TS Fuzzy Controller to Stochastic Systems

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Abstract—This paper proposes the analysis and design of robust fuzzy control to Stochastic Parametrics Uncertain Linear systems. This system type to be controlled is partitioned into several linear sub-models, in terms of transfer function, forming a convex polytope, similar to LPV (Linear Parameters Varying) system. Once defined the linear sub-models of the plant, these are organized into fuzzy Takagi-Sugeno (TS) structure. From the Parallel Distributed Compensation (PDC) strategy, a mathematical formulation is defined in the frequency domain, based on the gain and phase margins specifications, to obtain robust PI sub-controllers in accordance to the Takagi-Sugeno fuzzy model of the plant. The main results of the paper are based on the robust stability conditions with the proposal of one Axiom and two Theorems.

Keywords—Fuzzy Systems; Robust Stability, Stochastic Control, Stochastic Process

I. INTRODUCTION

THE ultimate goal of a control-system is to build a system that will work in the real environment. Since the real environment may change with time (parametric variations and nonlinearity) or operating conditions may vary (noise and disturbance), the control system must be able to withstand these variations [14]. This fact has motivated, since $1980's$, the proposal of new methodologies for design of robust controllers. In this context, fuzzy systems have been widely used due to flexibility of its structure to incorporate linguistic information (knowledge expert) with numerical information (sensors and actuators measurements), as well as its functional efficiency as universal approximator capable of treat adequately uncertainties, parametric variations and nonlinearity of the plant to be controlled ([8], [15], [21], [17]).

The Fuzzy Logic made a great advance in mid 1970s with some successful results of laboratory experiments after it was initially introduced by Zadeh in 1965 [22]. Mamdani and Assilian [11], controlled a steam engine with fuzzy techniques in 1975 that formed a fundamental frame for fuzzy controllers of the future. In 1985, Takagi and Sugeno [18], brought out a new rule-based modeling technique, which was named after them. These works inspired researchers to develop many fuzzy control applications. Among these, it has the following: Çetin e Demir [2], proposed two input fuzzy PID controller structure with coupled rules for two-degrees of freedom nonlinear quarter car model. The aim of the controller is reduce vehicle body motion and to ensure the comfort of passengers. Simulation results showed that the two input fuzzy PID controller structure is able to good tracking performance in the nonlinear

quarter car model so that ride comfort can be guaranteed. Park et *al.* [12], contributed experimental study on the attitude control of spacecraft using a rotational simulator. For the reaction wheel actuator test, the proposed fuzzy controller was implemented and their performance was evaluated by simulation and experiments. Experimental results revealed the superiority of the proposed fuzzy controller in the presence of unmodeled dynamics, disturbance and nonlinearities such as bearing friction and payload vibration. Cheng et *al.* [4], proposed a fuzzy PID controller with closed-loop optimal fuzzy reasoning (COFR) for wind turbine, which is a strongly nonlinear system that has multivariable and uncertainty. Simulation results showed that fuzzy PID controller with COFR has better performances than PID controller when errors exist, especially robustness performance. Ahmed Rubaai et *al.* [3], implemented and demonstrated in the laboratory a fuzzy PID controller, and its effectiveness in tracking application has also been verified. Experimental results have shown excellent tracking performance of the proposed fuzzy PID controller and have demonstrated the usefulness of the proposed fuzzy PID controller in motor drives with uncertainties. The efficacy of the fuzzy PID controller has been demonstrated by its positive results, when compared with those of the classical PID controller.

In this paper a theoretical approach of robust fuzzy control design based on gain and phase margins specifications for linear systems with stochastic variations in the parameters, as a LPV system, in the continuous time domain, is proposed. A mathematical formulation based on Takagi-Sugeno fuzzy model structure as well as the PDC strategy is presented. Analytical formulas are deduced for the sub-controllers parameters, in the robust fuzzy controller rules base, according to the fuzzy model parameters of the LPV plant to be controlled. Results for the necessary and sufficient conditions for the fuzzy controller design, from the proposed robust methodology, with one axiom and two theorems are presented. The paper is organized as follows: In section II, it is introduced firstly the Linear Parameters Varying Systems in terms of transfer function; secondly the Parallel Distributed Compesation (PDC) Strategy is presented; thirdly the Takagi-Sugeno Fuzzy Systems is presented. In section III, the robust fuzzy PI control design and tuning formulas, based on gain and phase margins specifications, as well as the robust stability analysis of the fuzzy controller, are proposed. In section IV, the robust fuzzy control design and tuning formulas, based on gain and phase margins specifications, as well as the robust stability analysis of the fuzzy controller, are proposed. Finally, conclusions are drawn in section V.

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A. Problem Formulation

This section presents some concepts for the formulation and development of the proposal methodology.

1) Linear Parameters Varying Systems: The notion of LPV systems was first introduced by Shamma and Athans [16], and have gained some attention during the last decade. This class of systems is different from standard linear time-varying counterpart due to the causal dependence of its controller gains on the variations of the plant dynamics. LPV systems are characterized as linear systems that depend on time-varying real parameters. These parameters are assumed to be a *priori* unknown exogenous signal. However, it can be measured or estimated upon operation of the system [2]. The study of LPV systems is motivated by the gain-scheduling control design methodology. This strategy is a popular engineering method used to design controllers for systems with widely varying nonlinear and-or parameter dependent dynamics, i.e. systems for which a single linear time-invariant model is insufficient. However, in spite of numerous successful applications, the construction of the overall control structure invariably calls for the engineering insights of the designer and, more critically, the resulting control laws do not provide any guarantees in the face of rapid changes in the scheduled variables. These difficulties have been the main motivation for the development of modern gain-scheduling control techniques, and have led to some challenging research in the area of the analysis and synthesis of LPV systems [9]. The main advantages brought about by LPV techniques are that the stability and performance of the controlled system are guaranteed, and the interpolation and realization problems associated with conventional gain-scheduling methods are simplified [6]. The application of LPV techniques can be found in wide range of physical systems, such as Missile [13], Aircraft Control [10], Underwater Vehicle [6], Suspension Systems [7], etc.

In this paper, the general form of a LPV system, in terms of transfer function, is described as

$$G(s,\theta) = \frac{b_m(\theta)s^m + b_{m-1}(\theta)s^{m-1} + \cdots + b_1(\theta)s + b_0(\theta)}{a_n(\theta)s^n + a_{n-1}(\theta)s^{n-1} + \cdots + a_1(\theta)s + a_0(\theta)} \quad (1)$$

where $n \geq m$ and $\theta = (\theta_1, \cdots, \theta_k)$ is a parameters set that evolves continuously over time and its range is limited to a compact subset $\Theta \in \Re^k$. In addition, its time derivative is bounded and satisfies the constraint $\underline{\nu}_i \leq \dot{\theta}_i \leq \overline{\nu}_i$, $i = 1, 2, \cdots, k$. For notational purpose, denote $\Omega = \{\upsilon : \underline{\nu}_i \leq \upsilon \leq \overline{\nu}_i, i = 1, 2, \cdots, k\}$, where Ω is a given convex polytope in $\Re^k$ that contains the origin. Given the sets Θ and Ω, we define the parameter ν-variation set as

$$F_\Theta^\Omega = \left\{ \theta \in \ell^1\left(\Re_+, \Re^k\right) : \theta(t) \in \Theta, \dot{\theta} \in \Omega, \forall t \geq 0 \right\} \quad (2)$$

In this paper, this formulation is very efficient to find a LPV control law, which guarantees robust stability to the LPV plant, $G(s,\theta)$, to be controlled.

2) Parallel Distributed Compesation (PDC) Strategy: The history of the so-called parallel distributed compensation (PDC) began with a model-based design procedure proposed by Wang et *al.* [20]. The PDC offers a procedure to design a fuzzy controller from a given T-S fuzzy model. To realize the PDC, a controlled plant is first represented by a T-S fuzzy model. In the PDC design, each control rule is designed from the corresponding rule of a T-S fuzzy model. The designed fuzzy controller shares the same fuzzy sets with the fuzzy model in the premise parts. The figure 2 shows the concept of PDC design.

3) Takagi-Sugeno Fuzzy Systems: The TS fuzzy model, originally proposed by Takagi and Sugeno [18], is composed of a fuzzy **IF-THEN** rule base that partitions a space - usually called the *universe of discourse* - into fuzzy regions described by the *rule antecedents*. The *consequent* of each rule i is a simple functional expression of model inputs and that all fuzzy terms are monotonic functions. In this case, specifically, the TS fuzzy model can be regarded as a mapping from the antecedent (input) space to a convex region (polytope) in the local sub-models space into the consequent, defined by the variants consequents parameters of the plant to be controlled. This property simplifies the analysis of the TS fuzzy model in a context of robust time-variant and linear system for design of controllers with desired characteristics of the closed loop control system or stability analysis.

The $i|^{[i=1,2,\cdots,l]}$-th TS rule, without loss of generality, the following structure:

$$R^{(i)} : \text{ IF } \tilde{x}_1 \text{ is } F^i_{j|\tilde{x}_1} \text{ AND } \cdots \text{ AND } \tilde{x}_n \text{ is } F^i_{j|\tilde{x}_n} \quad (3)$$
$$\text{THEN } \tilde{y}_i = f_i(\tilde{\mathbf{x}})$$

where

$$\tilde{\mathbf{x}}^T = [\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n],$$
$$\tilde{\mathbf{y}}^T = [\tilde{y}_1, \tilde{y}_2, \cdots, \tilde{y}_n],$$

l is the number of fuzzy **IF-THEN** rules. The vector $\tilde{\mathbf{x}} \in \Re^n$ contains the antecedent linguistic variables. Each linguistic variable has its own universe of discourse $U_{\tilde{x}_1}, \cdots, U_{\tilde{x}_n}$ partitioned by fuzzy sets representing the linguistic terms. The variable $\tilde{x}_t|^{[t=1,2,\cdots,n]}$ belongs to the fuzzy set $F^i_{j|\tilde{x}_t}$ with a value $\mu^i_{F_{j|\tilde{x}_t}}$ defined by a membership function $\mu^i_{\tilde{x}_t} : \Re \to [0,1]$, with $\mu^i_{F_{j|\tilde{x}_t}} \in \mu^i_{F_{1|\tilde{x}_t}}, \mu^i_{F_{2|\tilde{x}_t}}, \mu^i_{F_{3|\tilde{x}_t}}, \cdots, \mu^i_{F_{p_{\tilde{x}_t}|\tilde{x}_t}}$, where $p_{\tilde{x}_t}$ is the number of partitions of the universe of discourse associated to the linguistic variable $\tilde{\mathbf{x}}$. The activation degree of h_i for the rule i, is given by:

$$h_i(\tilde{\mathbf{x}}) = \mu^i_{F_{j|\tilde{x}_1^*}} \otimes \mu^i_{F_{j|\tilde{x}_2^*}} \otimes \cdots \mu^i_{F_{j|\tilde{x}_n^*}} \quad (4)$$

where $\tilde{x}_t^*$ is some point in $U_{\tilde{x}_t}$. The normalized activation degree for the rule i, is given by:

$$\gamma_i(\tilde{\mathbf{x}}) = \frac{h_i(\tilde{\mathbf{x}})}{\sum_{\lambda=1}^{l} h_\lambda(\tilde{\mathbf{x}})} \quad (5)$$

where it is assumed that

$$\sum_{\lambda=1}^{l} h_\lambda(\tilde{\mathbf{x}}) > 0,$$

$$h_\lambda(\tilde{\mathbf{x}}) \geq 0, \quad i = 1, 2, \cdots, l$$

And, this normalization implies that

$$\sum_{i=1}^{l} \gamma_i(\tilde{\mathbf{x}}) = 1 \tag{6}$$

The TS fuzzy model response is a weighted sum of the consequent parameters, i.e., a convex linear combination of the local functions (models) f_i, which reads

$$y(\tilde{\mathbf{x}}) = \sum_{i=1}^{l} \gamma_i(\tilde{\mathbf{x}}) f_i(\tilde{\mathbf{x}}) \tag{7}$$

Each linear component $f_i(\tilde{\mathbf{x}})$ is called a *subsystem*. This model can be seen as a Linear Parameters Varying (LPV) System, as defined previously. This property simplifies the analysis of the TS fuzzy model in a context of robust time-variant and linear system for design of controllers with desired characteristics of the closed loop control system or stability analysis.

In this paper is presented an fuzzy robust model based control scheme from the TS fuzzy model structure, the PDC strategy and gain and phase margins robust specifications. In the proposed methodology, the fuzzy controller parameters, with TS structure, are obtained through analytical formulas from the definition of gain and phase margins specifications. The robust fuzzy controller designed and the TS fuzzy model of the LPV plant, with stochastic variations, to be controlled shares the same fuzzy sets, in the antecedents. In the fuzzy inference engine the sub-controller is selected based on the plant dynamic behavior and the gain and phase margins robust specifications. The dynamic system class under analysis for the fuzzy control design, is defined as linear parameters varying, and the structure of the robust control is proposed with the objective to obtain the above robustness characteristics, from generalized analytical formulas.

II. ROBUST FUZZY PI CONTROL

A. TS fuzzy model for a first-order LPV plant

The TS fuzzy inference system for a first-order LPV plant, $G_p(s)$, presents in the $i|^{[i=1,2,...,l]}$-th rule, without loss of generality, the following structure:

$$R^{(i)}: \text{ IF } \tilde{\tau} \text{ is } F_{k|\tilde{\tau}}^{i} \text{ AND } \tilde{K}_p \text{ is } G_{k|\tilde{K}_p}^{i} \tag{8}$$

$$\text{THEN } G_p^i(s) = \frac{K_i^p}{1 + s\tau^i} e^{-sL}$$

where time constant $\tilde{\tau}$ and the gain $\tilde{K}_p$ represent the linguistic variables of the antecedents of the fuzzy model. The activation degree of h_i for the rule i, is given by:

$$h_i(\tilde{\tau}, \tilde{K}_p) = \mu_{F_{k|\tilde{\tau}^*}}^{i} \otimes \mu_{G_{k|\tilde{K}_p^*}}^{i} \tag{9}$$

The normalized activation degree for the rule i, is given by:

$$\gamma_i(\tilde{\tau}, \tilde{K}_p) = \frac{h_i(\tilde{\tau}, \tilde{K}_p)}{\sum_{\lambda=1}^{l} h_\lambda(\tilde{\tau}, \tilde{K}_p)} \tag{10}$$

And, this normalization implies

$$\sum_{i=1}^{l} \gamma_i(\tilde{\tau}, \tilde{K}_p) = 1 \tag{11}$$

Therefore, the TS fuzzy model, $G_p(\tilde{\tau}, \tilde{K}_p, s)$, of the LPV plant is a weighted sum of first order linear sub-models, as follow:

$$G_p(\tilde{\tau}, \tilde{K}_p, s) = \sum_{i=1}^{l} \gamma_i(\tilde{\tau}, \tilde{K}_p) \cdot \frac{K_p^i}{1 + s\tau^i} e^{-sL} \tag{12}$$

B. TS fuzzy model for a PI-LPV Controller

The TS fuzzy inference system proposed for the PI-LPV controller, $G_c(s)$, whereas the definition of parallel distributed compensation, presents in the $j|^{[j=1,2,...,l]}$-th rule, without loss of generality, is given by:

$$R^{(j)}: \text{ IF } \tilde{\tau} \text{ is } F_{k|\tilde{\tau}}^{j} \text{ AND } \tilde{K}_p \text{ is } G_{k|\tilde{K}_p}^{j} \tag{13}$$

$$\text{THEN } G_c^j(s) = K_c^j \left(1 + \frac{1}{sT_I^j}\right)$$

The activation degree h_j for the rule j, is given by:

$$h_j(\tilde{\tau}, \tilde{K}_p) = \mu_{F_{k|\tilde{\tau}^*}}^{j} \otimes \mu_{G_{k|\tilde{K}_p^*}}^{j} \tag{14}$$

where $\tilde{\tau}^*$ and $\tilde{K}_p^*$ are some point in $U_{\tilde{\tau}}$ and $U_{\tilde{K}_p}$, respectively. The normalized activation degree for the rule j, is given by:

$$\gamma_j(\tilde{\tau}, \tilde{K}_p) = \frac{h_j(\tilde{\tau}, \tilde{K}_p)}{\sum_{\lambda=1}^{l} h_\lambda(\tilde{\tau}, \tilde{K}_p)} \tag{15}$$

And, this normalization implies

$$\sum_{j=1}^{l} \gamma_j(\tilde{\tau}, \tilde{K}_p) = 1 \tag{16}$$

Therefore, the TS fuzzy model for the fuzzy PI-LPV controller, $G_c(\tilde{\tau}, \tilde{K}_p, s)$, is a weighted sum of the local PI sub-controllers, as follows:

$$G_c(\tilde{\tau}, \tilde{K}_p, s) = \sum_{j=1}^{l} \gamma_j(\tilde{\tau}, \tilde{K}_p) \cdot K_c^j \left(1 + \frac{1}{sT_I^j}\right) \tag{17}$$

The compensated open-loop fuzzy model, according to the PDC strategy, with the controller and the plant, from the equations 12 and 17, respectively, is

$$G_c(\tilde{\tau}, \tilde{K}_p, s) G_p(\tilde{\tau}, \tilde{K}_p, s) = \sum_{j=1}^{l} \sum_{i=1}^{l} \gamma_j(\tilde{\tau}, \tilde{K}_p) \gamma_i(\tilde{\tau}, \tilde{K}_p) \times \tag{18}$$

$$\times \frac{K_c^j K_p^i \left(1 + sT_I^j\right)}{sT_I^j (1 + s\tau^i)} e^{-sL}$$

C. Robust Stability Based on Gain and Phase Margins

Gain and phase margins have always served as important measures of robustness. It is known that phase margin is related to the damping of the control system, and the gain margin is related to how the control system is stable, this is, how many the gain of the plant to be controlled can vary so that the control system goes to instability ([1],[19]). Denote the process and the controller transfer function by $G_p(s)$ and $G_c(s)$, and the specified gain and phase margins by A_m and ϕ_m, respectively, The formulas for gain margin and phase margin are as follows::

$$arg\left[G_c(\tilde{\tau}, \tilde{K}_p, j\omega_p)G_p(\tilde{\tau}, \tilde{K}_p, j\omega_p)\right] = -\pi \quad (19)$$

$$A_m = \frac{1}{\left|G_c(\tilde{\tau}, \tilde{K}_p, j\omega_p)G_p(\tilde{\tau}, \tilde{K}_p, j\omega_p)\right|} \quad (20)$$

$$\left|G_c(\tilde{\tau}, \tilde{K}_p, j\omega_g)G_p(\tilde{\tau}, \tilde{K}_p, j\omega_g)\right| = 1 \quad (21)$$

$$\phi_m = arg\left[G_c(\tilde{\tau}, \tilde{K}_p, j\omega_g)G_p(\tilde{\tau}, \tilde{K}_p, j\omega_g)\right] + \pi \quad (22)$$

where the gain margin is given by equations 19 and 20, and the phase margin is given by equations 21 and 22, respectively. The frequency ω_p, in which the Nyquist curve has a phase $-\pi$ is the phase crossover frequency, and the frequency ω_g, in which the curve Nyquist has a amplitude of 1 is the gain crossover frequency. Replacing the equation 18 in 19-22, it has:

$$l\left[\sum_{i=1}^{l}\left(\arctan\left(\omega_p T_I^i\right) - \arctan\left(\omega_p \tau^i\right)\right) - \frac{\pi}{2} - \omega_g L\right] = -\pi \quad (23)$$

$$A_m = \frac{1}{\sum_{i=1}^{l}\sum_{j=1}^{l}\gamma_i\left(\tilde{\tau}, \tilde{K}_p\right)\gamma_j\left(\tilde{\tau}, \tilde{K}_p\right)\left(\frac{K_c^j K_p^i}{\omega_p T_I^j}\right)\left(\sqrt{\frac{\left(\omega_p T_I^j\right)^2+1}{\left(\omega_p \tau^i\right)^2+1}}\right)} \quad (24)$$

$$\sum_{i=1}^{l}\sum_{j=1}^{l}\gamma_i\left(\tilde{\tau}, \tilde{K}_p\right)\gamma_j\left(\tilde{\tau}, \tilde{K}_p\right)\left(\frac{K_c^j K_p^i}{\omega_g T_I^j}\right)\left(\sqrt{\frac{\left(\omega_g T_I^j\right)^2+1}{\left(\omega_g \tau^i\right)^2+1}}\right) = 1 \quad (25)$$

$$\phi_m = l\left[\sum_{i=1}^{l}\left(\arctan\left(\omega_g T_I^i\right) - \arctan\left(\omega_g \tau^i\right)\right) - \frac{\pi}{2} - \omega_g L\right] + \pi \quad (26)$$

For a given linear sub-model, $G_p^i(s, \tilde{K}_p^i, \tilde{\tau}^i)$, and gain and phase margins specifications (A_m, ϕ_m), the equations 23-26 can be used to determine the parameters of the PI sub-controllers, $G_c^j(s, K_c^j, T_I^j)$, in the crossover frequency (ω_p, ω_g) numerically, but not analytically, due to presence of the nonlinear $\arctan$ function. However, an analytical solution can be obtained approximating the $\arctan$ function, as follows:

$$\arctan x \approx \begin{cases} \frac{1}{4}\pi x & (|x| \leq 1), \\ \frac{1}{2}\pi - \frac{\pi}{4x} & (|x| > 1) \end{cases} \quad (27)$$

The numerical solution of the equations 23-26 shows that for $\tau^i > 3L$, $x \gg 1$ where x is one of the $\omega_p T_I^j$, $\omega_p \tau^i$, $\omega_g T_I^j$ or $\omega_g \tau^i$. Therefore, using the approximation of $\arctan$

function in the case $|x| > 1$, the equations 24 and 25 are given by

$$\sum_{i=1}^{l}\sum_{j=1}^{l}\gamma_i\left(\tilde{\tau}, \tilde{K}_p\right)\gamma_j\left(\tilde{\tau}, \tilde{K}_p\right) \times \frac{A_m}{\omega_p}\left(\frac{K_c^j K_p^i}{\tau^i}\right) = 1 \quad (28)$$

$$\sum_{i=1}^{l}\sum_{j=1}^{l}\gamma_i\left(\tilde{\tau}, \tilde{K}_p\right)\gamma_j\left(\tilde{\tau}, \tilde{K}_p\right) \times \left(\frac{K_c^j K_p^i}{\omega_g \tau^i}\right) = 1 \quad (29)$$

respectively. Using the same approach, the equations 23 and 26 are given by:

$$l\left[\sum_{i=1}^{l}\left(\frac{\pi}{4\omega_p \tau^i} - \frac{\pi}{\omega_p T_I^i} - \frac{\pi}{2}\right) - \omega_p L\right] = -\pi \quad (30)$$

$$\phi_m = l\left[\sum_{i=1}^{l}\left(\frac{\pi}{4\omega_g \tau^i} - \frac{\pi}{\omega_g T_I^i} - \frac{\pi}{2}\right) - \omega_g L\right] + \pi \quad (31)$$

respectively. The dimensionless quantity L/τ, which is useful for characterizing processes, is defined in [1] as the *normalized dead time*. Therefore, the analytical solution for the tuning of the PI sub-controllers parameters, $G_c^i(s)\big|^{[i=1,2,...,l]}$, according to equations 23 - 26, is given by:

$$\begin{bmatrix} \sum_{i=1}^{l}\gamma_i\left(\tilde{\tau}, \tilde{K}_p\right)\left(\frac{K_p^i}{\tau^i}\right) & \cdots & \sum_{i=1}^{l}\gamma_i\left(\tilde{\tau}, \tilde{K}_p\right)\left(\frac{K_p^i}{\tau^i}\right) \\ \sum_{i=1}^{l}\gamma_i\left(\tilde{\tau}, \tilde{K}_p\right)\left(\frac{K_p^i}{\tau^i}\right) & \cdots & \sum_{i=1}^{l}\gamma_i\left(\tilde{\tau}, \tilde{K}_p\right)\left(\frac{K_p^i}{\tau^i}\right) \end{bmatrix} \times$$

$$\times \begin{bmatrix} \gamma_1\left(\tilde{\tau}, \tilde{K}_p\right) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \gamma_i\left(\tilde{\tau}, \tilde{K}_p\right) \end{bmatrix} \times \quad (32)$$

$$\times \begin{bmatrix} K_c^1 \\ \vdots \\ K_c^i \end{bmatrix} = \begin{bmatrix} \frac{\omega_p}{A_m} \\ \omega_g \end{bmatrix}$$

and

$$\begin{bmatrix} l\frac{\pi}{\omega_p} & \cdots & l\frac{\pi}{\omega_p} \\ l\frac{\pi}{\omega_g} & \cdots & l\frac{\pi}{\omega_g} \end{bmatrix} \begin{bmatrix} \left(T_I^1\right)^{-1} \\ \left(T_I^2\right)^{-1} \\ \vdots \\ \left(T_I^l\right)^{-1} \end{bmatrix} = \quad (33)$$

$$= \begin{bmatrix} l\left\{\sum_{i=1}^{l}\left(\frac{\pi}{4\omega_p \tau^i}\right) - \frac{\pi}{2} - \omega_p L\right\} + \pi \\ l\left\{\sum_{i=1}^{l}\left(\frac{\pi}{4\omega_g \tau^i}\right) - \frac{\pi}{2} - \omega_g L\right\} - \phi_m + \pi \end{bmatrix}$$

where ω_p is given by

$$\omega_p = \frac{A_m \phi_m + \frac{1}{2}\pi A_m(A_m - 1)}{(A_m^2 - 1)L} \quad (34)$$

1) Robustness and Stability Analysis: For the design of fuzzy PI controller, from equations 32-34, respectively, based on the gain and phase margins specifications, the following Axiom and Theorems are proposed:

The linear sub-models, $G_p^i(s)\big|^{[i=1,2,...,l]}$, of the LPV plant, are necessarily of minimum phase, i.e., all poles of the characteristic equation are placed in the left half-plane of the complex plane.

Theorem 1: *Each robust PI sub-controller, $G_c^j(s)\big|^{[j=1,2,...,l]}$, guarantee the gain and phase margins specifications for the linear sub-model, $G_p^i(s)\big|^{[i=1,2,...,l]}$ with $i = j$, of the LPV plant to be controlled.*

Proof: The normalized activation degree, in a given operating point, on the rules base of the robust PI fuzzy controller, satisfies the following condition:

$$\sum_{i=1}^{l} \gamma_j\left(\tilde{\tau},\tilde{K}_p\right) = 1 \tag{35}$$

The total normalized activation degree, for a simple p-th rule activated , where $1 \le p \le l$, is given by

$$0 + \ldots + \gamma_p\left(\tilde{\tau},\tilde{K}_p\right) + 0 + \ldots + 0 = 1 \tag{36}$$

$$\gamma_p\left(\tilde{\tau},\tilde{K}_p\right) = 1 \tag{37}$$

Based on the Parallel Distributed Compensation strategy, in which the robust fuzzy PI controller and the fuzzy model of the plant to be controlled have the same antecedent, it has

$$\begin{bmatrix} \gamma_p\left(\tilde{\tau},\tilde{K}_p\right)\left(\frac{K_p^p}{\tau^p}\right) & \cdots & \gamma_p\left(\tilde{\tau},\tilde{K}_p\right)\left(\frac{K_p^p}{\tau^p}\right) \\ \gamma_p\left(\tilde{\tau},\tilde{K}_p\right)\left(\frac{K_p^p}{\tau^p}\right) & \cdots & \gamma_p\left(\tilde{\tau},\tilde{K}_p\right)\left(\frac{K_p^p}{\tau^p}\right) \end{bmatrix} \times$$

$$\times \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \gamma_1\left(\tilde{\tau},\tilde{K}_p\right) & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ K_c^p \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\omega_p}{A_m} \\ \omega_g \end{bmatrix} \tag{38}$$

Solving the equation 38 for K_c, it has

$$\gamma_p\left(\tilde{\tau},\tilde{K}_p\right)\left(\frac{K_p^p}{\tau^p}\right)\gamma_p\left(\tilde{\tau},\tilde{K}_p\right)(K_c^p) = \frac{\omega_p}{A_m} \tag{39}$$

and

$$\gamma_p\left(\tilde{\tau},\tilde{K}_p\right)\left(\frac{K_p^p}{\tau^p}\right)\gamma_p\left(\tilde{\tau},\tilde{K}_p\right)(K_c^p) = \omega_g \tag{40}$$

Isolating K_c^p, the equation 39, is given by:

$$K_c^p = \left(\frac{\tau^p}{K_p^p}\right)\left(\frac{\omega_p}{A_m}\right)\left(\frac{1}{\gamma_p\left(\tilde{\tau},\tilde{K}_p\right)^2}\right) \tag{41}$$

To obtain the parameter T_I^p, in a given time, as defined previously, it has:

$$\begin{bmatrix} l\frac{\pi}{\omega_p} & \cdots & l\frac{\pi}{\omega_p} \\ l\frac{\pi}{\omega_g} & \cdots & l\frac{\pi}{\omega_g} \end{bmatrix} \begin{bmatrix} 0 \\ (T_I^p)^{-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \tag{42}$$

$$= \begin{bmatrix} l\left(\frac{\pi}{4\omega_p\tau^p} - \frac{\pi}{2} - \omega_p L\right) + \pi \\ l\left(\frac{\pi}{4\omega_g\tau^p} - \frac{\pi}{2} - \omega_g L\right) + \pi - \phi_m \end{bmatrix}$$

which results in

$$l\frac{\pi}{\omega_p}\frac{1}{T_I^p} = l\left(\frac{\pi}{4\omega_p\tau^p} - \frac{\pi}{2} - \omega_p L\right) + \pi \tag{43}$$

and

$$l\frac{\pi}{\omega_g}\frac{1}{T_I^p} = l\left(\frac{\pi}{4\omega_g\tau^p} - \frac{\pi}{2} - \omega_g L\right) + \pi - \phi_m \tag{44}$$

Isolating ϕ_m, the equation 44, is given by:

$$\phi_m = l\left(\frac{\pi}{4\omega_g\tau^p} - \frac{\pi}{\omega_g}\frac{1}{T_I^p} - \frac{\pi}{2} - \omega_g L\right) + \pi \tag{45}$$

Substituting the equation 39 in 18, it has:

$$\gamma_p\left(\tilde{\tau},\tilde{K}_p\right)\gamma_p\left(\tilde{\tau},\tilde{K}_p\right)\left(\frac{K_p^p A_m}{\tau^p\omega_p}\right)\left(\frac{\tau^p\omega_p}{K_p^p A_m}\right) \times \tag{46}$$

$$\times \left(\frac{1}{\gamma_p\left(\tilde{\tau},\tilde{K}_p\right)\gamma_p\left(\tilde{\tau},\tilde{K}_p\right)}\right) = 1$$

therefore

$$A_m = A_m \tag{47}$$

Assuming, in a given time, the total activation of a simple rule p, as defined previously, in equation 36, we have:

$$\phi_m = l\left(\frac{\pi}{4\omega_g\tau^p} - \frac{\pi}{\omega_g T_I^p} - \frac{\pi}{2} - \omega_g L\right) + \pi \tag{48}$$

Comparing the equation 48 with 45, it has

$$\phi_m = \phi_m \tag{49}$$

Theorem 2: *Each robust PI sub-controller, $G_c^j(s)\big|^{[j=1,2,...,l]}$, guarantee the stability for all linear sub-models, $G_p^i(s)\big|^{[i=1,2,...,l]}$, of the LPV plant to be controlled.*

Proof: The closed-loop transfer function is given by:

$$G_{MF}\left(s,\tilde{\tau},\tilde{K}_p\right) = \sum_{j=1}^{l}\sum_{i=1}^{l}\gamma_j\left(\tilde{\tau},\tilde{K}_p\right)\gamma_i\left(\tilde{\tau},\tilde{K}_p\right) \times \tag{50}$$

$$\times \frac{K_c^j K_p^i\left(1 + sT_I^j\right)}{\left[sT_I^j\left(1 + s\tau^i\right) + K_c^j K_p^i\left(1 + sT_I^j\right)\right]}e^{-sL}$$

For the stability condition, the characteristic equation of the closed-loop transfer function, given in equation 50, must have roots (poles) in the left half-plane of the complex plane (negative real part). Therefore, it has

$$\sum_{i=1}^{l}\sum_{j=1}^{l}\gamma_i\left(\tilde{\tau},\tilde{K}_p\right)\gamma_j\left(\tilde{\tau},\tilde{K}_p\right)\times \quad (51)$$

$$\times\left[\tau^i T_I^j s^2 + \left(T_I^j + K_p^i K_c^j T_I^j\right)s + \left(K_p^i K_c^j\right)\right] = 0$$

By application of the Routh Stability Criterion [5] in 51, it has

$$
\begin{array}{c|cc}
s^2 & \tau^i T_I^j & K_p^i K_c^j \\
s^1 & (T_I^j + K_p^i K_c^j T_I^j) & 0 \\
s^0 & K_p^i K_c^j &
\end{array}
\quad (52)
$$

And, it is necessary that all terms of the first column are positive:

$$\tau^i T_I^j > 0 \quad (53)$$

$$(T_I^j + K_p^i K_c^j T_I^j) > 0 \quad (54)$$

$$K_p^i K_c^j > 0 \quad (55)$$

Since the parameters of the stable sub-models of the LPV plant to be controlled (τ^i e K_p^i), according to **Axiom 1**, are positive as well as the gain and phase margins specifications (A_m e ϕ_m), from Equations 32-34, the values of the robust gain scheduled fuzzy PI controller parameters (K_c^j e T_I^j) are positive. Therefore, the inequalities, in equations 53-55, are satisfied, and each robust PI sub-controller guarantee the stability for all sub-models of the LPV plant to be controlled.

III. Conclusion

This paper presented a proposal for analysis and design of robust fuzzy control, with PI structure, based on robust gain and phase margins specifications for linear systems with stochastic variations in the parameters, as a LPV plant. From the proposed analysis and design, it has the following final remarks: A mathematical formulation from the Takagi-Sugeno fuzzy model structure, based on parallel distributed compensation strategy was presented; the TS fuzzy model, due to the flexibility to incorporate in its structure the linear sub-models of the LPV plant made possible, via PDC strategy, the design of robust PI and PID sub-controllers; the proposed Axiom and Theorems guaranteed the robust stability, since all formulation and analysis were made in the frequency domain, based on gain and phase margins specifications.

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