

Some Remarks About Riemann-Liouville and Caputo Impulsive Fractional Calculus

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Abstract—This paper establishes some closed formulas for Riemann- Liouville impulsive fractional integral calculus and also for Riemann- Liouville and Caputo impulsive fractional derivatives.

Keywords—Rimann- Liouville fractional calculus, Caputo fractional derivative, Dirac delta, Distributional derivatives, High-order distributional derivatives.

I. INTRODUCTION

FRACTIONAL calculus has been used in a set of applications, mainly, to deal with modelling errors in differential equations and dynamic systems. There are also applications in Signal Processing and sampling and hold algorithms, [1-3]. Fractional integrals and derivatives can be of non-integer orders and even of complex order. This facilitates the description of some problems which are not easily descxribed by ordinary calculus due to modelling errors, [1-5]. There are several approaches for the integral fractional calculus, the most popular ones being the Riemann-Liouville fractional integral. There is also a fractional Riemann- Liouville derivative. However, the well-known Caputo fractional derivative are less involved since the associated integral operator manipulates the derivatives of the primitive function under the integral symbol. This paper extends the basic fractional differ-integral calculus to impulsive functions described through the use of Dirac distributions and Dirac distributional derivatives, [5], of real fractional orders. In the general case, it is admitted a presence of infinitely many impulsive terms at certain isolated point of the relevant function domains. Control Theory topics in [6-9] could be reformulated under thefractional formalism considered in this paper.

II. GENERALIZED RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL

Let us denote the set of positive real numbers by $\mathbf{R}_+ = \{r \in \mathbf{R} : r > 0\}$ and left-sided and right-sided Lebesgue integrals, respectively, as:

$$\int_0^x g(\tau) d\tau := \lim_{t \rightarrow x^-} \int_0^t g(\tau) d\tau \quad (\text{the identification}$$

$x \equiv x^-$ is used for all x in order to simplify the notation), and

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$$\int_0^x g(\tau) d\tau := \lim_{t \rightarrow x^+} \int_0^t g(\tau) d\tau$$

Now, consider real functions $f, \bar{f} : \mathbf{R}_+ \rightarrow \mathbf{R}$, such that $\int_0^x (x-t)^{\mu-1} \bar{f}(t) dt$ exists, $\forall x \in \mathbf{R}_+$, fulfilling:

$$f(x) = \bar{f}(x) + \sum_{x_i \in IMP} K_i \delta(x-x_i) = \bar{f}(x) + \sum_{i \in I(\infty)} K_i \delta(x-x_i)$$

$\delta(x)$ denotes the Dirac delta distribution, $K_i \delta(0) = f(x_i^+) - f(x_i)$ with $K_i \in \mathbf{R}$; $\forall i \in I(\infty) \subset \mathbf{Z}_+$, [5], and $IMP := \bigcup_{x \in \mathbf{R}_+} IMP(x) = \bigcup_{x \in \mathbf{R}_+} IMP(x^+)$ of indexing set

$I(\infty)$ is the whole impulsive set defined via empty or non-empty) partial impulsive strictly ordered denumerable sets:

$$IMP(x) := \{x_i \in \mathbf{R}_+ : f(x_i^+) - f(x_i) = K_i \delta(0), x_i < x\} \quad (1)$$

of indexing set $I(x) := \{i \in \mathbf{Z}_{0+} : x_i \in IMP(x)\} \subset I(x^+) \subset \mathbf{Z}_+$, for each $x \in \mathbf{R}_+$; and

$$IMP(x) \subset IMP(x^+) := \{x_i \in \mathbf{R}_+ : f(x_i^+) - f(x_i) = K_i \delta(0), x_i \leq x^+\} \subset \mathbf{R} \quad (2)$$

of indexing set

$$I(x) \subset I(x^+) := \{i \in \mathbf{Z}_{0+} : x_i \in IMP(x^+)\} \subset \mathbf{Z}_+, \quad \text{for each } x \in \mathbf{R}_+ \text{ with the indexing set of } IMP \text{ being}$$

$$I(\infty) = \bigcup_{x \in IMP(x)} I(x) = \bigcup_{x \in IMP(x^+)} I(x^+).$$

If we are interested in studying the fractional derivative of the impulsive function

$f : \mathbf{R}_+ \rightarrow \mathbf{R}$ then $\bar{f} : \mathbf{R}_+ \rightarrow \mathbf{R}$ is non- uniquely defined as

$$\bar{f}(x) = f(x) \quad \text{for } x \in \mathbf{R}_+ \setminus IMP, \quad \text{and } f(x_i) = \bar{f}(x_i),$$

$$f(x_i^+) = f(x_i) + K_i \delta(0) = \bar{f}(x_i) + K_i \delta(0), \quad \text{for } x_i \in IMP$$

with $\bar{f}(x^+) \in \mathbf{R}$ (non-uniquely) defined being bounded arbitrary (for instance, being zero or $\bar{f}(x^+) = f(x)$) if $x \in IMP$. Note that IMP and $I(\infty)$ have infinite cardinals if there are infinitely many impulsive values of the function $f(t)$.

Note that the existence of $\int_0^x (x-t)^{\mu-1} \bar{f}(t) dt$ implies that of $\int_0^x (x-t)^{\mu-1} f(t) dt = \int_0^x (x-t)^{\mu-1} \bar{f}(t) dt$ if $x \notin IMP(x)$, since $\int_0^x (x-t)^{\mu-1} \bar{f}(t) dt$ exists, and that of

$$\int_0^{x_i} (x-t)^{\mu-1} f(t) dt = \int_0^{x_i} (x-t)^{\mu-1} \bar{f}(t) dt + (x-x_i)^{\mu-1} (f(x_i^+) - f(x_i))$$

$$\text{if } x_i \in IMP(x^+) \quad (3)$$

Theorem 2.1. The extended fractional Riemann- Liouville integrals by considering impulsive functions are defined for any fixed order $\mu \in \mathbf{R}_+$ and all $x \in \mathbf{R}_+$ by

$$\begin{aligned} (J^\mu f)(x) &:= \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t) dt \\ &= \frac{1}{\Gamma(\mu)} \left(\int_0^x (x-t)^{\mu-1} \bar{f}(t) dt + \sum_{i \in I(x)} (x-x_i)^{\mu-1} (f(x_i^+) - f(x_i)) \right) \\ &= \frac{1}{\Gamma(\mu)} \sum_{i \in I(x) \cup \{0\}} \int_{x_i^+}^{x_{i+1}^+} (x-t)^{\mu-1} f(t) dt \\ &+ \frac{1}{\Gamma(\mu)} \int_{x_{n(x)}^+}^x (x-t)^{\mu-1} f(t) dt + \sum_{i \in I(x)} (x-x_i)^{\mu-1} (f(x_i^+) - f(x_i)) \end{aligned} \tag{4}$$

$$\begin{aligned} (J^\mu f)(x^+) &:= \frac{1}{\Gamma(\mu)} \int_0^{x^+} (x-t)^{\mu-1} f(t) dt \\ &= \frac{1}{\Gamma(\mu)} \left(\int_0^x (x-t)^{\mu-1} \bar{f}(t) dt + \sum_{i \in I(x^+)} (x-x_i)^{\mu-1} (f(x_i^+) - f(x_i)) \right) \\ &= \frac{1}{\Gamma(\mu)} \sum_{i \in I(x^+) \cup \{0\}} \int_{x_i^+}^{x_{i+1}^+} (x-t)^{\mu-1} f(t) dt \\ &+ \frac{1}{\Gamma(\mu)} \sum_{i \in I(x^+)} (x-x_i)^{\mu-1} (f(x_i^+) - f(x_i)) \end{aligned} \tag{5}$$

$$(J^0 f)(x^+) \equiv (J^0 f)(x) := f(x)$$

where $\Gamma: \mathbf{R}_{0+} \rightarrow \mathbf{R}_+$ is the Γ -function, [1-5] and $n: IMP \rightarrow \mathbf{Z}_+$ is defined by $n(x) = \text{card } I(x) = \text{card } IMP(x)$. \square

Note that if $x \in IMP$ then

$$\begin{aligned} (J^\mu f)(x^+) &= \frac{1}{\Gamma(\mu)} \sum_{i \in I(x^+) \cup \{0\}} \int_{x_i^+}^{x_{i+1}^+} (x-t)^{\mu-1} f(t) dt \\ &+ \frac{1}{\Gamma(\mu)} \sum_{i \in I(x^+)} (x-x_i)^{\mu-1} (f(x_i^+) - f(x_i)) \\ &= (J^\mu f)(x) + (x-x_{n(x)})^{\mu-1} (f(x_{n(x)}^+) - f(x_{n(x)})) \end{aligned}$$

$$\begin{aligned} \neq (J^\mu f)(x) &= \frac{1}{\Gamma(\mu)} \sum_{i \in I(x) \cup \{0\}} \int_{x_i^+}^{x_{i+1}^+} (x-t)^{\mu-1} f(t) dt \\ &+ \frac{1}{\Gamma(\mu)} \sum_{i \in I(x)} (x-x_i)^{\mu-1} (f(x_i^+) - f(x_i)) \end{aligned} \tag{6}$$

and if $x \notin IMP$, since $I(x^+) = I(x)$, then

$$(J^\mu f)(x^+) = (J^\mu f)(x).$$

Assume that $f \in C^{m-1}(\mathbf{R}_+, \mathbf{R})$ and its m -th derivative exists everywhere in \mathbf{R}_+ . Then, the Caputo fractional derivative of order $\mu \geq 0$ with $m-1 \leq \mu (\in \mathbf{R}_+) \leq m$, $m \in \mathbf{Z}_+$ is for any $x \in \mathbf{R}_+$:

$$\begin{aligned} (D^\mu f)(x) &:= \left(\frac{d}{dx} \right)^m (J^{m-\mu} f)(x) \\ &= \frac{1}{\Gamma(m-\mu)} \left(\frac{d}{dx} \right)^m \left(\int_0^x (x-t)^{m-\mu-1} f(t) dt \right) \end{aligned} \tag{7}$$

The following particular cases follow from this formula for $\mu = m-1$:

- (a) $\mu = -1; m = 0$ yields $(D^{-1} f)(x) = \int_0^x f(t) dt$ which is the standard integral of the function f . This case does not verify the "derivative constraint" $0 \leq m-1 \leq \mu (\in \mathbf{R}_+) < m$ leading to an integral result.
- (b) $\mu = 0; m = 1$ yields $(D^0 f)(x) = f(x)$ which so that $D^0 f$ is the identity operator
- (c) $\mu = 1; m = 2$ yields $(D^1 f)(x) = f^{(1)}(x)$
- (d) $\mu = 2; m = 3$ yields $(D^2 f)(x) = f^{(2)}(x)$ which is the standard first- derivative of the function f .

Compared to the parallel cases with the Caputo fractional derivative, note that the Riemann- Liouville fractional derivative, compared to the Caputo corresponding one, does not depend on the conditions at zero of the function and its derivatives. Define the Kronecker delta $\delta(a,b)$ of any pair of real numbers (a,b) as $\delta(a,b) = 1$ if $a = b$ and $\delta(a,b) = 0$ if $a \neq b$ and then evaluate recursively the Riemann - Liouville fractional derivative of order $\mu \geq 0$ from the above formula by using Leibniz's differentiation rule by noting that, since $\mu \neq m-j; \forall j (\in \mathbf{Z}_+) > 1$, only the differential part corresponding to the differentiation of the integrand is non zero for $j > m-\mu$. This yields the following result:

Theorem 3.1. Assume that $f \in C^{m-2}(\mathbf{R}_+, \mathbf{R})$ and $f^{(m-1)}$ exists everywhere in \mathbf{R}_+ and that $f(t)$ is integrable on \mathbf{R}_+ , then:

$$\begin{aligned} (D^\mu f)(x) &= \frac{1}{\Gamma(m-\mu)} \left(\frac{d}{dx} \right)^m \left(\int_0^x (x-t)^{m-\mu-1} f(t) dt \right) \\ &= \frac{1}{\Gamma(m-\mu)} \left(\frac{d}{dx} \right)^{m-1} \left[\int_0^x (m-\mu-1)(x-t)^{m-\mu-2} f(t) dt + f(x) \delta(\mu, m-1) \right] \\ &= \frac{1}{\Gamma(m-\mu)} f^{(m-1)}(x) \delta(\mu, m-1) \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\Gamma(m-\mu)} \left(\frac{d}{dx} \right)^{m-1} \left(\int_0^x (m-\mu-1)(x-t)^{m-\mu-2} f(t) dt \right) \\
 & = \frac{1}{\Gamma(m-\mu)} f^{(m-1)}(x) \delta(\mu, m-1) \\
 & + \frac{1}{\Gamma(m-\mu)} \left(\sum_{i=1}^{m-2} \prod_{j=i+1}^{m-1} [j-\mu] \right) f^{(i)}(x) \delta(\mu, m-i) \\
 & + \frac{1}{\Gamma(m-\mu)} \left[\prod_{j=0}^{m-1} [j-\mu] \right] \left(\int_0^x (x-t)^{-(\mu+1)} f(t) dt \right) \quad (8)
 \end{aligned}$$

If $f \in PC^k(\mathbf{R}_+, \mathbf{R})$ with $f^{(k)}(x)$ being discontinuous of first class then $f^{(m-1)}(x) = \delta^{(j(x))}(x)$ with $j(x) = m-1-k(x)$, one uses to obtain the right value of (8) the perhaps high-order distributional derivatives formula:

$$\begin{aligned}
 & |f^{(m-1)}(x^+) - f^{(m-1)}(x)| = \\
 & \frac{(-1)^k k!}{x^k} |f^{(m-1-k)}(x^+) - f^{(m-1-k)}(x)| \delta(0) = \infty \quad (9)
 \end{aligned}$$

to yield

$$\begin{aligned}
 & (D^\mu f)(x^+) = \frac{1}{\Gamma(m-\mu)} \\
 & \left[\frac{(-1)^{k(x)} k(x)!}{x^{k(x)}} |f^{(m-1-k(x))}(x^+) - f^{(m-1-k(x))}(x)| \delta(0) \delta(\mu, m-1) \right. \\
 & + \left. \left(\sum_{i=1}^{m-2} \prod_{j=i+1}^{m-1} [j-\mu] \right) f^{(i)}(x) \delta(\mu, m-i) \right. \\
 & \left. + \left[\prod_{j=0}^{m-1} [j-\mu] \right] \left(\int_0^x (x-t)^{-(\mu+1)} f(t) dt \right) \right] \quad (10)
 \end{aligned}$$

If $\mu = m-1$ then

$$\begin{aligned}
 & (D^{m-1} f)(x) \\
 & = f^{(m-1)}(x) + \left[\prod_{j=0}^{m-1} [j-\mu] \right] \left(\int_0^x (x-t)^{-m} f(t) dt \right) \quad (11)
 \end{aligned}$$

provided that $\left(\int_0^x (x-t)^{-(\mu+1)} f(t) dt \right)$ exists for $x \in \mathbf{R}_+$ (which is guaranteed if $f(t)$ is Lebesgue-integrable on \mathbf{R}_+), $f \in C^{m-2}(\mathbf{R}_+, \mathbf{R})$ and $f^{(m-1)}$ exists everywhere in \mathbf{R}_+ . The correction (10) applies when the derivative does not exist. \square

If $\mu \neq m-1$ with $m-1 \leq \mu \in \mathbf{R}_+ \leq m$ then after defining the impulsive sets, its associated indexing sets and the function $\bar{f}: \mathbf{R}_+ \rightarrow \mathbf{R}$ as for the extended Riemann-Liouville fractional integral, one gets:

$$\begin{aligned}
 & = \frac{1}{\Gamma(m-\mu)} \left[\prod_{j=0}^{m-1} [j-\mu] \right] \left(\int_0^x (x-t)^{-(\mu+1)} f(t) dt \right) \\
 & = \\
 & = \frac{1}{\Gamma(m-\mu)} \left[\prod_{j=0}^{m-1} [j-\mu] \right] \sum_{i \in I(x) \cup \{0\}} \int_{x_i^+}^{x^{i+1}} (x-t)^{-(\mu+1)} f(t) dt \\
 & + \frac{1}{\Gamma(m-\mu)} \left[\prod_{j=0}^{m-1} [j-\mu] \right] \int_{x_n^+(x)}^x (x-t)^{-(\mu+1)} f(t) dt \\
 & + \frac{1}{\Gamma(m-\mu)} \left[\prod_{j=0}^{m-1} [j-\mu] \right] \sum_{i \in I(x)} (x-x_i)^{-(\mu+1)} (f(x_i^+) - f(x_i)) \\
 & (D^\mu f)(x^+) = \frac{1}{\Gamma(m-\mu)} \left[\prod_{j=0}^{m-1} [j-\mu] \right] \\
 & \times \left(\int_0^x (x-t)^{-(\mu+1)} \bar{f}(t) dt + \sum_{i \in I(x)} (x-x_i)^{-(\mu+1)} (f(x_i^+) - f(x_i)) \right) \quad (12)
 \end{aligned}$$

IV. GENERALIZED CAPUTO FRACTIONAL DERIVATIVE

Assume that $f \in C^{m-1}(\mathbf{R}_+, \mathbf{R})$ and its m -th derivative exists everywhere in \mathbf{R}_+ . Then, the Caputo fractional derivative of order $\mu \geq 0$ with $m-1 \leq \mu \in \mathbf{R}_+ < m$, $m \in \mathbf{Z}_+$ is for any $x \in \mathbf{R}_+$:

$$\begin{aligned}
 & (D_*^\mu f)(x) := (J^{m-\mu} f^{(m)})(x) \\
 & = \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^{(m)}(t) dt \quad (14)
 \end{aligned}$$

; $m-1 \leq \mu < m$, $m \in \mathbf{Z}_+$, $x \in \mathbf{R}_+$

The following particular cases occur with $\mu = m-1$ leading to

$$(D_*^{m-1} f)(x) = \int_0^x f^{(m)}(t) dt = f^{(m-1)}(x) - f^{(m-1)}(0^+) \quad (15)$$

(a) $\mu = -1$; $m = 0$ yields $(D_*^{-1} f)(x) = f^{(-1)}(x) - f^{(-1)}(0^+)$ which is an integral result f . Note that this case does not verifies the "derivative constraint" $0 \leq \mu \in \mathbf{R}_+ < m$ leading to an integral result.

(b) $\mu = 0$; $m = 1$ yields

$$(D_*^0 f)(x) = f^{(0)}(x) - f^{(0)}(0^+) = f(x) - f(0^+)$$

(c) $\mu = 1$; $m = 2$ yields $(D_*^1 f)(x) = f^{(1)}(x) - f^{(1)}(0^+)$

(d) $\mu = 2$; $m = 3$ yields $(D_*^2 f)(x) = f^{(2)}(x) - f^{(2)}(0^+)$

We can extend the above formula to real functions with impulsive m -th derivative as follows. Assume that $f \in C^{m-2}(\mathbf{R}_+, \mathbf{R})$ with bounded piecewise $(m-1)$ -th

derivative existing everywhere in \mathbf{R}_+ and

$$f^{(m)}(x) \equiv \frac{d^m f(x)}{dx^m} \text{ being impulsive with}$$

$$f^{(m)}(x_i) = K_i \delta(0) = (f^{(m-1)}(x_i^+) - f^{(m-1)}(x_i)) \delta(0)$$

; $\forall x_i \in IMP$, equivalently, $\forall i \in I(\infty)$, at the eventual discontinuity points $x_i > 0$ at the impulsive set $IMP := \bigcup_{x \in \mathbf{R}_+} IMP(x)$, where the partial impulsive sets are re-defined as follows:

$$IMP(x) := \{x_i \in \mathbf{R}_+ : f^{(m-1)}(x_i^+) - f^{(m-1)}(x_i) = K_i, x_i < x\} \subset IMP(x^+) \tag{16}$$

$$IMP(x^+) := \{x_i \in \mathbf{R}_+ : f^{(m-1)}(x_i^+) - f^{(m-1)}(x_i) = K_i, x_i \leq x^+\} \subset IMP(x^+) \tag{17}$$

Now, consider $f \in C^{m-1}(0, \infty)$ with $f^{(m)}(x) \equiv \frac{d^m f(x)}{dx^m}$ being almost everywhere piecewise

continuous in \mathbf{R}_+ except possibly on a non-empty discrete impulsive set IMP . Define a non-impulsive real function $\bar{f} : \mathbf{R}_+ \rightarrow \mathbf{R}$ defined as $\bar{f}^{(m)}(x) = f^{(m)}(x)$ for $x \in \mathbf{R}_+ \setminus IMP$, and $f^{(m)}(x_i) = \bar{f}^{(m)}(x_i)$, $f^{(m)}(x_i) = \bar{f}^{(m)}(x_i) + K_i \delta(0)$ for $x_i \in IMP$ with $\bar{f}^{(m)}(x^+) = f^{(m)}(x)$; $x \in IMP$ (defined being bounded arbitrary (for instance, zero) if $x \in IMP$). Through a similar reasoning as that used for Riemann- Liouville fractional integral by replacing the function $f : \mathbf{R}_+ \rightarrow \mathbf{R}$ by its m -th derivative, one obtains the following result:

Theorem 4.1. The Caputo fractional derivative of order $\mu \in \mathbf{R}_+$ satisfying $m-1 < \mu \leq m$; $m \in \mathbf{Z}_+$ and all $x \in \mathbf{R}_+$ is given below:

$$\begin{aligned} (D_*^\mu f)(x) &:= \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^{(m)}(t) dt \\ &= \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} \bar{f}^{(m)}(t) dt \\ &+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (x-x_i)^{m-\mu-1} (f^{(m-1)}(x_i^+) - f^{(m-1)}(x_i)) \delta(x-x_i) \\ &= \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x) \cup \{0\}} \int_{x_i^+}^{x_{i+1}} (x-t)^{m-\mu-1} \bar{f}^{(m)}(t) dt \\ &+ \frac{1}{\Gamma(m-\mu)} \int_{x_{n(x)}^+}^x (x-t)^{m-\mu-1} \bar{f}^{(m)}(t) dt \\ &+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (x-x_i)^{m-\mu-1} (f^{(m-1)}(x_i^+) - f^{(m-1)}(x_i)) \delta(x-x_i) \end{aligned} \tag{18}$$

$$(D_*^\mu f)(x^+) := \frac{1}{\Gamma(m-\mu)} \int_0^{x^+} (x-t)^{m-\mu-1} f^{(m)}(t) dt$$

$$\begin{aligned} &= \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} \bar{f}^{(m)}(t) dt \\ &+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x^+)} (x-x_i)^{m-\mu-1} (f^{(m-1)}(x_i^+) - f^{(m-1)}(x_i)) \delta(x-x_i) \end{aligned} \tag{19}$$

where $n : IMP \rightarrow \mathbf{Z}_+$ is a discrete function defined by $n(x) = card I(x) = card IMP(x)$. \square

Note that if $x \in IMP$ then

$$\begin{aligned} (D_*^\mu f)(x^+) &= \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x^+) \cup \{0\}} \int_{x_i^+}^{x_{i+1}} (x-t)^{m-\mu-1} \bar{f}^{(m)}(t) dt \\ &+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x^+)} (x-x_i)^{m-\mu-1} (f^{(m-1)}(x_i^+) - f^{(m-1)}(x_i)) \delta(x-x_i) \\ &= (D_*^\mu f)(x) + (x-x_{n(x)})^{m-\mu-1} (f^{(m-1)}(x_{n(x)}^+) - f^{(m-1)}(x_{n(x)})) \delta(0) \\ &\neq (D_*^\mu f)(x) = \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x) \cup \{0\}} \int_{x_i^+}^{x_{i+1}} (x-t)^{m-\mu-1} \bar{f}^{(m)}(t) dt \\ &+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (x-x_i)^{m-\mu-1} (f^{(m-1)}(x_i^+) - f^{(m-1)}(x_i)) \delta(x-x_i) \end{aligned}$$

and if $x \notin IMP$, since $I(x^+) = I(x)$, then $(D_*^\mu f)(x^+) = (D_*^\mu f)(x)$. The above formalism applies when $f^{(m-1)} : \mathbf{R}_+ \rightarrow \mathbf{R}$ is piecewise continuous with isolated first- class discontinuity points, that is $f \in PC^{m-1}(\mathbf{R}_+, \mathbf{R})$ implying that $f \in C^{m-2}(\mathbf{R}_+, \mathbf{R})$. A more general situation arises when the discontinuities can point-wise arise for points of the function itself of for any successive derivative up- till order m . This would lead to a more general description than that given as follows. Define partial sets of positive integers as $\bar{k} := \{1, 2, \dots, k\}$

Assume that $f \in PC^j(\mathbf{R}_+, \mathbf{R})$ and x is a discontinuity point of first class of $f^{(j)}(x)$ for some $j \in \overline{m-1} \cup \{0\}$. Then, $f^{(j+\ell)}(x)$ are impulsive for $\ell \in \overline{m-j}$ of high order being increasing with ℓ . Define the $(j+1)$ - th impulsive sets of the function f on $(0, x) \subset \mathbf{R}_+$ as:

$$IMP_{j+1}(x) := \left\{ z \in \mathbf{R}_+ : z < x, 0 < \left| f^{(j)}(z^+) - f^{(j)}(z) \right| < \infty \right\}; \tag{20}$$

$$j \in \overline{m-1} \cup \{0\}, x \in \mathbf{R}_+$$

This leads directly the definition of the following impulsive sets:

$$\begin{aligned} IMP_{j+1} &:= \left\{ x \in \mathbf{R}_+ : 0 < \left| f^{(j)}(x^+) - f^{(j)}(x) \right| < \infty \right\} \\ &\equiv \bigcup_{x \in \mathbf{R}_+} IMP_{j+1}(x) \end{aligned} \tag{21}$$

$$\begin{aligned} IMP &:= \left\{ x \in \mathbf{R}_+ : 0 < \left| f^{(j)}(x^+) - f^{(j)}(x) \right| < \infty, \text{ some } j \in \overline{m-1} \cup \{0\} \right\} \\ &\equiv \bigcup_{x \in \mathbf{R}_+} \left(\bigcup_{j \in \overline{m-1} \cup \{0\}} IMP_{j+1}(x) \right) \end{aligned} \tag{22}$$

which can be empty. Thus, if $z \in IMP_{j+1}$ then $f^{(j-1)}(x^+) = f^{(j-1)}(x)$ exists with identical left and right

limits, $f^{(j)}(x^+) - f^{(j)}(x) = K = K(x) \neq 0$ and $f^{(j)}(x) = K\delta(0)$ with successive higher-order derivatives represented by higher-order Dirac distributional derivatives

The above definitions yield directly the following simple results:

Assertion 5.2. $x \in IMP \Rightarrow x \in IMP_j$ for a unique $j = j(x) \in \overline{m}$.

Proof: Proceed by contradiction. Assume that $x \in (IMP_{i+1} \cap IMP_{j+1})$ for $i, j (j \neq i) \in \overline{m-1} \cup \{0\}$. Then:

$$0 < \left| f^{(i)}(x^+) - f^{(i)}(x) \right| < \infty ;$$

$$0 < \left| f^{(j)}(x^+) - f^{(j)}(x) \right| < \infty$$

Assume with no loss of generality that $j = i + k > i$ for some $k (\leq m - i - 1) \in \mathbf{Z}_+$. Then,

$$\begin{aligned} \left| f^{(j)}(x^+) - f^{(j)}(x) \right| &= \left| f^{(i+k)}(x^+) - f^{(i+k)}(x) \right| \\ &= \frac{(-1)^k k!}{x^k} \left| f^{(i)}(x^+) - f^{(i)}(x) \right| \delta(0) = \infty \end{aligned}$$

with $x \in \mathbf{R}_+$. If $\left| f^{(i)}(x^+) - f^{(i)}(x) \right| \neq 0$ which contradicts

$$0 < \left| f^{(i)}(x^+) - f^{(i)}(x) \right| < \infty \text{ so that } i = j. \quad \square$$

Assertion 5.3. $x \in IMP \Rightarrow$

$$\left(x \in IMP_j \Leftrightarrow \exists \text{ a unique } j = j(x) = \max_{i \in \overline{m}} \left| f^{(i-1)}(x^+) - f^{(i-1)}(x) \right| < \infty \right)$$

Furthermore, such a unique $j = j(x)$ satisfies $\left| f^{(j-1)}(x^+) - f^{(j-1)}(x) \right| > 0$.

Proof: The existence is direct by contradiction. If $\neg \exists j = j(x) \in \overline{m-1} \cup \{0\}$ such that

$$\left| f^{(j)}(x^+) - f^{(j)}(x) \right| < \infty \text{ then } x \notin IMP. \text{ Now, assume}$$

there exist two nonnegative integers $i = i(x) = \left| f^{(i-1)}(x^+) - f^{(i-1)}(x) \right| < \infty$ and

$$j = j(x) = i + k = \left| f^{(i+k-1)}(x^+) - f^{(i+k-1)}(x) \right| < \infty ; \text{ for some}$$

$k \in \overline{m-i}$. But for $x > 0$,

$$\infty = \frac{(-1)^k k!}{x^k} \left| f^{(i-1)}(x^+) - f^{(i-1)}(x) \right| \delta(0)$$

$$= \left| f^{(i+k-1)}(x^+) - f^{(i+k-1)}(x) \right| < \infty$$

which is a contradiction. Then,

$$x \in IMP_j \Rightarrow \exists j = j(x) = \max_{i \in \overline{m}} \left| f^{(i-1)}(x^+) - f^{(i-1)}(x) \right| < \infty$$

which is unique. Also, from the definition of the impulsive sets $IMP_i(x)$,

$$\text{Vol. 5, No. 9, 2011} \quad \left| f^{(j-1)}(x^+) - f^{(j-1)}(x) \right| < \infty \Rightarrow x \in \bigcup_{i \in \overline{j} \cup \{0\}} IMP_i(x)$$

Now, assume that $x \in \bigcup_{i \in \overline{j-1} \cup \{0\}} IMP_i(x)$. Thus,

$$0 < \left| f^{(j-1)}(x^+) - f^{(j-1)}(x) \right| < \infty \Rightarrow \left| f^{(j)}(x^+) - f^{(j)}(x) \right| = \infty$$

from the definition of the impulsive sets. Then, $x \in IMP_j(x)$. The opposite logic implication

$$j = j(x) = \max_{i \in \overline{m}} \left| f^{(i-1)}(x^+) - f^{(i-1)}(x) \right| < \infty \Rightarrow x \in IMP_j$$

is proved. Then, it has been fully proved that $x \in IMP \Rightarrow$

$$\left(x \in IMP_j \Leftrightarrow \exists \text{ a unique } j = j(x) = \max_{i \in \overline{m}} \left| f^{(i-1)}(x^+) - f^{(i-1)}(x) \right| < \infty \right)$$

Now, establish again a contradiction by assuming that

$$j = j(x) = \left| f^{(k-1)}(x^+) - f^{(k-1)}(x) \right| = \max$$

$$\left| f^{(i-1)}(x^+) - f^{(i-1)}(x) \right| = 0 < \infty ; \forall k \in \overline{m}$$

what contradicts $x \in IMP$. This proves that the unique $j = j(x)$ implying and being implied by $x \in IMP_j$ satisfies

$$\left| f^{(j-1)}(x^+) - f^{(j-1)}(x) \right| > 0. \quad \square$$

Using the necessary - high order distributional derivatives, one gets that

$$x \in IMP \Rightarrow f^{(m)}(x) = \frac{(-1)^{m-j} (m-j)!}{x^{m-j}} \left(f^{(j)}(x^+) - f^{(j)}(x) \right) \delta(0)$$

; with $j \in \overline{m-1} \cup \{0\}$ being uniquely defined so that

$$0 < \left| f^{(j)}(x^+) - f^{(j)}(x) \right| < \infty. \text{ Thus, the } m\text{-th distributional}$$

derivative of $f: \mathbf{R}_+ \rightarrow \mathbf{R}$ can be represented as:

$$\begin{aligned} f^{(m)}(x) &= \bar{f}^{(m)}(x) \\ &+ \sum_{x_i \in IMP_{j+1}} \frac{(-1)^{j_i} (m-j_i)!}{x_i^{m-j_i}} \left(f^{(j_i)}(x_i^+) - f^{(j_i)}(x_i) \right) \delta(x - x_i) \end{aligned}$$

, $x \in \mathbf{R}_+$

with $j_i = j_i(x_i)$ being uniquely defined for each $x_i \in IMP$ so

that $x_i \in IMP_{j_i}$, where $\bar{f} \in C^{m-1}(\mathbf{R}_+, \mathbf{R})$ with everywhere

continuous first-derivative defined as $\bar{f}^{(j)}(x) = f^{(j)}(x)$;

$x \in \mathbf{R}_+$, $\bar{f}(0) = f(0)$. The above formula is applicable if

$f \notin PC^m(\mathbf{R}_+, \mathbf{R})$ but it is also applicable if

$f \in PC^m(\mathbf{R}_+, \mathbf{R})$ yielding:

$$\begin{aligned} f^{(m)}(x^+) &= f^{(m)}(x) = \bar{f}^{(m)}(x) \text{ if } x \notin IMP \\ f^{(m)}(x) &= \bar{f}^{(m)}(x) \end{aligned}$$

$$f^{(m)}(x^+) = f^{(m)}(x) + \frac{(-1)^{m-j} (m-j)!}{x^{m-j}} \left(f^{(j)}(x^+) - f^{(j)}(x) \right) \delta(0)$$

if $x \in IMP$

$$f^{(m-1)}(x) = \bar{f}^{(m-1)}(x)$$

$$f^{(m-1)}(x^+) = f^{(m-1)}(x) + \frac{(-1)^{m-j} (m-1-j)!}{x^{m-1-j}} \left(f^{(j)}(x^+) - f^{(j)}(x) \right) \delta(0)$$

if $x \in IMP$ and $j < m-1$

$$f^{(m-1)}(x) = \bar{f}^{(m-1)}(x)$$

$$f^{(m-1)}(x^+) = f^{(m-1)}(x) + (f^{(m-1)}(x^+) - f^{(m-1)}(x)) \text{ if}$$

$$x \in IMP \text{ and } j = m - 1$$

$$= \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x^+) \cup \{0\}} \int_{x_i^+}^{x^{i+1}} (x-t)^{m-\mu-1} \bar{f}^{(m)}(t) dt$$

$$+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x^+)} (-1)^{m-j(x_i)-1} (x-x_i)^{m-\mu-1}$$

$$\times \frac{(m-j(x_i)-1)!}{(x-x_i)^{m-j(x_i)-1}} (f^{(j(x_i))}(x_i^+) - f^{(j(x_i))}(x_i)) \quad (24)$$

for a unique $j = j(x) \in \overline{m-1} \cup \{0\}$ from Assertion 1. Denote further sets related to impulses as follows:

$$IMP(x) := \{z \in IMP : z < x\}; \quad IMP(x^+) := \{z \in IMP : z \leq x\}$$

$$; \quad \forall x \in \mathbf{R}_+$$

being indexed by two subsets of integers of the same corresponding cardinals defined by:

$$I(x) = \bar{j} = \overline{j(x)}$$

indexing the members z_i of $IMP(x)$ in increasing order

$$I(x^+)$$

being either $I(x)$ or $I(x)+1$, indexing the members z_i of $IMP(x^+)$ in increasing order

The following result holds:

Theorem 5.4. The Caputo fractional derivative of $f : \mathbf{R}_+ \rightarrow \mathbf{R}$ of order $\mu \in \mathbf{R}_+$ satisfying $m-1 < \mu \leq m; m \in \mathbf{Z}_+$ and all $x \in \mathbf{R}_+$ is after using distributional derivatives becomes in the most general case:

$$(D_*^\mu f)(x) := \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} f^{(m)}(t) dt$$

$$= \frac{1}{\Gamma(m-\mu)} \left(\int_0^x (x-t)^{m-\mu-1} \bar{f}^{(m)}(t) dt \right.$$

$$+ \sum_{i \in I(x)} (-1)^{m-j(x_i)-1} (x-x_i)^{m-\mu-1}$$

$$\left. \frac{(m-j(x_i)-1)!}{(x-x_i)^{m-j(x_i)-1}} (f^{(j(x_i))}(x_i^+) - f^{(j(x_i))}(x_i)) \delta(x-x_i) \right)$$

$$= \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x) \cup \{0\}} \int_{x_i^+}^{x^{i+1}} (x-t)^{m-\mu-1} \bar{f}^{(m)}(t) dt$$

$$+ \frac{1}{\Gamma(m-\mu)} \int_{x_n^+(x)}^x (x-t)^{m-\mu-1} \bar{f}^{(m)}(t) dt$$

$$+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x)} (-1)^{m-j(x_i)-1} (x-x_i)^{m-\mu-1}$$

$$\times \frac{(m-j(x_i)-1)!}{(x-x_i)^{m-j(x_i)-1}} (f^{(j(x_i))}(x_i^+) - f^{(j(x_i))}(x_i)) \quad (23)$$

$$(D_*^\mu f)(x^+) := \frac{1}{\Gamma(m-\mu)} \int_0^{x^+} (x-t)^{m-\mu-1} f^{(m)}(t) dt$$

$$= \frac{1}{\Gamma(m-\mu)} \int_0^x (x-t)^{m-\mu-1} \bar{f}^{(m)}(t) dt$$

$$+ \frac{1}{\Gamma(m-\mu)} \sum_{i \in I(x^+)} (-1)^{m-j(x_i)-1} (x-x_i)^{m-\mu-1}$$

$$\times \frac{(m-j(x_i)-1)!}{(x-x_i)^{m-j(x_i)-1}} (f^{(j(x_i))}(x_i^+) - f^{(j(x_i))}(x_i))$$

Note that $\left| (D_*^\mu f)(x^+) \right| = \infty$ if $x = x_i \in IMP$, as expected. □

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REFERENCES

- [1] O. Y. Luchko and R. Gorenflo, "An operational method for solving fractional differential equations with the Caputo derivatives", Acta Mathematica Vietnamica, Vol. 24, No. 2, pp. 207-233, 1999.
- [2] Yu. F. Luchko and H.M. Srivastava, "Differential equations of fractional order by using operational calculus", Computers and Mathematics with Applications, Vol. 29, No. 8, pp. 73-85, 1995.
- [3] K. Oldham and J. Spanier, The fractional Calculus. Theory and Applications of Differentiation and Integration to Arbitrary Order, Dover Publications INC, Mineola, New York, 1974.
- [4] S. Das, Functional Fractional calculus for System Identification and Controls, Springer-Verlag, Berlin, 2008.
- [5] L.A. Zadeh and C. A. Desoer, Linear System Theory, McGraw- Hill, New York, 1963.
- [6] M. De la Sen, "A method for general design of positive real functions", IEEE Transaction on Circuits and Systems I-Fundamental Theory and Applications, Vol. 45, no. 7, pp. 764-769, 1998, doi:10.1109/81.703845.
- [7] M. Delasen, "A method for general design of positive real functions", International Journal of Control, Vol. 41, no. 5, pp. 1189-1205, 1985, doi: 10.1080/0020718508961191.
- [8] M. Delasen, "A method for general design of positive real functions", International Journal of Systems Science, Vol. 14, no. 4, pp. 367-383, 1983, doi: 10.1080/00207728308926464.
- [9] N. Luo, J. Rodellar, J. Vehi and M. De la Sen, "Composite semiactive control of seismically excited structures", Journal of the Franklin Institute- Engineering and Applied Mathematics, Vol. 338, No. 2-3, pp. 764-769, 2001, doi: 10.1016/S0016-0032(00)00081-8.