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Solution of Density Dependent Nonlinear Reaction-Diffusion Equation Using Differential Quadrature Method

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Abstract—In this study, the density dependent nonlinear reaction-diffusion equation, which arises in the insect dispersal models, is solved using the combined application of differential quadrature method(DQM) and implicit Euler method. The polynomial based DQM is used to discretize the spatial derivatives of the problem. The resulting time-dependent nonlinear system of ordinary differential equations(ODE's) is solved by using implicit Euler method. The computations are carried out for a Cauchy problem defined by a one-dimensional density dependent nonlinear reaction-diffusion equation which has an exact solution. The DQM solution is found to be in a very good agreement with the exact solution in terms of maximum absolute error. The DQM solution exhibits superior accuracy at large time levels tending to steady-state. Furthermore, using an implicit method in the solution procedure leads to stable solutions and larger time steps could be used.

Keywords—Density Dependent Nonlinear Reaction-Diffusion Equation, Differential Quadrature Method, Implicit Euler Method.

I. Introduction

SOLVING the nonlinear reaction-diffusion equation

$$\dot{u} = \frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right) + f(u) \quad (\dot{u} = \frac{\partial u}{\partial t}) \tag{1}$$

is a demanding task among researchers since the equation arises in more and more modelling situations in many areas, such as biology, chemistry, medicine and ecology. For instance, if D is space-dependent then the model has biomedical importance or if D is constant and $f(u) = ru(1-u/K)$ (r is the linear reproduction rate and K is the carrying capacity of the environment) the resulting equation (Fisher-Kolmogoroff equation) models the spread of an adventageous gene in a population [1]. An extension to the above mentioned cases is the insect dispersal model which includes an increase in diffusion due to the population pressure. Such a model has growth terms and population depended diffusion coefficient $D(u)$ [1], and the resulting equation is called the density dependent nonlinear reaction-diffusion equation. The density-dependent nonlinear reaction-diffusion equation is rather complicated because of the stronger nonlinearity and most often only the numerical solutions are available.

In [2], Petrov-Galerkin method is used for the solution of one-dimensional nonlinear reaction-diffusion equation and the convergence of the method is discussed. Later, the combined application of DQM with an explicit finite difference method(FDM) is presented for the solution of nonlinear reaction-diffusion equation in one- and two-dimensions [3]. There, since an explicit time integration method is used, a relaxation parameter is proposed, in order to overcome the stability problems. However, it is observed that it is harder to find the parameter when the problem gets harder. Then in [4] both nonlinear reaction-diffusion equation and wave equation are solved using DQM with three different time integration schemes (FDM with a relaxation parameter, least squares method(LSM), finite element method (FEM)) and all three methods are compared in terms of accuracy and computational cost. In both [3] and [4] the nonlinearity is evaluated at the previous known level, in order to obtain a linear system of equations.

On the other hand, Painlevé analysis is applied to get several explicit solutions for the density dependent nonlinear reaction-diffusion equation for the case $D(u) = u$ by Satsuma [5]. Later in [6] necessary and sufficient conditions for the existence of travelling wave solutions for the nonlinear degenerate reaction-diffusion equation, which is a special form of density dependent nonlinear reaction-diffusion equation, is investigated. Moreover, solution of the Cauchy problem defined by the nonlinear degenerate reaction-diffusion equation found to be approaching 1 as $t \to \infty$ for any bounded initial condition.

In this study, the combined application of DQM and Implicit Euler method is used to solve the Cauchy problem defined by the density dependent nonlinear reaction-diffusion equation. The differential quadrature method, which was first proposed by Bellman and his associates [7], [8] in the early 1970's, approximates the solution of a partial differential equation using high order polynomial approximation or using Fourier series expansion. The spatial derivatives in the density dependent nonlinear reaction-diffusion equation are discretized using polynomial based DQM. One of the advantage of DQM is that it is also applicable in the absence of boundary conditions which is not the case for other domain discretization methods. The other advantage of the method is that the method leads to accurate numerical solutions using considerably small number of grid points [9]. For the time discretization of the system of ordinary differential equations obtained after the DQM discretization, implicit Euler method is applied. Then Newton method is made use of to solve the resulting nonlinear system of equations for the required time level starting from the initial condition.

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The proposed method is tested on an example problem. The numerical solution is seen to be in a very good agreement with the exact solution in terms of maximum absolute error with a small number of discretization points and the solution shows superior accuracy at large time levels tending to steady-state. Since an implicit method is used, the solution does not depend on the time increment and comparing to the previous studies [3], [4] larger time increments could be used.

II. Problem Definition

The one-dimensional density dependent nonlinear reaction-diffusion equation modelling the insect dispersal, is given in the form [1]

$$\dot{u} = \frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right) + f(u). \tag{2}$$

In equation (2) the upper dot is used for the time derivative, $D(u) = D_0 u^m$ ($D_0,\ m$ are positive constants) is the diffusion coefficient which depends on the population u and $f(u) = ku^p(1-u^q)$ ($p,\ q$ are positive constants) represents the growth term. After a suitable rescaling of t and x, (2) takes the following general form [1]

$$\dot{u} = \frac{\partial}{\partial x}\left(u^m\frac{\partial u}{\partial x}\right) + u^p(1-u^q) \tag{3}$$

or

$$\dot{u} = u^m\frac{\partial^2 u}{\partial x^2} + mu^{m-1}\left(\frac{\partial u}{\partial x}\right)^2 + u^p(1-u^q). \tag{4}$$

Equations of the form (4) are complicated to solve and most often only the numerical solutions are available. In the following sections, a numerical procedure using the combination of DQM and implicit Euler method for the solution of the Cauchy problem defined by (4), i.e.,

$$\begin{cases} \dot{u} = u^m\dfrac{\partial^2 u}{\partial x^2} + mu^{m-1}\left(\dfrac{\partial u}{\partial x}\right)^2 + u^p(1-u^q) \\ \qquad\qquad\qquad\qquad x\in\mathbb{R},\quad t>0, \\ u(x,0) = u_0(x) \qquad\quad x\in\mathbb{R} \end{cases} \tag{5}$$

is proposed. In (5), $u_0(x)$ is the given initial condition depending on the space variable x.

III. DQM Formulation

For the DQM discretization of the spatial derivatives of density dependent nonlinear reaction-diffusion equation given in Section II polynomial based DQM is used. To this end, one should assume that N-th degree polynomials are used to approximate the first and second order spatial derivatives of the solution. Then the DQM approach at a grid point x_i reads as

$$u(x_i) = \sum_{j=1}^{N} r_j(x_i)\,u(x_j), \tag{6}$$

$$u_x(x_i) = \sum_{j=1}^{N} r_j^{(1)}(x_i)\,u(x_j), \tag{7}$$

$$u_{xx}(x_i) = \sum_{j=1}^{N} r_j^{(2)}(x_i)\,u(x_j), \tag{8}$$

where $i = 1,...,N$, N is the number of grid points in the whole domain and $r_j(x)$'s are the Lagrange interpolated polynomials. In equations (7) and (8), $r_j^{(n)}(x_i)\quad (n=1,2)$ are the weighting coefficients at the grid points $x = x_i$ $(i = 1,2,...,N)$ to be determined by DQM by using a practical notation [9] and are given as follows;

$$r_j^{(1)}(x_i) = \frac{M^{(1)}(x_i)}{(x_i - x_j)\,M^{(1)}(x_j)} \quad i\neq j,\quad i,j = 1,2,...,N, \tag{9}$$

$$r_i^{(1)}(x_i) = -\sum_{j=1,i\neq j}^{N} r_j^{(1)}(x_i), \tag{10}$$

$$M^{(1)}(x_j) = \prod_{k=1,k\neq j}^{N}(x_j - x_k), \tag{11}$$

$$r_j^{(2)}(x_i) = 2r_j^{(1)}(x_i)\left(r_i^{(1)}(x_i) - \frac{1}{x_i - x_j}\right) \quad \begin{matrix} i\neq j, \\ i,j = 1,2,...,N, \end{matrix} \tag{12}$$

$$r_i^{(2)}(x_i) = -\sum_{j=1,i\neq j}^{N} r_j^{(2)}(x_i). \tag{13}$$

When the DQM is applied to discretize the spatial derivatives of the density dependent nonlinear reaction-diffusion equation, then one has the following nonlinear system of ODEs

$$\dot{u}_i = u_i^m\sum_{j=1}^{N} r_{ij}^{(2)}u_j + mu_i^{m-1}\left(\sum_{j=1}^{N} r_{ij}^{(1)}u_j\right)^2 + u_i^p(1-u_i^q) \tag{14}$$

where $r_{ij}^{(2)} = r_j^{(2)}(x_i)$, $u_i = u(x_i)$, $i = 1,2,...,N$.

IV. Time Integration and the Solution Procedure

For the discretization of the time derivative in equation (14) implicit Euler method [10] is used, i.e.,

$$\frac{u_{i,n+1} - u_{i,n}}{\Delta t} = u_{i,n+1}^m\sum_{j=1}^{N} r_{ij}^{(2)}u_{j,n+1} + mu_{i,n+1}^{m-1}\left(\sum_{j=1}^{N} r_{ij}^{(1)}u_{j,n+1}\right)^2 + u_{i,n+1}^p(1-u_{i,n+1}^q) \tag{15}$$

where $'n'$ stands for the n-th time level and $t_n = n\Delta t$, Δt being the time step. Equation (15) can be reorganized as

$$\varphi_{i,n+1} = \varphi_i(u_{1,n+1}, u_{2,n+1}, ..., u_{N,n+1}) = 0 \tag{16}$$

with

$$\varphi_i(u_{1,n+1},u_{2,n+1},...,u_{N,n+1}) = u_{i,n} - u_{i,n+1}$$
$$+\Delta t u_{i,n+1}^m \sum_{j=1}^{N} r_{ij}^{(2)} u_{j,n+1} + m\Delta t u_{i,n+1}^{m-1} \left(\sum_{j=1}^{N} r_{ij}^{(1)} u_{j,n+1} \right)^2$$
$$+\Delta t u_{i,n+1}^p (1 - u_{i,n+1}^q). \tag{17}$$

In order to solve the nonlinear system of equations (16), Newton's method [10] is applied. To this end, the Jacobian matrix is constructed as

$$\mathbf{J}_n = \begin{bmatrix} \frac{\partial \varphi_{1,n}}{\partial u_{1,n}} & \frac{\partial \varphi_{1,n}}{\partial u_{2,n}} & ... & \frac{\partial \varphi_{1,n}}{\partial u_{N,n}} \\ \frac{\partial \varphi_{2,n}}{\partial u_{1,n}} & \frac{\partial \varphi_{2,n}}{\partial u_{2,n}} & ... & \frac{\partial \varphi_{2,n}}{\partial u_{N,n}} \\ ... & ... & ... & ... \\ \frac{\partial \varphi_{N,n}}{\partial u_{1,n}} & \frac{\partial \varphi_{N,n}}{\partial u_{2,n}} & ... & \frac{\partial \varphi_{N,n}}{\partial u_{N,n}} \end{bmatrix}. \tag{18}$$

where

$$(\mathbf{J}_n)_{ii} = -1 + \Delta t u_{i,n}^m r_{ii}^{(2)} + m\Delta t u_{i,n}^{m-1} \sum_{j=1}^{N} r_{ij}^{(2)} u_{j,n}$$
$$+2m\Delta t u_{i,n}^{m-1} \left(\sum_{j=1}^{N} r_{ij}^{(1)} u_{j,n} \right) r_{ii}^{(1)}$$
$$+m(m-1)\Delta t u_{i,n}^{m-2} \left(\sum_{j=1}^{N} r_{ij}^{(1)} u_{j,n} \right)^2$$
$$+\Delta t p u_{i,n}^{p-1} - \Delta t (p+q) u_{i,n}^{p+q-1}$$
$$(\mathbf{J}_n)_{ij} = \Delta t (r_{ij}^{(2)} u_{i,n}^m) + 2m\Delta t u_{i,n}^{m-1} \left(\sum_{j=1}^{N} r_{ij}^{(1)} u_{j,n} \right) r_{ij}^{(1)},$$
$$i \neq j, \quad i,j = 1,2,...,N. \tag{19}$$

Then the solution of the density dependent nonlinear reaction-diffusion equation is found at each time step via solving the linear system of equations

$$\mathbf{J_n} \Delta \mathbf{u}_n = -\mathbf{\Phi}_n \tag{20}$$

for $\Delta \mathbf{u}_n$ where

$$\Delta \mathbf{u}_n = \mathbf{u}_{n+1} - \mathbf{u}_n \tag{21}$$

starting with the initial condition u_0 given in (5). Here $\mathbf{u}_n$ is the vector containing the solution at the discretized points for the n-th time level and $\mathbf{\Phi}$ is the $N \times 1$ vector with components $\varphi_{i,n}$ $(i = 1,2,...,N)$.

V. Numerical Results

We consider the Cauchy problem defined by the density dependent nonlinear reaction-diffusion equation modelling the insect dispersal including logistic population growth with $p = q = m = 1$ [1] i.e.,

$$\begin{cases} \dot{u} = \dfrac{\partial}{\partial x}\left[u \dfrac{\partial u}{\partial x}\right] + u(1-u) & x \in (-\infty,\infty),\ t > 0 \\ \\ u(x,0) = 1 - e^{x/\sqrt{2}}, & x \in (-\infty,\infty). \end{cases} \tag{22}$$

With this choice of the reaction term ($f(u) = u(1-u)$), the population disperses more rapidly to the regions of lower density as the population gets more crowded [1].

The exact solution to this problem is [1]

$$u(x,t) = 1 - e^{(x-ct)/\sqrt{2}}, \quad c = \frac{1}{\sqrt{2}}. \tag{23}$$

To measure the quality of the numerical solution maximum absolute error τ_n for the n-th time level

$$\tau_n = \max_{1 \leq i \leq N} |u_{exact}(x_i,t_n) - u_{DQM}(x_i,t_n)| \tag{24}$$

is used. In equation (24) $u_{exact}(x_i,t_n)$ and $u_{DQM}(x_i,t_n)$ denote the exact and the numerical solutions obtained by the method proposed in this paper at the grid point x_i (i=1,2,...,N) for the n-th time level, respectively.

In order to compute the solution one has to use a finite interval, which is chosen here as $[-1,1]$. In [9], it is indicated that the use of the nonuniform mesh in the polynomial based DQM gives rise to more stable results. In this study to construct a nonuniform mesh Chebyshev-Gauss-Lobatto (CGL) points are used to discretize the spatial domain. The CGL points are the points with the property $|T_N(x_n)| = 1, \quad n = 1,2,...,N$ where $T_N(x)$ is the N-th degree Chebyshev polynomial and the CGL points are given in [9] as

$$x_n = \cos\left(\frac{(n-1)\pi}{N-1}\right) \qquad n = 1,...,N \tag{25}$$

for the interval $[-1,1]$.

In the computations the advantage of using an implicit scheme has been once more observed. Stability problems are not encountered due to the use of implicit time integration step and larger time increments can be used. Especially, for time levels through steady-state considerably large time steps can be used, e.g. for $t = 30$, $\Delta t = 3.0$ can be taken.

Table I shows the maximum absolute errors for a fixed time ($t = 30.0$) for various numbers of grid points. The accuracies by using $N = 5,8,11$ are almost the same and there is a drop for $N = 15$. From the table, DQM is observed to give very good accuracy with a small number of grid points. For $N = 15$, the drop of accuracy is due to the ill-conditioned Vandermonde-system obtained after the DQM discretization, which is the known nature of DQM for large N.

Table II and III give the comparison of the DQM solution with the exact solution in terms of maximum absolute error given in (24) for small time levels and for the times tending

TABLE I
MAXIMUM ABSOLUTE ERRORS FOR DIFFERENT NUMBER OF GRID POINTS

$t = 30.0$	$N = 5$	$N = 8$	$N = 11$	$N = 15$
τ_n	6.0×10^{-7}	4.7×10^{-7}	1.2×10^{-5}	3.1×10^{-2}

to steady-state, respectively. The computations are carried out with $N = 5$ and it is seen to be enough to obtain the solution with eleven digits accuracy at steady-state.

TABLE II
MAXIMUM ABSOLUTE ERRORS FOR SMALL TIME LEVELS

$N = 5$	$t = 0.01$	$t = 0.1$	$t = 0.5$	$t = 1.0$
τ_n	1.9×10^{-4}	3.6×10^{-3}	5.3×10^{-2}	9.7×10^{-2}

TABLE III
MAXIMUM ABSOLUTE ERRORS FOR INCREASING TIMES

$N = 5$	$t = 5.0$	$t = 12.0$	$t = 20.0$	$t = 50.0$
τ_n	8.4×10^{-2}	2.5×10^{-3}	6.7×10^{-5}	2.8×10^{-11}

Fig. 1 and Fig. 2 exhibit the behaviour of the solution for small times and for the times tending to steady-state, respectively. The steady-state value which is 1 is obtained around $t = 16$. The agreement between the exact and DQM solutions in terms of graphics is very well especially at steady-state.

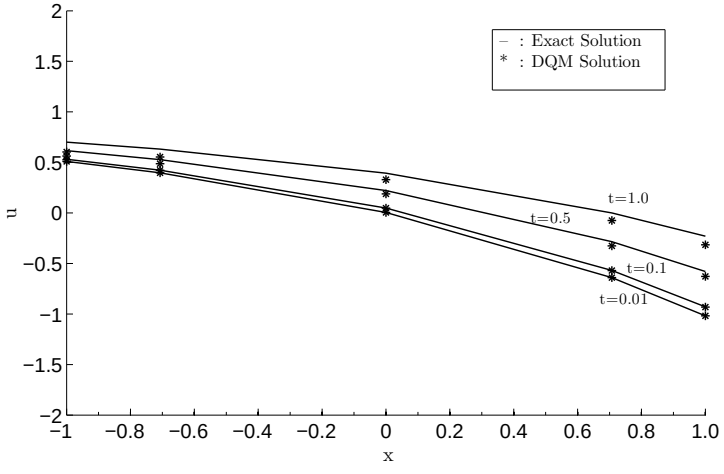

Fig. 1. Solution at small time levels

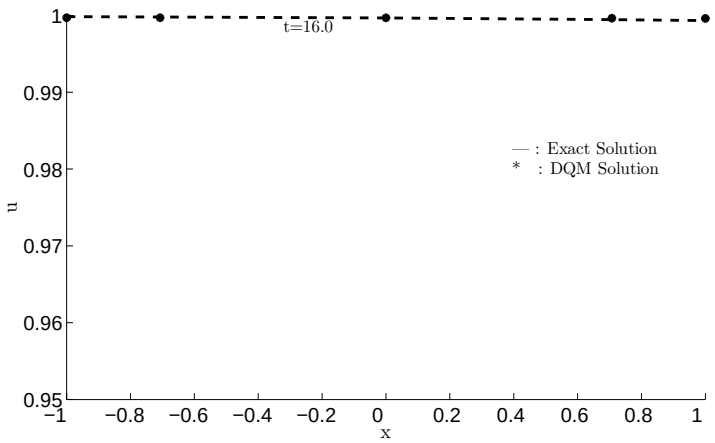

Fig. 2. Solution at steady-state

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