International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:5, No:4, 2011

Translation Surfaces in Euclidean 3-Space

Muhammed Çetin, Yılmaz Tunçer, and Nejat Ekmekçi

***Abstract*—**In this paper, the translation surfaces in 3-dimensional Euclidean space generated by two space curves have been investigated. It has been indicated that Scherk surface is not only minimal translation surface.

***Keywords*—**Minimal surface, Surface of Scherk, Translation surface

I. Introduction

As is well-known, the theory of translation surfaces is always one of interesting topics in Euclidean space. Translation surfaces have been investigated from the various viewpoints by many differential geometers. L. Verstraelen, J. Walrave and S. Yaprak have investigated minimal translation surfaces in n-dimensional Euclidean spaces [3]. H. Liu has given the classification of the translation surfaces with constant mean curvature or constant Gauss curvature in 3-dimensional Euclidean space E^3 and 3-dimensional Minkowski space E_1^3 [2]. D. W. Yoon has studied translation surfaces in the 3-dimensional Minkowski space whose Gauss map G satisfies the condition $\Delta G = AG,\ A \in Mat(3, IR)$, where Δ denotes the Laplacian of the surface with respect to the induced metric and $Mat(3, IR)$ the set of 3x3 real metrics [1]. M. I. Munteanu and A. I. Nistor have studied the second fundamental form of translation surfaces in E^3 [4]. They have given a non-existence result for polynominal translation surfaces in E^3 with vanishing second Gauss curvature K_{II}. They have classified those translation surfaces for which K_{II} and H are proportional.

In this paper, the translation surfaces in 3-dimensional Euclidean space by using non-planar space curves have been investigated and some differential geometric properties for both translation surfaces and minimal translation surfaces have been given. Furthermore, a classification of minimal translation surfaces with examples have been given.

II. Translation Surfaces with Space Curves

Let $M(u,v)$ be a translation surface in 3-dimensional Euclidean space. Then $M(u,v)$ is parametrized by

$$M(u,v) = \alpha(u) + \beta(v)$$

where α and β being unit-speed space curves of the arc-length parameters u and v, respectively. Let $\{T_\alpha, N_\alpha, B_\alpha\}$ be the Frenet frame field of α with curvature κ_α and torsion τ_α. Also, let $\{T_\beta, N_\beta, B_\beta\}$ be the Frenet frame field of β with curvature κ_β and torsion τ_β.

A surface that can be generated from two space curves by translating either one of them parallel to itself in such a way that each of its points describes a curve that is a translation of the other curve. Let $M(u,v)$ be a translation surface in 3-dimensional Euclidean space.

$$M(u,v) = \alpha(u) + \beta(v)$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)$. Then

$$M(u,v) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3).$$

The unit normal of translation surface can be defined by

$$U(u,v) = \frac{1}{\sin\varphi}\left(T_\alpha \wedge T_\beta\right)$$

where $\varphi(u)$ is the angle between tangent vectors of $\alpha(u)$ and $\beta(v)$. The first fundamental form I of the surface $M(u,v)$ is

$$I = du^2 + 2\cos\varphi\, du dv + dv^2$$

and the second fundamental form II is

$$II = \kappa_\alpha \cos\theta_\alpha du^2 + \kappa_\beta \cos\theta_\beta dv^2$$

where θ_α and θ_β are the angels between U and N_α, N_β, respectively.

Theorem 1: If α is an asymptotic line of translation surface, then α is a planar curve.

Proof : Since $\cos\theta_\alpha = \langle U, N_\alpha \rangle$, then

$$\cos\theta_\alpha = \frac{-1}{\sin\varphi}\langle B_\alpha, T_\beta \rangle. \tag{1}$$

Differentiating (1) with respect to u, so

$$\theta_\alpha' \sin\theta_\alpha = \varphi' \cot\varphi \cos\theta_\alpha - \tau_\alpha \sin\theta_\alpha.$$

α is an asymptotic line, so $\cos\theta_\alpha = 0$, $\sin\theta_\alpha = \pm 1$ and θ_α should be a real constant. Hence $\theta_\alpha' = -\tau_\alpha$ and $\tau_\alpha = 0$. Thus α is a planar curve.

Corollary 1: Let's suppose that β is not a geodesic of surface, then β is a planar curve if and only if the θ_β is constant.

Proof : Since $\cos\theta_\beta = \langle U, N_\beta \rangle$, then

$$\cos\theta_\beta = \frac{-1}{\sin\varphi}\langle T_\alpha, B_\beta \rangle. \tag{2}$$

Differentiating (2) with respect to v, so

M. Çetin is with the Instutition of Science and Technology, Msc Student, Uşak University, Uşak, TURKEY (e-mail: mat.mcetin@hotmail.com).

Y. Tunçer is with the Department of Mathematics, Science and Art Faculty, Uşak University, Uşak, TURKEY (e-mail: yilmaz.tuncer@usak.edu.tr).

N. Ekmekçi is with the Department of Mathematics, Science Faculty, Ankara University, Ankara, TURKEY (e-mail: ekmekci@science.ankara.edu.tr).

International Journal of Engineering, Mathematical and Physical Sciences
ISSN: 2517-9934
Vol:5, No:4, 2011

$$\sin\theta_\beta(\theta_\beta' + \tau_\beta) = 0.$$

β is not a geodesic of surface, so $\theta_\beta' + \tau_\beta = 0$. Hence θ_β is a real constant if and only if β is a planar curve.

On the other hand the shape operator of translation surface

$$S = \frac{1}{\sin^2\varphi}\begin{bmatrix} \kappa_\alpha\cos\theta_\alpha & -\kappa_\alpha\cos\varphi\cos\theta_\alpha \\ -\kappa_\beta\cos\varphi\cos\theta_\beta & \kappa_\beta\cos\theta_\beta \end{bmatrix}.$$

Then the Gauss curvature K and mean curvature H are

$$K = \frac{\kappa_\alpha\kappa_\beta\cos\theta_\alpha\cos\theta_\beta}{\sin^2\varphi} \tag{3}$$

and

$$H = \frac{\kappa_\alpha\cos\theta_\alpha + \kappa_\beta\cos\theta_\beta}{2\sin^2\varphi}. \tag{4}$$

Theorem 2: Gauss curvature of a translation surface generating by space curves is zero if and only if at least one of generator curves is an asiymptotic line of surface.

Proof : Let Gauss curvature be zero, then from (3)

$$\kappa_\alpha\kappa_\beta\cos\theta_\alpha\cos\theta_\beta = 0.$$

the generator curves of translation surface is not line, so $\kappa_\alpha \neq 0$, $\kappa_\beta \neq 0$ and $\cos\theta_\alpha\cos\theta_\beta = 0$. Hence $\cos\theta_\alpha = 0$ or $\cos\theta_\beta = 0$. If $\cos\theta_\alpha$ is zero, then $\theta_\alpha = (2k+1)\frac{\pi}{2}$, $k \in Z$, hence α is an asymptotic line. Similarly, If $\cos\theta_\beta$ is zero, then $\theta_\beta = (2k+1)\frac{\pi}{2}$, $k \in Z$, hence β is an asymptotic line. Conversely, let α or β be an asymptotic line of surface. If α is an asymptotic line of surface, then $\theta_\alpha = (2k+1)\frac{\pi}{2}$, $k \in Z$ and $K = 0$ or If β is an asymptotic line of surface, then $\theta_\beta = (2k+1)\frac{\pi}{2}$, $k \in Z$ and $K = 0$.

Example 1: Let $M(u,v)$ be the translation surface given by

$$M(u,v) = (m_1, m_2, m_3)$$

where

$$m_1 = \sin\frac{u}{2} + \cos\frac{v}{3} - 1,$$

$$m_2 = \cos\frac{u}{2} + \sin\frac{v}{3} - 1,$$

$$m_3 = \frac{\sqrt{3}u}{2} + \frac{2\sqrt{2}v}{3}$$

with generator curves

$$\alpha(u) = \left(\sin\frac{u}{2}, \cos\frac{u}{2} - 1, \frac{\sqrt{3}u}{2}\right)$$

and

$$\beta(v) = \left(\cos\frac{v}{3} - 1, \sin\frac{v}{3}, \frac{2\sqrt{2}v}{3}\right).$$

The tangent and principal normal vectors of α are

$$T_\alpha = \left(\frac{1}{2}\cos\frac{u}{2}, -\frac{1}{2}\sin\frac{u}{2}, \frac{\sqrt{3}}{2}\right)$$

$$N_\alpha = \left(-\sin\frac{u}{2}, -\cos\frac{u}{2}, 0\right).$$

The curvature of α is

$$\kappa_\alpha = \frac{1}{4}.$$

Similarly, the tangent and principal normal vectors of β are

$$T_\beta = \left(-\frac{1}{3}\sin\frac{v}{3}, \frac{1}{3}\cos\frac{v}{3}, \frac{2\sqrt{2}}{3}\right)$$

$$N_\beta = \left(-\cos\frac{v}{3}, -\sin\frac{v}{3}, 0\right).$$

The curvature of β is

$$\kappa_\beta = \frac{1}{9}.$$

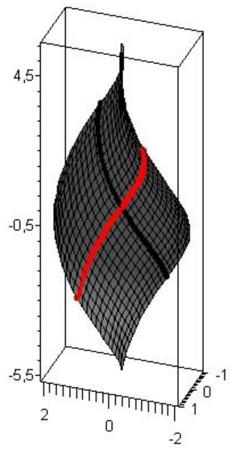

Fig. 1 Translation surface generated by two helices.

Theorem 3: Let α and β be space curves with nonzero curvatures and let α be an asymptotic line. Translation surface is minimal if and only if β is an asymptotic line of surface too.

Let $\kappa_g^\alpha, \tau_g^\alpha$ and κ_n^α be geodesic curvature, geodesic torsion and normal curvature along α of M, respectively, then

$$\kappa_g^\alpha = \kappa_\alpha\sin\theta_\alpha, \quad \kappa_n^\alpha = \kappa_\alpha\cos\theta_\alpha, \quad \tau_g^\alpha = \tau_\alpha + \theta_\alpha'.$$

Similarly, Let $\kappa_g^\beta, \tau_g^\beta$ and κ_n^β be geodesic curvature, geodesic torsion and normal curvature along β of M, respectively, then

$$\kappa_g^\beta = \kappa_\beta\sin\theta_\beta, \quad \kappa_n^\beta = \kappa_\beta\cos\theta_\beta, \quad \tau_g^\beta = \tau_\beta + \theta_\beta'.$$

If M is a translation surface with $K = 0$, then M is a ruled surface or at least one of generator curves of surface is asymptotic line.

Let the generator curves be asymptotic lines of M then shape operator, Gauss curvature and mean curvature are

$$S=\frac{1}{\sin^2\varphi}\begin{bmatrix}\kappa_n^\alpha & -\cos\varphi\kappa_n^\alpha\\ -\cos\varphi\kappa_n^\beta & \kappa_n^\beta\end{bmatrix}$$

$$K=\frac{\kappa_n^\alpha\kappa_n^\beta}{\sin^2\varphi},\quad H=\frac{\kappa_n^\alpha+\kappa_n^\beta}{2\sin^2\varphi}$$

and principal curvatures of M are

$$k_1=\frac{\kappa_n^\alpha+\kappa_n^\beta}{2\sin^2\varphi}+\sqrt{\left(\frac{\kappa_n^\alpha+\kappa_n^\beta}{2\sin^2\varphi}\right)^2-\frac{\kappa_n^\alpha\kappa_n^\beta}{\sin^2\varphi}}$$

$$k_2=\frac{\kappa_n^\alpha+\kappa_n^\beta}{2\sin^2\varphi}-\sqrt{\left(\frac{\kappa_n^\alpha+\kappa_n^\beta}{2\sin^2\varphi}\right)^2-\frac{\kappa_n^\alpha\kappa_n^\beta}{\sin^2\varphi}}$$

respectively. Thus, that followings are satisfies at umbilical points of M

$$\kappa_n^\alpha=\kappa_n^\beta,\quad \cos\varphi\kappa_n^\alpha=0,\quad \cos\varphi\kappa_n^\beta=0.$$

Hence, the following theorem can be given.

Theorem 4: Let M be a translation surface generated by its asymptotic lines then M is a minimal surface if and only if

$$\kappa_n^\alpha+\kappa_n^\beta=0$$

is satisfies.

The Gauss curvature of translation surface with respect to normal curvatures are

$$K=-\left(\frac{\kappa_n^\alpha}{\sin\varphi}\right)^2 \text{ and } K=-\left(\frac{\kappa_n^\beta}{\sin\varphi}\right)^2.$$

Thus $K\le 0$ along the generator curves and it is concluded that all the points of a translation surface is either flat or hyperbolic. In other words, there are no any umbilic points on a minimal translation surface generated by space curves. On the other hand, differentiating $\kappa_\alpha\cos\theta_\alpha+\kappa_\beta\cos\theta_\beta=0$ with respect to u, then

$$\kappa_\alpha'\cos\theta_\alpha-\kappa_\alpha\theta_\alpha'\sin\theta_\alpha=0$$

$$\frac{\kappa_\alpha'}{\kappa_\alpha}=\frac{\theta_\alpha'\sin\theta_\alpha}{\cos\theta_\alpha}$$

$$\ln\kappa_\alpha=-\ln\cos\theta_\alpha+\ln c_1$$

$$\kappa_\alpha\cos\theta_\alpha=c_1$$

$$\kappa_n^\alpha=c_1=\text{constant}.$$

Similarly, differentiating $\kappa_\alpha\cos\theta_\alpha+\kappa_\beta\cos\theta_\beta=0$ with respect to v, then

$$\kappa_\beta'\cos\theta_\beta-\kappa_\beta\theta_\beta'\sin\theta_\beta=0$$

$$\frac{\kappa_\beta'}{\kappa_\beta}=\frac{\theta_\beta'\sin\theta_\beta}{\cos\theta_\beta}$$

$$\ln\kappa_\beta=-\ln\cos\theta_\beta+\ln c_2$$

$$\kappa_\beta\cos\theta_\beta=c_2$$

$$\kappa_n^\beta=c_2=\text{constant}.$$

Hence, the following corollary can be given.

Corollary 2: Normal curvatures of a minimal translation surface are constant along generator curves.

III. Classification of Minimal Translation Surfaces

The classification of the minimal translation surfaces have been given in five cases here.

1. The case $\kappa_\alpha\neq 0$, $\kappa_\beta\neq 0$ and $\cos\theta_\alpha=\cos\theta_\beta=0.$

In this case, binormal lines of generator curves are linearly dependent, therefore $\{T_\beta,N_\beta\}$ rotates according to $\{T_\alpha,N_\alpha\}$ with the angle $\varphi.$ Thus

$$T_\beta=\sin\varphi N_\alpha+\cos\varphi T_\alpha \tag{5}$$

$$N_\beta=\cos\varphi N_\alpha-\sin\varphi T_\alpha \tag{6}$$

can be written. Differentiating (5) with respect to $u,$ then

$$-(\kappa_\alpha+\varphi')\sin\varphi T_\alpha+(\varphi'+\kappa_\alpha)\cos\varphi N_\alpha+\tau_\alpha\sin\varphi B_\alpha=0. \tag{7}$$

T_α,N_α and B_α are linearly independent, so

$$(\kappa_\alpha+\varphi')\sin\varphi=0$$

$$(\varphi'+\kappa_\alpha)\cos\varphi=0$$

$$\tau_\alpha\sin\varphi=0.$$

Then,

$$\kappa_\alpha=-\varphi',\quad \tau_\alpha=0.$$

Thus α is a planar curve. Differentiating (6) with respect to $u,$ similar results can be found. Similarly, differentiating (5) with respect to $v,$ then

$$\kappa_\beta N_\beta=0\ \Rightarrow\ \kappa_\beta=0.$$

It is contradiction with the case 1, so there isn't such minimal translation surfaces under these conditions. On the other hand, from

$$\cos\theta_\alpha=\frac{\kappa_n^\alpha}{\kappa_\alpha}\text{ and }\cos\theta_\beta=\frac{\kappa_n^\beta}{\kappa_\beta},$$

there is no any minimal translation surfaces with $\kappa_n^\alpha=0$ and $\kappa_n^\beta=0$ along generator curves. Consequently, the folowing corollary can be given.

Corollary 3: There is no any minimal translation surface which has admit asymptotic line with non-zero curvatures along generator curves.

2. The case $\kappa_\alpha=0$ and $\kappa_\beta=0.$

In this case, the surface is plane.

3. The case

i) $\kappa_\alpha=0$, $\kappa_\beta\neq 0$ and $\cos\theta_\beta=0.$

In this case, the surface is cylindrical. Since $\cos\theta_\beta=0$ then nomal vector field of surface and binormal vector field of β curve are linearly dependent. Hence, α lies in $\{T_\beta,N_\beta\}$ plane.

ii) $\kappa_\alpha\neq 0$, $\kappa_\beta=0$ and $\cos\theta_\alpha=0.$

It is similar to case 3.i.

4. The case $\kappa_\alpha=\kappa_\beta\neq 0$, $\cos\theta_\alpha=-\cos\theta_\beta\neq 0.$

This case has been investigated in two parts.

i) $\kappa_\alpha=\kappa_\beta\neq 0$, $\cos\theta_\alpha=-\cos\theta_\beta=1.$

In this case, principal normal lines of generator curves are linearly dependent, therefore $\{T_\beta, B_\beta\}$ rotates according to $\{T_\alpha, B_\alpha\}$ with the angle φ. Thus,

$$T_\alpha = \cos\varphi T_\beta - \sin\varphi N_\beta \quad (8)$$

$$N_\alpha = \sin\varphi T_\beta + \cos\varphi N_\beta \quad (9)$$

can be written. Differentiating (8) with respect to v, then

$$\kappa_\beta \sin\varphi T_\beta + \kappa_\beta \cos\varphi N_\beta - \tau_\beta \sin\varphi B_\beta = 0.$$

T_β, N_β and B_β are linearly independent, so

$$\kappa_\beta \sin\varphi = 0 \Rightarrow \kappa_\beta = 0$$

$$\tau_\beta \sin\varphi = 0 \Rightarrow \tau_\beta = 0.$$

It is contradiction with the case 4.i, so there isn't such minimal translation surfaces under these conditions.

ii) $\kappa_\alpha = \kappa_\beta \neq 0,\ \cos\theta_\alpha = -\cos\theta_\beta \neq 1.$

The Scherk surface is an example this case.

Example 2: Surface of Scherk is defined by

$$M(u,v) = \left(u, v, \frac{1}{a}\log\left(\frac{\cos av}{\cos au}\right)\right)$$

with the generator curves

$$\alpha(u) = \left(u, 0, -\frac{1}{a}\log(\cos au)\right)$$

$$\beta(v) = \left(0, v, \frac{1}{a}\log(\cos av)\right).$$

The tangent and principal normal vectors of α are

$$T_\alpha = \frac{\ln 10}{\sqrt{\ln^2 10 + \tan^2 au}}\left(1, 0, \frac{\tan au}{\ln 10}\right)$$

and

$$N_\alpha = \frac{1}{\sqrt{\ln^2 10 + \tan^2 au}}(-\tan au, 0, \ln 10). \quad (10)$$

The curvature of α is

$$\kappa_\alpha = \frac{a(1+\tan^2 au)\ln^2 10}{(\ln^2 10 + \tan^2 au)^{\frac{3}{2}}}. \quad (11)$$

Similarly, the tangent and principal normal vectors of β are

$$T_\beta = \frac{\ln 10}{\sqrt{\ln^2 10 + \tan^2 av}}\left(0, 1, -\frac{\tan av}{\ln 10}\right)$$

and

$$N_\beta = \frac{1}{\sqrt{\ln^2 10 + \tan^2 av}}(0, -\tan av, -\ln 10). \quad (12)$$

The curvature of β is

$$\kappa_\beta = \frac{a(1+\tan^2 av)\ln^2 10}{(\ln^2 10 + \tan^2 av)^{\frac{3}{2}}}. \quad (13)$$

Also, the unit normal vector of surface is

$$U = \frac{1}{\sqrt{\tan^2 au + \tan^2 av + \ln^2 10}}(-\tan au, \tan av, \ln 10) \quad (14)$$

and from (10), (12) and (14),

$$\cos\theta_\alpha = \frac{\tan^2 au + \ln^2 10}{\sqrt{\tan^2 au + \tan^2 av + \ln^2 10}\sqrt{\ln^2 10 + \tan^2 au}} \quad (15)$$

and

$$\cos\theta_\beta = \frac{-\tan^2 av - \ln^2 10}{\sqrt{\tan^2 au + \tan^2 av + \ln^2 10}\sqrt{\ln^2 10 + \tan^2 av}}. \quad (16)$$

Substituting (11), (13), (15) and (16) in (4), so $H = 0$.

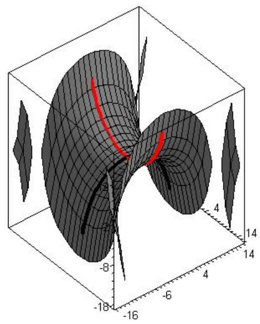

Fig. 2 Scherk surface is one of minimal translation surfaces.

The following example can be given for this case different from surface of Scherk.

Example 3: Let $M(u,v)$ be the translation surface given by

$$M(u,v) = (m_1, m_2, m_3)$$

where

$$m_1 = \sin\frac{u}{2} - \sin\frac{v}{2}$$

$$m_2 = \cos\frac{u}{2} - \cos\frac{v}{2}$$

$$m_3 = \frac{\sqrt{3}u}{2} + \frac{\sqrt{3}v}{2}$$

with the generator curves

$$\alpha(u) = \left(\sin\frac{u}{2}, \cos\frac{u}{2} - 1, \frac{\sqrt{3}u}{2}\right)$$

and

$$\beta(v) = \left(-\sin\frac{v}{2}, -\cos\frac{v}{2} + 1, \frac{\sqrt{3}v}{2}\right).$$

The tangent and principal normal vectors of α are

$$T_\alpha = \left(\frac{1}{2}\cos\frac{u}{2}, -\frac{1}{2}\sin\frac{u}{2}, \frac{\sqrt{3}}{2}\right)$$

and

$$N_\alpha = \left(-\sin\frac{u}{2}, -\cos\frac{u}{2}, 0\right). \quad (17)$$

The curvature of α is

$$\kappa_\alpha = \frac{1}{4}. \quad (18)$$

Similarly, the tangent and principal normal vectors of β are

$$T_{\beta} = \left(-\frac{1}{2}\cos\frac{v}{2}, \frac{1}{2}\sin\frac{v}{2}, \frac{\sqrt{3}}{2} \right)$$

and

$$N_{\beta} = \left(\sin\frac{v}{2}, \cos\frac{v}{2}, 0 \right). \tag{19}$$

The curvature of β is

$$\kappa_{\beta} = \frac{1}{4}. \tag{20}$$

The unit vector of surface is

$$U = (u_1, u_2 u_3) \tag{21}$$

where

$$u_1 = -\frac{\sqrt{3}}{4\rho}\left(\sin\frac{u}{2} - \sin\frac{v}{2} \right)$$

$$u_2 = -\frac{\sqrt{3}}{4\rho}\left(\cos\frac{u}{2} - \cos\frac{v}{2} \right)$$

$$u_3 = \frac{1}{4\rho}\sin\left(\frac{v-u}{2} \right)$$

and

$$\rho = \sqrt{\frac{3}{8} + \frac{3}{8}\cos\left(\frac{u-v}{2} \right) + \frac{1}{16}\sin^2\left(\frac{u-v}{2} \right)}.$$

From (17), (19) and (21),

$$\cos\theta_{\alpha} = \frac{\sqrt{3}}{4\rho}\left(1 + \cos\left(\frac{u-v}{2} \right) \right) \tag{22}$$

and

$$\cos\theta_{\beta} = -\frac{\sqrt{3}}{4\rho}\left(1 + \cos\left(\frac{u-v}{2} \right) \right). \tag{23}$$

Finally, by using (4), (18), (20), (22) and (23), so $H = 0$. Another case $\kappa_{\alpha} \neq 0$, $\kappa_{\beta} \neq 0$ $(\kappa_{\alpha} \neq \kappa_{\beta})$ and $\cos\theta_{\alpha} \neq 0$, $\cos\theta_{\beta} \neq 0$ $(\cos\theta_{\alpha} \neq \cos\theta_{\beta})$ is shown that Scherk surface is not only minimal translation surface in 3-Euclidean space.

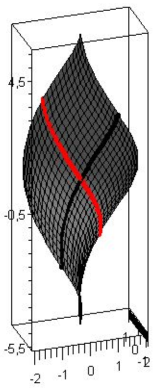

Fig. 2 Another minimal translation surface generated by two helices.

References

[1] D. W. Yoon, “On the Gauss map of translation surfaces in Minkowski 3-space,” *Taiwanese Journal of Mathematics*, vol. 6, no. 3, pp. 389-398, 2002.

[2] H. Liu, “Translation surfaces with constant mean curvature in 3-dimensional spaces,” *Journal of Geometry*, vol. 64, pp. 141-149, 1999.

[3] L. Verstraelen, J. Walrave, and S. Yaprak, “The minimal translation surfaces in Euclidean space,” *Soochow Journal of Mathematics*, vol. 20, no. 1, pp. 77-82, 1994.

[4] M. Munteanu, and A. I. Nistor, “On the geometry of the second fundamental form of translation surfaces in E^3, *arXiv:0812.3166v1 [math.DG]* 16 Dec. 2008.