$ of a left zero band I and a monoid T is called a left monoid, and the direct product $I \times T$ of a left zero band I and a unipotent semigroup T is called a left unipotent semigroup. It is well known that a right normal band Λ can be expressed as a strong semilattice of right zero bands, that is, $\Lambda = [Y; \Lambda_\alpha; \varphi_{\alpha,\beta}]$. By using the above results,

He[7] have proved that the following results:

Lemma 2.7 The following statements are equivalent for a semigroup S :

- (i) $S(U)$ is a left C-Ehresmann semigroup for some $U \subseteq E(S)$;
- (ii) S is a semilattice Y of left monoids $S_\alpha = I_\alpha \times T_\alpha$ and

$U = \{(i, 1_\alpha) | i \in I_\alpha\}$ is a subsemigroup of S where $\alpha \in Y, 1_\alpha$ is the identical element in T_α .

By virtue of above lemma, a left C-Ehresmann semigroup $S(U) = [Y; S_\alpha = I_\alpha \times T_\alpha]$ may be defined as a semilattice of left monoids $S_\alpha = I_\alpha \times T_\alpha$, and the set $U = \{(i, 1_\alpha) | i \in I_\alpha, \alpha \in Y\}$ is also a subsemigroup of $S(U)$.

The following lemma have been recently proved by Li-Shum [2].

Lemma 2.8 Let S be a semigroup. Then $S(U)$ is a left C-Ehresmann cyber group for some $U \subseteq E(S)$ if and only if S is isomorphic to a spined product $S_1 \times_Y \Lambda$ of a left C-Ehresmann semigroup $S_1 = [Y; I_\alpha \times T_\alpha]$ and a right normal band $\Lambda = [Y; \Lambda_\alpha; \varphi_{\alpha,\beta}]$ with respect to the semilattice Y .

By using Lemma 2.7, we can easily follow that S is a left C-full Ehresmann semigroup if and only if S is a semilattice of left unipotent semigroups. Hence we can denote a left C-full Ehresmann semigroup S by $S = [Y; S_\alpha = I_\alpha \times T_\alpha]$ (see[7]).

By using Lemma 2.8, we can easily imply that S is a left C-full Ehresmann cyber group if and only if S is isomorphic to a spined product $S_1 \times_Y \Lambda$ of a left C-full Ehresmann semigroup $S_1 = [Y; I_\alpha \times T_\alpha]$ and a right normal band $\Lambda = [Y; \Lambda_\alpha; \varphi_{\alpha,\beta}]$ with respect to the semilattice Y (see [2]).

III. THE STRUCTURE OF WEAKLY LEFT C-WRPP SEMIGROUPS

In this section, the concept of weakly left C-wrpp semigroups is introduced. We shall prove that a structure theorem for weakly left C-wrpp semigroups. First, we introduce the concept of weakly left C-wrpp semigroups.

Definition 3.1 A semigroup S is called a weakly left C-wrpp semigroup, if S is a strong wrpp semigroup and satisfy identity $exey = exy$ for all $e \in E(S)$ and $x, y \in S$.

According to [5], we know that the left C-wrpp semigroup is a special case of the weakly left C-wrpp semigroup.

Lemma 3.2 Let S be a strongly wrpp semigroup. Then S is a full abundant semigroup with $a^\circ = a^+$, for all $a \in S$.

Proof. Let S be a strongly wrpp. To prove S is a full abundant semigroup, we only need to prove $a^\circ = a^+$ for all $a \in S$. Let $I_a = \{e | ea = ae = a\}$. For all $e \in I_a$, since

$(a^+e)^+ a = (a^+e)^+ (a^+e)a = (a^+e)a = a(a^+e)(a^+e)^+ = a(a^+e)^+$, and $a = aeL^{**} a^+ eL^{**} (a^+e)^+$. So we have $(a^+e)^+ \in L_a^{**} \cap I_a$. This implies that $(a^+e)^+ = a^+$. Thus, we have $a^+ea^+ = a^+e$ and whence $a^+e \in E(S)$. Consequently, we obtain that $a^+e = (a^+e)^+ = a^+$. On the other hand, we can easily verify that $ea^+ \in L_a^{**} \cap I_a$.

Therefore, $ea^+ = a^+$, and so $e \in I_{a^+}$. Thus, it follows that $I_a \subseteq I_{a^+}$. Clearly, $I_{a^+} \subseteq I_a$ and whence $I_a = I_{a^+}$. This means that $(a, a^+) \in \tilde{Q}$. Therefore, S is a full abundant semigroup with $a^\circ = a^+$. The proof is completed.

Lemma 3.3 Let S be a weakly left C-wrpp semigroup. Then

S is a left C-full Ehresmann cyber group.

Proof. Let S be a weakly left C-wrpp semigroup. Then we have $exey = exy$ for all $e \in E(S), x, y \in S$, and hence $E(S)$ is a left quasi-normal band. According to Lemma 3.2, we know that S is a full abundant semigroup with $a^\circ = a^+$. To prove that S is a left C-full Ehresmann cyber group, it suffices to prove $S(E(S))$ satisfying $(ab)^+ D(E(S)) a^\circ b^\circ$ for all $a, b \in S$.

Now, Let $a, b \in S$. For all $x, y \in S^1$, we can infer that

$$\begin{aligned} (ab)^+ xR (ab)^+ y &\Leftrightarrow abxR aby \text{ (R is a left congruence)} \\ \Leftrightarrow a^+ ba^+ b^+ xR a^+ ba^+ b^+ y \text{ (E(S) is a left quasi-normal band)} \\ \Leftrightarrow aba^+ b^+ xR aba^+ b^+ y \text{ (R is a left congruence)} \\ \Leftrightarrow (ab)^+ a^+ b^+ xR (ab)^+ a^+ b^+ y. \end{aligned}$$

This means that $(ab)^+ L^*(ab)^+ a^+ b^+$. Since $(ab)^+, (ab)^+ a^+ b^+ \in E(S)$, we can verify that $(ab)^+ L(ab)^+ a^+ b^+$, and whence $(ab)^+ =$

$(ab)^+ a^+ b^+$. For all $x, y \in S$, we have

$$\begin{aligned} (ab)^+ a^+ xR (ab)^+ a^+ y &\Leftrightarrow aba^+ xR aba^+ y \\ \Leftrightarrow a^+ ba^+ xR a^+ ba^+ y \\ \Leftrightarrow b^+ a^+ ba^+ xR b^+ a^+ ba^+ y \text{ (R is a left congruence)} \\ \Leftrightarrow (ba)^+ b^+ a^+ ba^+ xR (ba)^+ b^+ a^+ ba^+ y \text{ (R is a left congruence)} \\ \Leftrightarrow ba^+ xR ba^+ y &\Leftrightarrow b^+ a^+ xR b^+ a^+ y \\ \Leftrightarrow a^+ b^+ a^+ xR a^+ b^+ a^+ y \text{ (R is a left congruence)} \\ \Leftrightarrow (ab)^+ a^+ b^+ a^+ xR (ab)^+ a^+ b^+ a^+ y \text{ (R is a left congruence)} \\ \Leftrightarrow (ab)^+ a^+ xR (ab)^+ a^+ y. \end{aligned}$$

So we have $(ab)^+ a^+ L^*(ab)^+ a^+ b^+$, and hence $(ab)^\circ = (ab)^+ R (ab)^+ a^+ L a^+ b^+ a^+ R a^+ b^+ = a^\circ b^\circ$. The proof is completed.

We now characterize the weakly left C-wrpp semigroups.

Theorem 3.4 A semigroup S is a weakly left C-wrpp semigroup if and only if S is isomorphic to a spined product $S_1 \times_Y \Lambda$ of a left C-wrpp semigroup $S_1 = [Y; S_\alpha = I_\alpha \times T_\alpha]$ and a right normal band $\Lambda = [Y; \Lambda_\alpha; \varphi_{\alpha,\beta}]$ with respect to semilattice Y .

Proof. Necessity. Let S be a weakly left C-wrpp semigroup. Then by Lemma 3.3, S is a left C-full Ehresmann cyber group with $a^\circ = a^+$ for all $a \in S$. According to Lemma 2.8, we know that S can express as $S_1 \times_Y \Lambda$, where $S_1 = [Y; S_\alpha = I_\alpha \times T_\alpha]$ is a left C-full Ehresmann semigroup and $\Lambda = [Y; \Lambda_\alpha; \varphi_{\alpha,\beta}]$ is a right normal band. We only need to show that S_1 is a strongly wrpp semigroup and T_α is a left-R cancellative monoid.

For all $(i, a) \in S_\alpha, (j, b) \in S_\beta$ and $(k, c) \in S_\gamma$, we have

$$\begin{aligned} (i, a)(j, b)R (i, a)(k, c) &\Rightarrow (i, a)(i, 1_\alpha)(j, b)R (i, a)(i, 1_\alpha)(k, c) \\ \Rightarrow ((i, a), \lambda)((i, 1_\alpha)(j, b), \mu)R &((i, a), \lambda)((i, 1_\alpha)(k, c), \mu) \text{ (} \lambda \in \Lambda_\alpha, \\ \mu \in \Lambda_{\alpha\beta} = \Lambda_{\alpha\gamma} \text{)} \\ \Rightarrow ((i, 1_\alpha), \lambda)((i, 1_\alpha)(j, b), \mu)R &((i, 1_\alpha), \lambda)((i, 1_\alpha)(k, c), \mu) \text{ (} \lambda \in \Lambda_\alpha, \\ \mu \in \Lambda_{\alpha\beta} = \Lambda_{\alpha\gamma} \text{)} &\Rightarrow (i, 1_\alpha)(j, b)R (i, 1_\alpha)(k, c) \end{aligned}$$

since $((i, a), \lambda)^+ = ((i, a), \lambda)^\circ = ((i, 1_\alpha), \lambda)$, where 1_α is the identity in $T_\alpha (\alpha \in Y)$. Similarly, we can deduce that $(i, a)(j, b)$

$R (i, a) \Rightarrow (i, 1_\alpha)(j, b)R (i, 1_\alpha)$. Thus $(i, a)L^{**}(i, 1_\alpha)$. By $(i, 1_\alpha)$ being the unique element in $L^{**} \cap I_{(i,a)}$, we observe that S_1 is a strongly wrpp semigroup with $(i, a)^+ = (i, 1_\alpha)$. If $a, b, c \in T_\alpha$ such that $abR ac$, then $(i, a)(i, b)R (i, a)(i, c)$ for all $i \in I_\alpha$. By $(i, a)L^{**}(i, 1_\alpha)$, we have $(i, 1_\alpha)(i, b)R (i, 1_\alpha)(i, c)$, and whence $bR c$, this means that T_α is a left-R cancellative monoid.

Sufficiency. Assume that $S = S_1 \times_Y \Lambda$, where $S_1 = [Y; S_\alpha = I_\alpha \times T_\alpha]$ is a left C-wrpp semigroup and $\Lambda = [Y; \Lambda_\alpha; \varphi_{\alpha,\beta}]$ is a right normal band, then $(i, a)^+ = (i, 1_\alpha)$ for $(i, a) \in S_\alpha$, where 1_α is the identity in T_α . We easily verify that S is a strongly wrpp semigroup with $((i, a), \lambda)^+ = ((i, 1_\alpha), \lambda)$, and whence we can also check that S is a weakly left C-wrpp semigroup.

Weakly left C-semigroups were first investigated by Guo[6] in 1996, and weakly left C-rpp semigroups were investigated by Cao [9] in 2000. It is clear that weakly left C-semigroups and weakly left C-rpp semigroups are special weakly left C-wrpp. As applications of Theorem 3.4, we have the following corollaries:

Corollary 3.5 A semigroup S is a weakly left C-rpp semigroup if and only if S is isomorphic to a spined product $S_1 \times_Y \Lambda$ of a left C-rpp semigroup $S_1 = [Y; S_\alpha = I_\alpha \times T_\alpha]$ and a right normal band $\Lambda = [Y; \Lambda_\alpha; \varphi_{\alpha,\beta}]$ with respect to semilattice Y .

Corollary 3.6 A semigroup S is a weakly left C-semigroup if and only if S is isomorphic to a spined product $S_1 \times_Y \Lambda$ of a left C-semigroup $S_1 = [Y; S_\alpha = I_\alpha \times T_\alpha]$ and a right normal band $\Lambda = [Y; \Lambda_\alpha; \varphi_{\alpha,\beta}]$ with respect to semilattice Y .

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